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ANALYSIS OF A RESIDUAL ERROR ESTIMATOR IN GOAL-ORIENTED ADAPTIVITY

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Abstract. When solving a boundary problem numerically, it is necessary to control the quality of the approximate solution. This control can be achieved by an adaptive procedure of the mesh to efficiently approximate the numerical solution of the problem. While in the past an energy measure was generally used to define the error, at present, it is preferred that the error measure is in quantities of interest. The error representation in these quantities of interest involves the solutions of the original problem and an associated adjoint problem, which can be combined in different ways, therefore, the error representation is not unique. In this article we analyze three representations of the error combined with an error estimator based on local problem solving using submeshes that discretize each of the elements. An optimal remeshing criterion for quantities of interest is used to define an adaptive procedure. An optimal remeshing criterion is used to define a goal-oriented adaptivity procedure. Numerical experimentation shows the efficiency of the estimator and the convergence properties of the adaptive procedure.

1. Introduction

It is common to use the finite element method (FEM) to obtain numerical approximations solution for many mathematical models that arise from engineering or science problems to be obtained from the finite element method (FEM). However, mainly due to the complexity of the problems, the numerical solution must be certified, i.e. it must be endowed by an error bound. This control can be achieved through an adaptive procedure of the mesh to efficiently approximate the numerical solution of the problem. While, in the past, this control was achieved by using an energy norm of error, today, it is preferred that the measurement of error be based on quantities of interest. The quantities of interest represent physical quantities of the solution such as averages, flow rates, velocities, or shear stress at a point. Mathematically, the quantity of interest is characterized by a linear functional $\mathcal{J}(u)$ on the space of functions to which the solution belongs.

The error representation is given by a combination of the solution of the original (primal) problem and the solution of a problem attached (dual) to the primal problem. In the dual problem, the linear functional $\mathcal{J}(u)$ appears on the right-hand side of the weak equation and plays the role of an external load. The combination

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of error estimates in energy norm for the primal and dual problems allows us to estimate error bounds on the quantity of interest [1, 3, 4, 5, 6].

There are two alternatives to estimate the error in the energy norm, which correspond to two large families of methods. First, the error estimators by post-processing, originally introduced by Zienkiewicz and Zhu [7] to estimate stresses in the energy norm. The behavior of smoothing techniques to define error estimates in quantities of interest is analyzed in Calderón and Díez [8], in this case, displacements were estimated instead of stresses. The second alternative is the residual type error estimators, introduced by Babuška and Rheinboldt [9]. These approximate the error by solving local problems, where the source term is given by the weak residual, see among others [10, 11, 12, 13].

In general, residual estimators approximate the boundary conditions of local problems by distributing the flow jumps between adjacent elements, achieving a certain balance for each local problem [14, 10]. However, Díez et al. [15, 16], manage to avoid the calculation of the flow jump and all its drawbacks. Following these ideas, this paper generalizes this residual estimator to quantities of interest.

Once the error has been estimated, a fundamental component within an adaptive process is given by the remeshing criterion. Therefore, it is necessary to define and analyze remeshing criteria for the adaptive process in goal-oriented error estimation. In this context, each author has followed his own recipe, often based on heuristic considerations and without any requirement on the design and optimality of the mesh, or on the representation of the error used [5, 17, 18]. Calderón and Díez in [19], define and analyze remeshing criteria for quantities of interest and optimize one of the proposed criteria. On the other hand, in [2], the authors designed a goal-oriented adaptive algorithm for time-dependent PDEs employing explicit methods in time.

In this work, we derive three error representation of the quantity of interest and develop a goal-oriented adaptive process for elliptic boundary value problems following the ideas presented in [2]. Although the elliptic problems considered in this paper can be seen as a particular case to those studied in [2], here we derive three representations of the error in the quantity of interest and develop and analyze an adaptive process for those error representations. Also, to estimate the error, we developed a residual error estimator by solving local problems, instead of using reference solutions obtained by solving global problems on meshes with elements split in half, the efficiency of the residual error estimator on the quantity of interest is analyzed. The meshes generated by the adaptive process are constructed from an optimal remeshing criterion for quantities of interest.

The outline of this article is the following: section 2 describes the primal and dual problems and their discretizations, also defines the error equation and derives three error representation. In section 3 we introduce the residual error estimator and discuss its implementation. In section 4 we develop the optimal remeshing criterion that we are going to use in our goal-oriented adaptive process. Section 5 defines the algorithm for the adaptive process. In section 6 we present the numerical results in $2D$ space obtained with the goal-oriented adaptive process. Finally, section 7 summarizes the conclusions and possible extensions of this work.

2. Problem statement

In this section, we consider the weak formulation of a boundary value problem in general to obtain the error equation, the error in the quantity of interest, and dual problem.

2.1. Model problem and primal problem. Let $\Omega \subset \mathbb{R}^d$, where $d \in \{1, 2, 3\}$, an open bounded domain. For the weak formulation, we need defined the following spaces:

$$\begin{aligned} \mathscr{V} &:= H^1(\Omega) = \left\{ u \in L^2(\Omega) : \nabla u \in L^2(\Omega) \right\} \\ \mathscr{V}_0 &:= H^1_0(\Omega) = \left\{ u \in L^2(\Omega) : \nabla u \in L^2(\Omega), u = 0 \text{ on } \partial\Omega \right\}. \end{aligned}$$

So we use the following weak formulation for a boundary value problem (with homogeneous boundary values)

$$\begin{cases} \text{Find } u \in \mathscr{V} \text{ such that} \\ \mathcal{B}(u, v) = \ell(v), \quad \forall v \in \mathscr{V}_0, \end{cases} \tag{2.1}$$

where $\mathcal{B}(\cdot, \cdot)$ is a continuous coercive bilinear form on $\mathscr{V}_0$ and ℓ is a linear form on $\mathscr{V}_0$ that belongs to the dual space $\mathscr{V}'$, and we use the energy norm. So, by the Lax-Milgram theorem, the weak formulation (2.1) has a unique solution. We define the problem (2.1) as *primal problem*.

2.2. Discrete problem and error equation. In order to approximate the solution of (2.1) we use finite elements for the discretization in Ω. We denote by Ω_k, $k = 1, \ldots, \mathtt{n_{elem}}$ the elements of the mesh. Related to the discretization of (2.1), we need defined the followings space:

$$\begin{aligned} \mathscr{V}_{\mathrm{H}} &:= \left\{ u \in \mathscr{V} : u_{|_{\Omega_k}} \in P_r\left(\overline{\Omega_k}\right), \forall k = 1, \ldots \mathtt{n_{elem}} \right\} \\ \mathscr{V}_{\mathrm{H0}} &:= \{ u \in \mathscr{V}_{\mathrm{H}} \text{ such that } u = 0 \text{ on } \partial\Omega \} \end{aligned}$$

where $P_r(\cdot)$ is the space of all polynomials with degree less than or equal to r on Ω_k with values in $\mathbb{R}$. $\mathscr{V}_{\mathrm{H}}$ is a finite-dimensional subspace of $\mathscr{V}$ generated by the continuous, piecewise functions defined over each finite element Ω_k, and $\mathscr{V}_{\mathrm{H0}}$ is the corresponding discrete subspace of $\mathscr{V}_0$. Then, we define the discrete problem of (2.1) as

$$\begin{cases} \text{Find } u_{\mathrm{H}} \in \mathscr{V}_{\mathrm{H}} \text{ such that} \\ \mathcal{B}(u_{\mathrm{H}}, v) = \ell(v), \quad \forall v \in \mathscr{V}_{\mathrm{H0}} \subset \mathscr{V}_0, \end{cases} \tag{2.2}$$

The error of the primal problem $e := u - u_{\mathrm{H}}$, belongs to space $\mathscr{V}_0$, and satisfies the error equation

$$\mathcal{B}(e, v) = \ell(v) - \mathcal{B}(u_{\mathrm{H}}, v) =: \mathcal{R}^{\mathtt{P}}(v), \quad \forall v \in \mathscr{V}_0, \tag{2.3}$$

where $\mathcal{R}^{\mathtt{P}}$ defines *the weak residual* associate with the approximate solution u_{H} of the primal problem (2.1). $\mathcal{R}^{\mathtt{P}}$ is a linear functional in $\mathscr{V}_0'$, it depends on the given date and the finite elements solution u_{H}.

2.3. Quantity of interest and dual problem. We want to control the quality of the numerical solution u_{H} by a quantity of interest, which is represented by a linear functional $\mathcal{J} : \mathscr{V}_0 \longrightarrow \mathbb{R}$. The precision of $\mathcal{J}(u_{\text{H}})$ can be estimate in terms of the error $\mathcal{J}(e) = \mathcal{J}(u) - \mathcal{J}(u_{\text{H}})$.

At first sight, to obtain $\mathcal{J}(e)$, we should calculate e by (2.3) and then obtain $\mathcal{J}(e)$. However, the problem (2.3) is practically impossible to solve computationally, even in its discrete version. Another way to get $\mathcal{J}(e)$ without calculating e, is to try to find $z \in \mathscr{V}_0$ such that $\mathcal{J}(e) = \mathcal{R}^{\text{P}}(z)$. This is possible if we introduce the following *dual problem*

$$\begin{cases} \text{Find } z \in \mathscr{V}_0 \text{ such that} \\ \mathcal{B}(v, z) = \mathcal{J}(v), \quad \forall v \in \mathscr{V}_0. \end{cases} \tag{2.4}$$

Since $e \in \mathscr{V}_0$, in this problem we can replace v by e, then we obtain the following representation of the error in the quantity of interest.

$$\mathcal{J}(e) = \mathcal{B}(e, z) = \mathcal{R}^{\text{P}}(z). \tag{2.5}$$

Problems (2.1) and (2.4) have the same difficulty to find the exact solution respectively. Therefore we must define the discrete formulation of the problem (2.4) as

$$\begin{cases} \text{Find } z_{\text{H}} \in \mathscr{V}_{\text{H0}} \text{ such that} \\ \mathcal{B}(v, z) = \mathcal{J}(v), \quad \forall v \in \mathscr{V}_{\text{H0}}. \end{cases} \tag{2.6}$$

The error of the dual problem $\epsilon := z - z_{\text{H}}$, belongs to space $\mathscr{V}_0$ and satisfies the error equation

$$\mathcal{B}(v, \epsilon) = \mathcal{J}(v) - \mathcal{B}(v, z_{\text{H}}) =: \mathcal{R}^{\text{D}}(v), \quad \forall v \in \mathscr{V}_0, \tag{2.7}$$

where $\mathcal{R}^{\text{D}}$ defines the *weak residual* associate with the approximate solution z_{H}.

The different representations of the error in the quantity of interest are induced by the above mentioned, and they are summarized in the following theorem.

Theorem 2.1. *Let $u_{\text{H}} \in \mathscr{V}_{\text{H0}}$ and $z_{\text{H}} \in \mathscr{V}_{\text{H0}}$ be the finite elements approximations of the solutions of the primal and dual problems respectively (the solutions of the* (2.2) *and* (2.6) *problems). Then the representation of the error in the quantity of interest associate to these solutions $\mathcal{J}(e)$, can be represented as:*

$$\begin{aligned} \mathcal{J}(e) &= \mathcal{B}(e, z - \phi_{\text{H}}) && \text{(2.8a)} \\ &= \mathcal{R}^{\text{P}}(z - \phi_{\text{H}}) && \text{(2.8b)} \\ &= \mathcal{R}^{\text{D}}(u - \phi_{\text{H}}), && \text{(2.8c)} \end{aligned}$$

where $\mathcal{B}(\cdot, \cdot)$ is the bilinear form in (2.1)*, $\mathcal{R}^{\text{P}}$ and $\mathcal{R}^{\text{D}}$ are the residuals of the primal* (2.3) *and dual problems* (2.7) *respectably, and ϕ_{H} is any function in $\mathscr{V}_{\text{H0}}$.*

Proof. For any $\phi_{\text{H}} \in \mathscr{V}_{\text{H0}}$, by Galerkin orthogonality we have,

$$\mathcal{B}(e, \phi_{\text{H}}) = \mathcal{R}^{\text{P}}(\phi_{\text{H}}) = 0. \tag{2.9}$$

On the other hand, since $e \in \mathscr{V}_0$, we can replace v in (2.4) by e to obtain

$$\mathcal{J}(e) = \mathcal{B}(e, z). \tag{2.10}$$

By (2.9) and (2.10), we obtain the first equality: $\mathcal{J}(e) = \mathcal{B}(e,z) - \mathcal{B}(e,\phi_{\text{H}}) = \mathcal{B}(e, z-\phi_{\text{H}})$. Since the solution z of the dual problem (2.4) belongs to the space $\mathscr{V}_0$, we can replace v by z in (2.3) to obtain

$$\mathcal{B}(e,z) = \mathcal{R}^{\mathtt{P}}(z). \tag{2.11}$$

Now, if we use (2.10) and (2.11) together the Galerkin orthogonality (2.9), we obtain the second equality (2.8b).

Replacing v by $u-\phi_{\text{H}}$ in (2.7), we have

$$\mathcal{J}(u-\phi_{\text{H}}) - \mathcal{B}(u-\phi_{\text{H}}, z_{\text{H}}) = \mathcal{R}^{\mathtt{D}}(u-\phi_{\text{H}}),$$

since $\phi_{\text{H}} \in \mathscr{V}_{\text{H0}}$. We also have $u_{\text{H}} \in \mathscr{V}_{\text{H0}}$, therefore $\mathcal{J}(u_{\text{H}}) - \mathcal{B}(u_{\text{H}}, z_{\text{H}}) = 0$. By combining the last two equalities, we obtain

$$\mathcal{J}(e) - \mathcal{B}(e, z_{\text{H}}) = \mathcal{R}^{\mathtt{D}}(u-\phi_{\text{H}}).$$

And, by the Galerkin orthogonality we obtain the last equation

$$\mathcal{J}(e) = \mathcal{R}^{\mathtt{D}}(u-\phi_{\text{H}}).$$

□

Note that in (2.8) the error representation holds for any function $\phi_{\text{H}} \in \mathscr{V}_{\text{H0}}$, so different functions ϕ_{H} do not modify the global representation of $\mathcal{J}(e)$. Nevertheless, the local representations of the error are different depending on the choice of ϕ_{H}. A proposal for an optimal choice of ϕ_{H}, is given by Diez and Calderón [19]. In particular, in this work, $\phi_{\text{H}} = z_{\text{H}}$ is chosen for the representations in (2.8a) and (2.8b), and $\phi_{\text{H}} = u_{\text{H}}$ is chosen for the representation in (2.8c). Then the representations of the error in (2.8) are rewritten as

$$\mathcal{J}(e) = \mathcal{B}(e,\epsilon) = \mathcal{R}^{\mathtt{P}}(\epsilon) = \mathcal{R}^{\mathtt{D}}(e). \tag{2.12}$$

2.4. Local error representation.

In any adaptive process, a local distribution of error is necessary throghout the domain Ω. This is generally achieved by restricting the integral operators on each element Ω_k of the mesh. By (2.12) we obtained

$$\mathcal{J}(e) = \sum_{k=1}^{\mathtt{n_{elem}}} \mathcal{B}_k(e,\epsilon) = \sum_{k=1}^{\mathtt{n_{elem}}} \mathcal{R}_k^{\mathtt{P}}(\epsilon) = \sum_{k=1}^{\mathtt{n_{elem}}} \mathcal{R}_k^{\mathtt{D}}(e), \tag{2.13}$$

where $\mathtt{n_{elem}}$ is the number of the elements of the mesh, $\mathcal{B}_k(\cdot,\cdot)$, $\mathcal{R}_k^{\mathtt{P}}(\cdot)$ and $\mathcal{R}_k^{\mathtt{D}}(\cdot)$ are the restrictions of the $\mathcal{B}(\cdot,\cdot)$, $\mathcal{R}^{\mathtt{P}}(\cdot)$ and $\mathcal{R}^{\mathtt{D}}(\cdot)$ to Ω_k, respectively. The energy norm of the error into each element Ω_k is denoted by $\|e\|_k^2 = \mathcal{B}_k(e,e)$. While the error representations in (2.8) are globally equivalent, the local error representations in (2.13) are not, i.e. $\mathcal{B}_k(e,\epsilon) \neq \mathcal{R}_k^{\mathtt{P}}(\epsilon) \neq \mathcal{R}_k^{\mathtt{D}}(e)$. For example, by Green identity, the primal residual representation of the error for elliptic problem is

$$\sum_{k=1}^{\mathtt{n_{elem}}} \mathcal{B}_k(e,v) = \sum_{k=1}^{\mathtt{n_{elem}}} \Big\{ \mathcal{R}_k^{\mathtt{P}}(v) + \int_{\partial\Omega_k} \boldsymbol{n} \cdot \nabla u_{\text{H}} v ds \Big\}.$$

So for any element Ω_k

$$\mathcal{B}_k(e,v) = \mathcal{R}_k^{\mathtt{P}}(v) + \frac{1}{2} \sum_{\Gamma_i} \int_{\Gamma_i} v [\![\nabla u_{\text{H}}]\!] \cdot \boldsymbol{n} ds, \tag{2.14}$$

where $[\![\nabla u_{\text{H}}]\!] := \nabla u_{\text{H}}|_{\Omega_1} - \nabla u_{\text{H}}|_{\Omega_2}$, and $\bigcup_i \Gamma_i = \partial\Omega_k$. If $\Gamma_i \subset \partial\Omega$, the integral is zero. We will use (2.14) in the numerical results in section 6.

3. Error estimation

In the error representations (2.12), primal e and dual ϵ errors are unknown, since the exact solutions u and z are unknown. Therefore, We have to propose a computationally accessible strategy to estimate these errors. A practical way to estimate these errors is through the reference solutions u_{h} and z_{h} instead of the exact solutions u and z respectively. Clearly, these reference solutions lead to the reference errors $e_{\text{ref}} := u_{\text{h}} - u_{\text{H}}$ and $\epsilon_{\text{ref}} := z_{\text{h}} - z_{\text{H}}$, which we could use in the error representations (2.12), but the global calculation of u_{h} and z_{h} in the space $\mathscr{V}_{\text{h0}}$ is computationally prohibitive, because the space $\mathscr{V}_{\text{h0}}$ is finer than $\mathscr{V}_{\text{H0}}$. Therefore, the error estimate must be local, i.e., we must estimate the error in each element of the mesh, using only the information contained in the element or a small neighborhood around it.
The local estimations of the reference errors e_{ref} and ϵ_{ref} is obtained using the residual equations (2.3) and (2.7) respectively. These local problems are solved with homogeneous Dirichlet boundary conditions, which allows to obtain an independent error estimation for each element. Obviously the error is not zero at the boundary $\partial\Omega_k$ of the each element Ω_k, however, a strategy to improve the estimation of the error at the boundary $\partial\Omega_k$ will be proposed in section 3.2. In the remainder of this section, results will be presented only with respect to error e of the primal problem. Analogously, the estimator and the same results can be obtained for the error ϵ of the dual problem.

Let us start by defining some spaces and functions that we will be using.

$$
\begin{aligned}
&\mathscr{V}_k = \{u \in \mathscr{V}_0 : u \text{ has compact support in } \Omega_k\}\\
&\mathscr{V}^* := \mathscr{V}_1 \oplus \mathscr{V}_2 \oplus \cdots \oplus \mathscr{V}_{\mathtt{n_{elem}}}\\
&e_k = \text{the projection of } e \text{ over } \mathscr{V}_k\\
&e^* := \sum_{k=1}^{\mathtt{n_{elem}}} e_k
\end{aligned}
$$

We can see that the subspaces $\mathscr{V}_k$ are orthogonal to each other with respect to $\mathcal{B}(\cdot,\cdot)$. e^* is a global approximation of e, in fact e^* is the projection of e over $\mathscr{V}$.

Lemma 3.1. *The projection e_k and the global approximation e^* satisfies the following bounds:*

$$\sum_{k=1}^{\mathtt{n_{elem}}} \|e_k\|^2 \leq \|e\|^2. \tag{3.1}$$

$$\|e_k\| \leq \|e\|_k. \tag{3.2}$$

$$\|e^*\|^2 \leq \|e\|^2. \tag{3.3}$$

Proof. Since e_k is the projection of e over $\mathscr{V}_k$ according to the global form $\mathcal{B}(\cdot,\cdot)$, the difference $e - e_k$ is orthogonal to e_k, i.e.

$$\mathcal{B}(e - e_k, e_k) = 0. \tag{3.4}$$

This orthogonality condition is also satisfied if we use the local form $\mathcal{B}_k(\cdot,\cdot)$, i.e. is orthogonal to e_k, i.e.

$$\mathcal{B}_k(e - e_k, e_k) = 0 \tag{3.5}$$

As the subspaces $\mathscr{V}_k$ are orthogonal to each other with respect to $\mathcal{B}(\cdot,\cdot)$, then by (3.4) and Bessel's inequality, we obtained the first bounded:

$$\sum_{k=1}^{\mathtt{n_{elem}}} \|e_k\|^2 = \sum_{k=1}^{\mathtt{n_{elem}}} \mathcal{B}(e_k, e_k) = \sum_{k=1}^{\mathtt{n_{elem}}} \mathcal{B}(e_k, e) \leq \mathcal{B}(e, e) = \|e\|^2.$$

By (3.4) we obtained the following equality:

$$\begin{aligned} \|e\|_k^2 &= \mathcal{B}_k(e, e) = \mathcal{B}_k([e - e_k] + e_k, [e - e_k] + e_k) \\ &= \mathcal{B}_k(e_k, e_k) + \mathcal{B}_k(e - e_k, e - e_k) = \|e_k\|_k^2 + \|e - e_k\|_k^2, \end{aligned}$$

from which we obtained the second bound. The third bound follows immediately from the definition of the e^* and by (3.1). □

We deduce that any family of subspaces $\mathscr{V}_k$ of $\mathscr{V}_0$, which are defined on disjoints subdomains, allows us to obtain local projections e_k of the exact error e, whose norm $\|e_k\|$ is a lower bound of the local norm of the exact error e (3.2). On the other hand, e^* is a global approximation of the exact error e, whose global norm $\|e^*\|$ is a lower bound of the global error measure $\|e\|$ (3.3). And finally, both the global approximation e^* and the local approximation e_k, improve as the space $\mathscr{V}^*$, which is generated by the subspaces $\mathscr{V}_k$, approaches $\mathscr{V}_0$.

The global approximation e^* is associated a family of subspaces $\mathscr{V}_k$, which is defined from a partition of Ω into subdomains Ω_k. The global approximation e^* is zero at the points on the boundary of each Ω_k that are in the interior of Ω. Those points are called *hidden points* of the global approximation because we are artificially forcing e^* to be zero at them. For this reason, the estimation e^* is called the *interior estimate* and is perfectly characterized by choosing elementary submeshes. On the other hand, the space $\mathscr{V}^*$ cannot be a good approximation of $\mathscr{V}_0$ if many points of the domain Ω remain hidden.

3.1. A local error estimator: interior residuals. Each subspace $\mathscr{V}_k$ is defined over an element Ω_k. The definition of the local subspaces $\mathscr{V}_k$ characterize the projections e_k, and each e_k is taken a local approximation of the error, i.e. as a approximation of the error inside each element Ω_k. To systematize the computations and to have a simple implementation, we defined an reference elementary submesh over a reference element. The elementary submeshes over each Ω_k are built applying the isoparametric transformation to this reference elementary submesh, see 1.

By assembly all the elementary submeshes, a refined mesh discretizing the whole domain Ω is obtained. Te characteristic size of the refined mesh is denoted by h. This mesh could be used in the computation of a reference solution u_{h}, which is more accurate than u_{H}. However, the computation of the reference solution must be avoided because of the large amount of degrees of freedom involved.

Now, with respect to the spaces $\mathscr{V}_{\mathrm{H0}}$, $\mathscr{V}_{\mathrm{h0}}$, $\mathscr{V}^*$ and $\mathscr{V}_{\mathrm{h0}}$, we have the following strict inclusions.

$$\mathscr{V}_{\mathrm{H0}} \subset \mathscr{V}_{\mathrm{h0}} \quad \text{and} \quad \mathscr{V}^* \subset \mathscr{V}_{\mathrm{h0}}.$$

In effect, the first inclusion is a consequence of the fact that the space $\mathscr{V}_{\mathrm{h0}}$ is obtained by the reference mesh, which is built from the original computational mesh. And the second inclusion is a consequence of the fact that the space $\mathscr{V}_{\mathrm{h0}}$ include all the spaces $\mathscr{V}_k$ generated by the elementary submeshes, and therefore also contains its sum $\mathscr{V}^*$ ($\mathscr{V}^* := \mathscr{V}_1 \oplus \mathscr{V}_2 \oplus \cdots \oplus \mathscr{V}_{\mathtt{n_{elem}}}$). This inclusion is strict and cannot be an equal because $\mathscr{V}_{\mathrm{h0}}$ contains the interpolation functions generating each $\mathscr{V}_k$, associated with the interior nodes, but also the interpolation functions associated with the nodes lying.

The reference error $e_{\mathrm{ref}} := u_{\mathrm{h}} - u_{\mathrm{H}}$ is the error of the solution u_{H} with respect to the reference solution u_{h}, and does not coincide with the error of the reference solution $u - u_{\mathrm{h}}$, which must be smaller than e or e_{ref}. In fact, the error e, is the sum of the reference error e_{ref} and the error of the reference solution $e = e_{\mathrm{ref}} + (u - u_{\mathrm{h}})$.

Lemma 3.2. *The norms $\|e_{ref}\|$ and $\|e^*\|$ satisfied the following inequalities*

$$\|e^*\| \leq \|e_{ref}\| \leq \|e\| \tag{3.6}$$

Proof. As $u_{\mathrm{h}}, u_{\mathrm{H}} \in \mathscr{V}_{\mathrm{h0}}$, by Galerkin orthogonality, we obtain that the reference error is the projection of $e = u - u_{\mathrm{H}}$ on $\mathscr{V}_{\mathrm{h0}}$. In effect,

$$B(e - (u_{\mathrm{h}} - u_{\mathrm{H}}), u_{\mathrm{h}} - u_{\mathrm{H}}) = B(u - u_{\mathrm{h}}, u_{\mathrm{h}}) - B(u - u_{\mathrm{h}}, u_{\mathrm{H}}) = 0.$$

Therefore, the norm $\|e_{\mathrm{ref}}\|$ is a lower bound of the global measure of the error $\|e\|$: $\|e_{\mathrm{ref}}\| \leq \|e\|$. The other inequality is obtained because e^* is the projection of e_{ref} on $\mathscr{V}^*$:

$$B(e_{\mathrm{ref}} - e^*, e^*) = B(u - u_{\mathrm{H}} - e^* - (u - u_{\mathrm{h}}, e^*) = B(e - e^*, e^*) - B(u - u_{\mathrm{h}}, e^*) = 0.$$

□

With the error estimator we want to approximate the local an global norms of e_{ref} without solving the global problem. In fact, e^* the projection of e_{ref} on $\mathscr{V}^*$, is already an initial estimate of e_{ref}. The global projection e^* is obtained by the local projection e_k on each subspace $\mathscr{V}_k$. Each projection e_k is computed solving a local problem, with a few degrees of freedom and a low computational cost.

The norm of e_k grows with the size of the associated subspace $\mathscr{V}_k$, i.e. with the number of degrees of freedom of the elementary submesh. A sequence of approximations based on gradually refined submeshes provides increasing estimates of $\|e^*\|$, but bounded by $\|e_{\mathrm{ref}}\|$. Therefore, if the submesh is refined, the norm of the projections, both global and local converge to a value that underestimates the reference error norm. This underestimation could be relatively far from $\|e\|$, and even from $\|e_{\mathrm{ref}}\|$, due to the fact that we are forcing the approximate function e^* to take zero values at the hidden points.

In the next section we introduce a new family of projections that retain the philosophy of the given projection. The new family of projections is based on a new partition of the domain Ω different from that provided by the elements Ω_k. This time the information contained in the flow jumps is taken into account

without having to compute these jumps directly. So, we define subspaces whose functions have supports that cover the edges of the elements Ω_k (the hidden nodes forgotten in the internal estimation).

3.2. Improvement of the estimation. (Enrichment of the estimation. Overlapped meshing). In this section new terms are added to the estimator so that the estimate obtained is as close as possible to the exact error, maintaining in any case the lower bound property.

Since e^* is the projection of the reference error e_{ref} on the space $\mathscr{V}^*$, the reference error can be expressed as the sum: $e_{\text{ref}} = e^* + e^{\perp}_{\text{ref}}$, where $e^{\perp}_{\text{ref}}$ is orthogonal to e^* and to the space $\mathscr{V}^*$. Then, as $\|e^*\|^2 = \|e_{\text{ref}}\|^2 - \|e^{\perp}_{\text{ref}}\|^2$, the norm $\|e^{\perp}_{\text{ref}}\|$ is the underestimation of the reference error associated with the global interior estimation.

The goal of the enrichment of the estimation is to approximate the forgotten part of the reference error $e^{\perp}_{\text{ref}}$, and add it to the interior estimates. The element $e^{\perp}_{\text{ref}}$ belong to $\mathscr{V}^{\perp}$, the orthogonal complement of the space $\mathscr{V}^*$. As we want to approximate $e^{\perp}_{\text{ref}}$, we must project e or e_{ref} on a space include in $\mathscr{V}^{\perp}$.

Let us consider a new partition of the domain Ω in a family of subdomains Λ_m, with $m = 1, \ldots, \mathtt{n}^{\Lambda}_{\mathtt{elem}}$. In order to understand the the difference between the two domain partitions, the subdomains that are defined by Λ_m are called *patches*, while Ω_k are called *elements* of the compute mesh. In the same way as with the elements, each patch Λ_m is discretized by a patch-submesh generating an interpolation subspace $\mathscr{W}_m$ in $\mathscr{V}_0(\Lambda_m)$. The subspaces $\mathscr{W}_m$ are generated following the same strategy used for the subspaces $\mathscr{V}_k$. Each path-submesh is chosen as part of the global reference mesh that generates $\mathscr{V}_{\text{h0}}$. Thus, submeshes that discretize overlapping elements and patches share nodes and elements of the refined global mesh generating $\mathscr{V}_{\text{h0}}$, which determines the geometry and mesh of the patches.

Since the subdomains Λ_m have zero measure intersections, the subspaces $\mathscr{W}_m$ are orthogonal to each other. The projections of e_{ref} or e on the subspaces $\mathscr{W}_m$ are a new family approximations to e_{ref}.

We define $\widetilde{\mathscr{W}_m}$ as the part of $\mathscr{W}_m$ orthogonal to e^*, i.e. $\widetilde{\mathscr{W}_m} := \mathscr{W}_m \cap span\{e^*\}^{\perp}$. As $\widetilde{\mathscr{W}_m}$ is a restriction of $\mathscr{W}_m$ subject to a single linear condition, the dimension of $\widetilde{\mathscr{W}_m}$ is equal to $\mathscr{W}_m$ minus one (or equal to the dimension of $\mathscr{W}_m$, if e^* is orthogonal to $\mathscr{W}_m$).

Let η_m the projection of the e_{ref} on $\widetilde{\mathscr{W}_m}$. Since e^* and each one of the estimates η_m are orthogonal, by Bessel's inequality

$$\|e^*\|^2 + \sum_{m=1}^{\mathtt{n}^{\Lambda}_{\mathtt{elem}}} \|\eta_m\|^2 = \sum_{k=1}^{\mathtt{n}_{\mathtt{elem}}} \|e_k\|^2 + \sum_{m=1}^{\mathtt{n}^{\Lambda}_{\mathtt{elem}}} \|\eta_m\|^2 \leq \|e\|^2. \tag{3.7}$$

As we are obtaining a lower bound of the exact error, it is preferable to impose as few constraints as possible. If too many restrictions are imposed on the problem, the estimate will be far from the value $\|e\|$. Therefore, only orthogonality is imposed on the overall interior estimate e^*, which adds the contributions of the patches to the interior estimate (3.7).

We have to estimate the local error locally to obtain an approximation to $\|e_{\text{ref}}\|_k^2$. Therefore, to the norm of the first estimate $\|e_k|_k^2$, we must add the contribution of all the patches Λ_m that overlap with the element Ω_k. This contribution is the measure of the restriction Λ_m in Ω_k, which is given by $\|\eta_m\|_k^2 = \mathcal{B}(\eta_m, \eta_m)$. Thus, an approximation to the local norm of the error $\|e_{\text{ref}}\|_k^2$ can be evaluate by

$$\|e_k\|_k^2 + \sum_m \|\eta_m\|_k^2 \,, \tag{3.8}$$

where the index m in the summation takes the values such that the patch Λ_m overleaps the element Ω_k.

In order to compute $\|\eta_m\|_k$ (the contributions to the element Ω_k), we must identify the elements that belong to both patch-submeshes (discretization of Λ_m) and the elementary-submeshes (discretization Ω_k). As this can be complex to organize the computations, a new approach is introduced to facilitate programming. Instead to compute $\|\eta_m\|_k$, we compute an equidistribution of $\|\eta_m\|^2$ between all the elements that overlap with the patch Λ_m. Therefore, the local quantities in (3.8), will be approximate by

$$\|e_k\|_k^2 + \sum_m \frac{\|\eta_m\|^2}{M_m} \,, \tag{3.9}$$

where M_m is the number of the elements that overlap with the patch Λ_m and the index m has the same range of variation as in equation (3.8). This approach may cause loss in the lower bound, particularly when elements with very different errors are adjacent.

The orthogonality condition to the global interior approximation is a linear restriction imposed at the same time with the boundary conditions. Of course, the boundary conditions of the problem on each patch are imposed according to the same criterion as given for imposing the boundary conditions in the elementary problems.

The orthogonality condition to the global interior approximation can be implemented in a simple way, if the nodes of the submeshes discretizing the patches and the elements coincide. Therefore the discretization of the patches is conditioned by the discretization of the elements. If this condition is applied, orthogonality becomes a linear restriction on the local problem over each patch. This constraint can be easily imposed using the Lagrange multiplier technique.

For the estimator definition to be complete, it is necessary to specify which partition corresponds to the patches and their corresponding discretization. A possible elemental submesh and associated patches are shown in Figure 1. The mesh must allow to choose a priori the remaining hidden points in the complete estimate. For example, these points can be the nodes of the original compute mesh. With this geometrical definition, the η_m projection includes the effect of the flow jumps through the side covering the patch Λ_m.

Instead of obtaining e_{ref} by solving a global problem, a first estimate e^* is proposed, which consists of solving local problems with a low computational cost. However, by adding η in the second phase of the estimation, the only remaining hidden points are at the intersections of the edges of the elements and the patches,

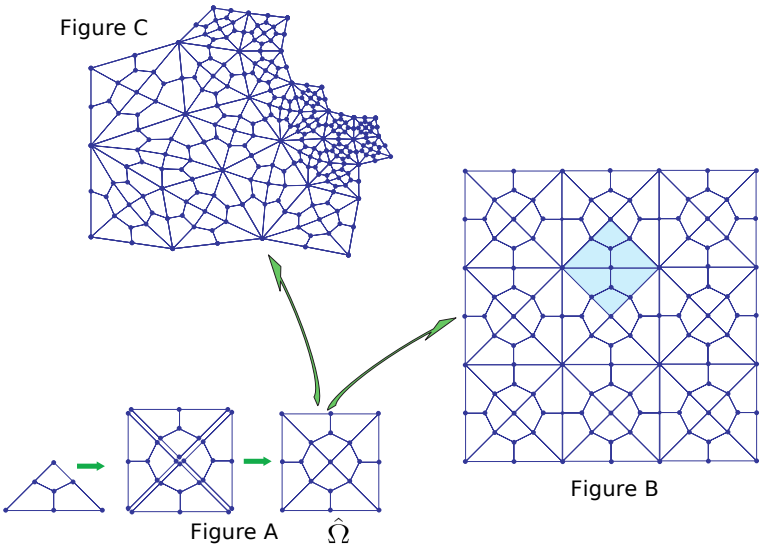

FIGURE 1. Reference elementary submesh over a reference element $\hat{\Omega}$ (Figure A), a induced elementary submeshes over a regular mesh (Figure B) and arbitrary meshes (Figure C), a patch-submeshs (shaded portion of Figure B).

thus reducing the number of hidden points considerably. Thus, the full estimate $e^* + \eta$ (which seeks to approximate e_{ref}) is zero at the remaining hidden points.

3.3. Implementation of the estimator in the quantity of interest. To avoid defining a new notation, in the primal problem we will use $e^* = e^* + \eta$ to define the sum of the estimated error in the interior of each element (interior estimate) and the enrichment Similarly, the estimated error ϵ^* is defined for the dual problem. Therefore, the estimated error, e^*, in the energy norm complies with the following bounds $\|e^*\| \leq \|e_{\text{ref}}\| \leq \|e\|$. Similarly, when we use e^* and ϵ^* in any of the representations of the error in (2.12),

$$\mathcal{J}(e^*) \approx \mathcal{S}_i, \quad i = 1, 2, 3, \tag{3.10}$$

where $\mathcal{S}_1 := \mathcal{B}(e, \epsilon)$, $\mathcal{S}_2 := \mathcal{R}^{\text{P}}(\epsilon)$ y $\mathcal{S}_3 := \mathcal{R}^{\text{D}}(e)$, its possible to obtain bounds for $\mathcal{J}(e)$ from [5] and [6]. Therefore, we can intuit that $\mathcal{J}(e^*)$ can approximate $\mathcal{J}(e_{\text{ref}})$. "appropriately", i.e., $\mathcal{J}(e_{\text{ref}}) \approx \mathcal{J}(e^*)$.

The control of the accuracy of the numerical solution u_{H} will be carried out by an adaptive process driven by the quantity of interest and not by the energy norm. In numerical experimentation, the results in energy norm will be used only to compare the effectiveness of the process.

4. Remeshing criterion

To obtain a solution that satisfies a required accuracy (*acceptable solution*) using a mesh with the least number of degrees of freedom (*optimal mesh*), an adaptive process is required in addition to the error representation and estimation. To

make this adaptive process it is necessary to define a *remeshing criterion*, which uses the results obtained from the error estimator to obtain an adequate spatial distribution of the elements leading to an optimal mesh.

Let $\mathscr{E}$ be the functional error that we want to control, in goal-oriented adaptivity $\mathscr{E}$ is $\mathcal{J}(e)$. $\mathscr{E}$ can be decomposed into the elemental contributions

$$\mathscr{E} = \sum_{k=1}^{\mathtt{n}_{\mathtt{elem}}} \mathscr{E}_k, \quad \text{with} \quad \mathscr{E}_k = \mathcal{B}_k, \quad \mathscr{E}_k = \mathcal{R}_k^{\mathtt{P}}(\epsilon), \quad \mathscr{E}_k = \mathcal{R}_k^{\mathtt{D}}(e).$$

Each element of this mesh is denoted as Ω_k. If the elements in the mesh are sufficiently regular, the size of the each element is taken as

$$\mathrm{H}_k = \big[\Omega_k\big]^{1/d},$$

where d is the dimension of the space and $\big[\Omega_k\big]$ is the size element.

The number of elements and their characteristic size in the new mesh are denoted by $\widehat{\mathtt{n}}_{\mathtt{elem}}$ and $\widehat{\mathrm{H}}$, respectively.

The goal is to derive an expression for $\widehat{\mathrm{H}}_k$ as a function of $\mathscr{E}_k$ y H_k, such that the new mesh meets the accurate requirement with a low computational cost. Following the the ideas in [19] we obtain

$$\widehat{\mathrm{H}}_k = \Big[\frac{\widehat{\mathscr{E}}}{\mathscr{E}_k\, \widehat{\mathtt{n}}_{\mathtt{elem}}}\Big]^{1/\alpha} \mathrm{H}_k,$$

where $\widehat{\mathscr{E}}$ is the error in the new mesh and $\widehat{\mathtt{n}}_{\mathtt{elem}}$ is given by

$$\widehat{\mathtt{n}}_{\mathtt{elem}} = \left[\frac{1}{\widehat{\mathscr{E}}^{d/\alpha}} \sum_{k=1}^{\mathtt{n}_{\mathtt{elem}}} (\mathscr{E}_k)^{d/\alpha}\right]^{\alpha/(\alpha-d)},$$

where α is the local convergence order. Thus, $\widehat{\mathtt{n}}_{\mathtt{elem}}$ can be evaluated once all the local errors $\mathscr{E}_k$ in the current mesh have been computed since $\widehat{\mathscr{E}}$ is a predefined amount.

5. Adaptive error control

In this section we propose an algorithm for error control based on quantities of interest, which consist of an iterative process that implements all the techniques developed in the previous sections. This strategy is very general and does not require information about the type of problem that we are dealing with. Taking as an example the representation of the error $\mathcal{J}(e) = \mathcal{R}^{\mathtt{P}}(\epsilon)$, the algorithm is described as follows.

Algorithm 1 Adaptive error control

1: Define the initial mesh and compute: u_{H} and z_{H}.
2: Compute the dual error ϵ^* using the residual estimator.
3: Compute the spatial distribution of the error by

$$\mathcal{S}_2 := \sum_{k=1}^{\mathtt{n}_{\mathtt{elem}}} \mathcal{R}^{\mathtt{p}}(\epsilon^*).$$

4: If $\mathcal{J}(e) \approx \mathcal{S}_2 \leq \text{TOL}$ (the tolerance is reached), the iterative procedure finishes. The solution u_{H} satisfies the required accuracy. If the tolerance is not achieved:
   (a) Compute each $\widehat{\text{H}}_k$ to define the new computational mesh.
   (b) Create the new mesh.
5: Define the initial mesh as the mesh obtained in step $4.b$ and go to step 1.

6. Numerical experimentation

In this section we present the numerical results obtained by applying the adaptive algorithm proposed above.The behavior and accuracy of the estimates together with the remeshing criterion are analyzed in elliptic problem over a bounded domain $\Omega \subset \mathbb{R}^2$. In these examples, the effectiveness of the estimator is justified in cases where the problem (primal or dual) has large changes in its derivatives. The computations of both the primal solution u_{H} and the dual solution z_{H} are done using bilinear ($p = 1$) finite elements (quadrilaterals) on all meshes of the adaptive process (h-type adaptivity).

The quality of the estimate is measured using the *global effectivity index*

$$\mathscr{I}^{\mathtt{eff}} := \frac{\mathcal{S}_i}{\mathcal{J}(e)}, \quad \text{for } i = 1, 2, 3,$$

when the exact error is available (which is the case of the examples to be analyzed). Otherwise, the exact error is replaced by the reference error and the $Ieff$ is replaced by a *reference effectivity index* $Iref$. Similarly, when we use energy norm, the effectiveness index is defined as $\mathscr{I}^{\mathtt{eff}}_{\mathtt{ener}} = \|e^*\|/\|e\|$. The local quality of the estimates is measured using the *local effectivity index*. $\mathscr{I}^{\mathtt{eff}}_k = \mathcal{S}_{i,k}/\mathcal{J}_k(e)$, where $\mathcal{S}_{i,k}$ and $\mathcal{J}_k(e)$ are the contributions of the element Ω_k to $\mathcal{S}_i$ and $\mathcal{J}(e)$.

Example 1: We solved the Problem (2.1) for the Poisson equation

$$\begin{cases} -\Delta u = & 2000(y(1-y) + x(1-x)), \text{ in } \Omega = (0,1) \times (0,1). \\ u = & 0 \quad \text{on } \partial\Omega \end{cases} \tag{6.1}$$

The exact solution for this problem is $u(x, y) = 1000xy(x - 1)(y - 1)$, see Figure 2 (left). The quantity of interest is defined by the functional

$$\mathcal{J}(u) := \int_\Omega u(x)\psi(x - x_0)\mathrm{d}\Omega,$$

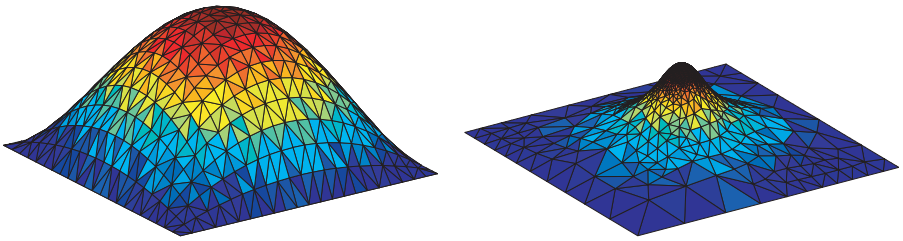

FIGURE 2. Example 1. Primal solution u_{h} (left) and dual solution z_{h} (right) problems.

with

$$\psi(x) := \begin{cases} C\exp(-\varepsilon^2/(\varepsilon^2 - |x|^2)) & \text{if} \quad |x| < \varepsilon. \\ 0 & \text{if} \quad |x| \geq \varepsilon, \end{cases} \tag{6.2}$$

$x_0 = (0.5\,,\, 0.5)$, $\varepsilon = 0.1$ and C is such that $\int_\Omega \psi(x - x_0)\mathrm{d}\Omega = 1$.

The adaptive process starts with a uniform mesh of 25 elements (36 nodes). With this mesh, the relative error in the quantity of interest is 4% but we want to obtain a solution with an error of less than 0.01%, i.e, with a tolerance of 10^{-4}.

Applying algorithm 1 with a tolerance of 10^{-4}, a succession or meshes is generated, the numerical results of which are shown in Table 1. and the global effectivity indexes. The last column shows the values of the effectivity index of the error estimator in the energy norm.

We show in Table 1 the exact relative error $J(e)/J(u)$, the relative estimated error of the different representations $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, and the global effectivity indexes. The last column shows the values of the effectivity index of the error estimator in the energy norm. It is observed that the effectiveness is around the average of 80%, in addition to the fact that $\|e^*\| \leq \|e\|$, as it was already justified theoretically.

Table 1 shows that to obtain a solution with a tolerance of 10^{-4}, the adaptive procedure driven by the error representations $\mathcal{B}(\cdot,\cdot)$, $\mathcal{R}^{\mathtt{P}}(\cdot)$ y $\mathcal{R}^{\mathtt{D}}(\cdot)$, requires meshes of 1677, 3424 and 2043 elements respectively. While in uniform meshes, to obtain the solution with the same tolerance, a mesh of 6241 elements is needed. This indicates that with the adaptive procedure the required solution is obtained with a lower computational cost. On the other hand, if the adaptive procedure were driven by the energy norm error estimator, i.e., if $\|e\| \leq \mathtt{TOL}$, then to achieve a tolerance of $\mathtt{TOL} = \mathtt{10}^{-3}$, meshes with more than 35 thousand elements are needed. This fact shows the effectiveness of the adaptive procedure when quantities of interest are used.

The global effectivity indexes in Table 1 show acceptable behaviors for the adaptive procedure, however, they present some type of noise or perturbation, which is not present in the effectivity indexes given in the energy norm (last column of the Table 1). To justify this phenomenon, we will use as an example mesh 3 obtained with the primal representation of the error $\mathcal{R}^{\mathtt{P}}(e^*)$. Figure 3 shows the effectivity indexes for the elements that lie inside the circle where the function $\psi(x) \neq 0$, see equation (6.2), and the elements that lie on the circumference that defines the circle. In most of the elements that are on the circumference, the estimator presents an erratic or very poor effectiveness, bringing as a consequence that the

TABLE 1. Example 1: number of mesh obtained by adaptivity process with TOL $= 10^{-4}$, nodes of the mesh, elements of the mesh, exact relative error of the quantity of interest, estimated relative error of the representations $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$, global effectivity indexes of the error estimator, effectiveness index in energy norm.

mesh	nodes	$\mathtt{n_{elem}}$	$\mathcal{J}(e)/\mathcal{J}(u)$	$\mathcal{S}_1/\mathcal{J}(u)$	$I^{\mathtt{eff}}$	$I^{\mathtt{eff}}_{\mathtt{ener}}$
1	36	25	0.41566×10^{-1}	0.40811×10^{-1}	0.982	0.898
2	451	417	0.93986×10^{-3}	0.80186×10^{-3}	0.853	0.813
3	1677	1630	0.68098×10^{-4}	0.67497×10^{-4}	0.991	0.826
4	3405	3360	0.13791×10^{-3}	0.10137×10^{-3}	0.735	0.811
5	6991	6932	0.23020×10^{-4}	0.45793×10^{-4}	1.989	0.831
6	7202	7147	0.50594×10^{-4}	0.70827×10^{-4}	1.400	0.809

mesh	nodes	$\mathtt{n_{elem}}$	$\mathcal{J}(e)/\mathcal{J}(u)$	$\mathcal{S}_2/\mathcal{J}(u)$	$I^{\mathtt{eff}}$	$I^{\mathtt{eff}}_{\mathtt{ener}}$
1	36	25	0.41566×10^{-1}	0.36236×10^{-1}	0.872	0.898
2	630	584	0.86034×10^{-3}	0.84065×10^{-3}	0.977	0.880
3	1621	1581	0.19842×10^{-3}	0.16801×10^{-3}	0.847	0.828
4	3424	3379	0.61742×10^{-4}	0.65388×10^{-4}	1.059	0.797
5	6938	6886	0.13927×10^{-5}	0.28707×10^{-4}	4.320	0.812

mesh	nodes	$\mathtt{n_{elem}}$	$\mathcal{J}(e)/\mathcal{J}(u)$	$\mathcal{S}_3/\mathcal{J}(u)$	$I^{\mathtt{eff}}$	$I^{\mathtt{eff}}_{\mathtt{ener}}$
1	36	25	0.41566×10^{-1}	0.33673×10^{-1}	0.810	0.898
2	223	204	0.30608×10^{-2}	0.19315×10^{-2}	0.631	0.794
3	2043	1979	0.42929×10^{-4}	0.10918×10^{-3}	2.543	0.828
4	3298	3246	0.84181×10^{-4}	0.86068×10^{-4}	1.022	0.818
5	6627	6573	0.47942×10^{-4}	0.76695×10^{-4}	1.560	0.809
6	6820	6761	0.43673×10^{-4}	0.37876×10^{-4}	0.867	0.847

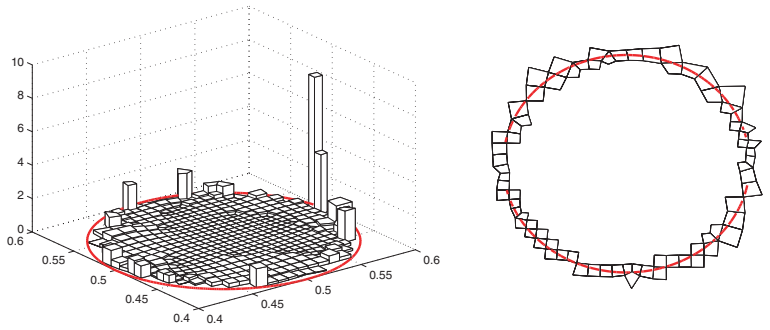

FIGURE 3. Example 1. Local effectivity indexes of mesh 3 for $\mathcal{R}^{\mathrm{P}}(e^*)$ (left). 547 elements lie inside the circle and 74 elements lie on circle (right).

global effectiveness index presents oscillating values, when in reality the meshes have changed very little. This behavior is due to the discontinuity presented by the local problems and the number of the points Gauss quadrature within the nonzero region when the integral is calculated. On the other hand, some elements that remain on the inner boundary of the circle also show low effectiveness, see Figure 3 (left), This is because the patch of elements used to define the enrichment of the estimator η is not completely contained in the circle.

It is important to note that the error representation $\mathcal{J}(e) = \mathcal{B}(e, \epsilon)$, requires to estimate the error in the primal and dual problems, resulting in disadvantage with the other two representations given in (2.12), which require only to estimate the error in one of the two problems. On the other hand, if the estimated errors are given by $e^* = e - \eta_1$ and $\epsilon^* = \epsilon - \eta_2$, with $\eta_1, \eta_2 \in \mathscr{V}_{\mathrm{H0}}$, then

$$\mathcal{J}(e) = \mathcal{B}(e, \epsilon) = \mathcal{B}(e^*, \epsilon^*) + \mathcal{B}(\eta_1, \eta_2).$$

Obviously, the term $\mathcal{B}(\eta_1, \eta_2)$ is not necessarily small and could deteriorate the quality of the estimate, for more details, see [8]. This fact can cause the oscillatory behavior of the relative error of $\mathcal{B}(\cdot, \cdot)$ shown in Figure 4.

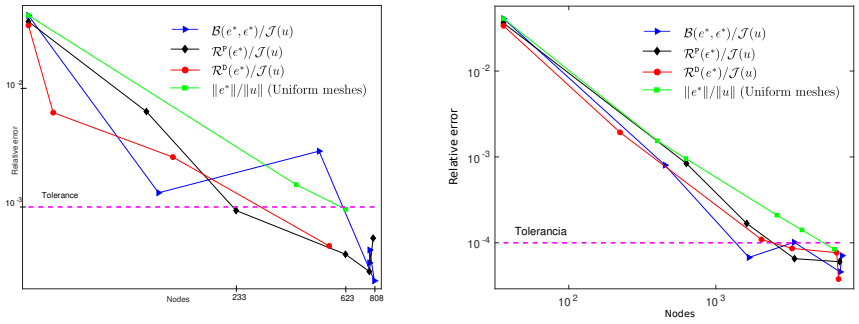

FIGURE 4. Example 1. Relative error using the adaptivity process for the three error representations $\mathcal{S}_i/\mathcal{J}(u)$, $i = 1 \ldots 3$, and the relative error $||e^*||/||u||$ using uniform meshes (without the adaptivity process). TOL $= 10^{-3}$ (left) and TOL $= 10^{-4}$ (right).

On the other hand, we show in Figure 4 the estimated relative errors for two tolerances: 10^{-3} y 10^{-4}. In order to show the efficiency of the adaptive process, a small modification was made in step 4 of Algorithm 1. Once the desired tolerance was reached, the algorithm was allowed to continue two or three more iterations, observing that the new meshes presented slight modifications in the number of elements and nodes. That is, the succession of meshes converges.

Example 2: We solved the Problem (2.1) for the Poisson equation

$$-\Delta u = f(x, y), \text{ in } \Omega = (0, 1) \times (0, 1). \tag{6.3}$$

with the following Dirichlet boundary conditions

$$\begin{cases} u(x,0) = & x\left(0.6 + e^{-10.5\left(3+16\left(x-\frac{1}{2}\right)^2\right)^2}\right), & x \in [0,1] \\ u(1,y) = & 0.6 + e^{-10.5\left(3+16\left(y-\frac{1}{2}\right)^2\right)^2}, & y \in [0,1] \\ u(x,1) = & x\left(0.6 + e^{-10.5\left(3+16\left(x-\frac{1}{2}\right)^2\right)^2}\right), & x \in [0,1] \\ u(0,y) = & 0, & y \in [0,1] \end{cases} \tag{6.4}$$

The exact solution is $u(x,y) = x(0.6 + e^{-10.5(1-16(x-\frac{1}{2})^2-16(y-\frac{1}{2})^2)^2})$, see Figure 5 (left). The source term f is obtained by substituting u in (6.3). A quantity of interest is defined to control the solution at the points where the derivatives have strong variations, i.e. on the circle, $(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{16}$. Thus, $\mathcal{J}(u) := \int_\Omega u(x)\psi(x-x_0)\mathrm{d}\Omega$, where $\psi(x-x_0)$ is defined in (6.2), $x_0 = (0.5\,,\,0.5)$, $\varepsilon = 0.3$ and C is such that $\int_\Omega \psi(x-x_0)\mathrm{d}\Omega = 1$.

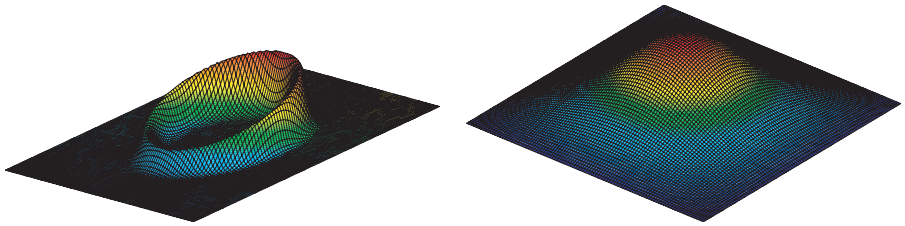

FIGURE 5. Example 2. Primal solution u_{h} (left) and dual solution z_{h} (right) problems.

The adaptive procedure is initiated on a uniform mesh of 25 elements (36 nodes) with a relative error in the quantity of interest of about 15%. ($\mathcal{J}(e_{\mathrm{ref}})/\mathcal{J}(u_{\mathrm{h}}) \approx 15\%$) and, we want to obtain two solutions with errors lower than 0.1% and 0.01%, and the conclusions that can be derived from the results obtained are the same as those obtained in example 1. For the two tolerances, the Figure 6 shows the graph of the relative error of the solution obtained on meshes generated by the adaptive procedure oriented by the three representations of the error, it also shows the relative error of the solution obtained on uniform meshes (without the adaptive process). In the Figure 6, we can see that the adaptive process defines meshes with much smaller degrees of freedom than in the uniform meshes, in which the numerical solution is obtained with the prescribed tolerance. This confirms the efficiency of the adaptive process for finding optimal meshes for solving problems with quantities of interest. The numerical results for the solution with tolerance 10^{-4} are shown in Table 2. The results show that the effectivity indexes for the energy norm have an effectiveness percentage above 80%. However, the indexes of effectiveness in the quantity of interest have significant losses in their effectiveness, but this does not affect the effectiveness of the adaptive process.

The meshes generated by adaptive procedure from an initial mesh of 36 nodes are shown in Figure 7. They show the expected qualitative behavior due to the gradient changes of the primal problem solution.

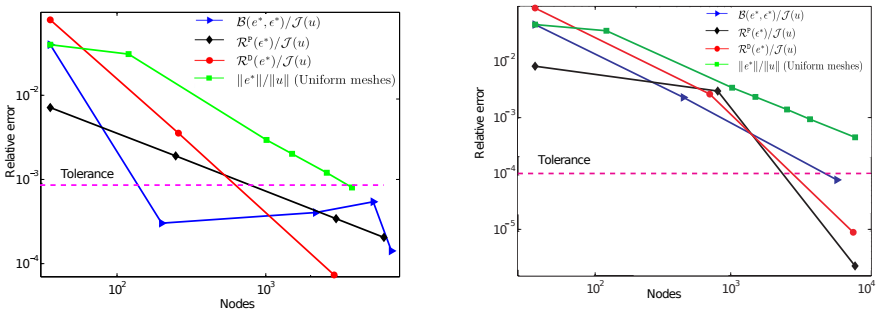

FIGURE 6. Example 2. Relative error using the adaptivity process for the three error representations $\mathcal{S}_i/\mathcal{J}(u)$, $i = 1 \ldots 3$, and the relative error $||e^*||/||u||$ using uniform meshes (without the adaptivity process). TOL $= 10^{-3}$ (left) and TOL $= 10^{-4}$ (right).

TABLE 2. Example 2: number of mesh obtained by adaptivity process with TOL $= 10^{-4}$, nodes of the mesh, elements of the mesh, exact relative error of the quantity of interest, estimated relative error of the representations $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$, global effectivity indexes of the error estimator, effectiveness index in energy norm.

mesh	nodes	$\mathtt{n_{elem}}$	$\mathcal{J}(e)/\mathcal{J}(u)$	$\mathcal{S}_1/\mathcal{J}(u)$	$\mathscr{I}^{\mathtt{eff}}$	$\mathscr{I}^{\mathtt{eff}}_{\mathtt{ener}}$
1	36	25	$0.14415 \times 10^{+0}$	0.46504×10^{-1}	0.323	0.920
2	447	420	0.63467×10^{-2}	0.23080×10^{-2}	0.364	0.772
3	6075	6064	0.40330×10^{-3}	0.76848×10^{-4}	0.191	0.803

mesh	nodes	$\mathtt{n_{elem}}$	$\mathcal{J}(e)/\mathcal{J}(u)$	$\mathcal{S}_2/\mathcal{J}(u)$	$\mathscr{I}^{\mathtt{eff}}$	$\mathscr{I}^{\mathtt{eff}}_{\mathtt{ener}}$
1	36	25	$0.14415 \times 10^{+0}$	0.83325×10^{-2}	0.058	0.920
2	802	751	0.55860×10^{-2}	0.29973×10^{-2}	0.537	0.789
3	8265	8246	0.19689×10^{-4}	0.22136×10^{-5}	0.112	0.817

mesh	nodes	$\mathtt{n_{elem}}$	$\mathcal{J}(e)/\mathcal{J}(u)$	$\mathcal{S}_3/\mathcal{J}(u)$	$\mathscr{I}^{\mathtt{eff}}$	$\mathscr{I}^{\mathtt{eff}}_{\mathtt{ener}}$
1	36	25	$0.14415 \times 10^{+0}$	0.91904×10^{-1}	0.638	0.920
2	700	662	0.60684×10^{-2}	0.26516×10^{-2}	0.437	0.792
3	8009	7998	0.35895×10^{-4}	0.88750×10^{-5}	0.247	0.805

Because of the effectiveness that the energy-norm estimator achieves, it will be appropriate to analyze the quantity of interest using bounds for the error made, as is done in [5] y [6]. In addition, a comparative analysis in terms of effectiveness and computational cost with other estimators (residual and postprocessing) should be performed. Mainly, with residual estimators using bubble functions, see [10]) , and post-processing or smoothing estimators such as those given in [8].

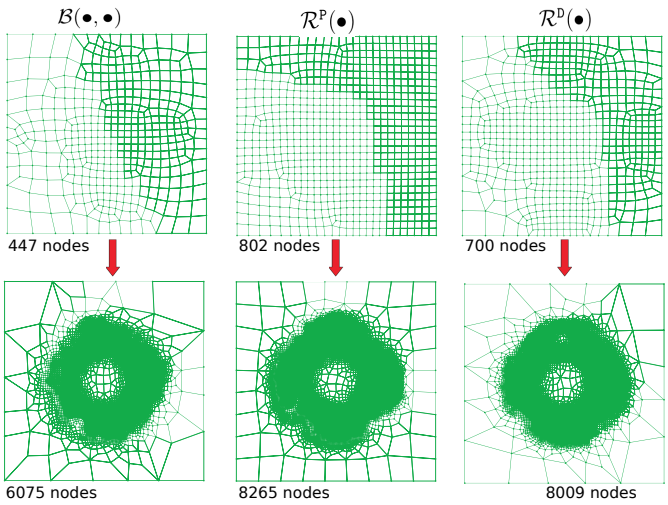

FIGURE 7. Example 2: Meshes generated by adaptive procedure starting from an initial uniform mesh of 36 nodes.

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