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SOME RESULTS ON NONLOCAL GENERALIZED FRACTIONAL INTEGRODIFFERENTIAL EQUATION

M. A. KAZI, V. V. KHARAT*, A. R. RESHIMKAR, M. T. GOPHANE, AND S. D. THIKANE

Abstract. This paper is about study existence and uniqueness of solutions for generalized fractional integrodifferential equations with nonlocal conditions. We prove the existence result for the problem using Kransnoseleskii fixed point theorem and using Banach fixed point theorem, we prove the uniqueness of solutions for the problem. The theory so developed is supported with an example at the end of the article.

1. Introduction

Fractional integrodifferential equations have been a part of interest for many researchers since the fractional calculus found its applications in many fields of engineering and science. Many authors have studied fractional differential equations with Caputo fractional derivative, Riemann Liouville fractional derivative, ψ-Hilfer fractional derivative etc. Researches have also explored different initial and boundary conditions. Byszewski was the first who studied fractional differential equations with nonlocal conditions, see [3, 4, 5].

In [11], authors studied fractional integrodifferential equations with Caputo fractional derivative with nonlocal conditions in Banach space and studied existence results for the problem. Many recent papers have dealt with the existence, uniqueness and other properties of solutions of special forms of the equations (1.1), see [9, 10, 12, 13, 14, 15, 18] and some of the references cited therein.

Recently, in an interesting paper [2], the authors have studied existence and stability results of solution for the initial value problem

$$\begin{cases} ({}^\delta D^{\mu,\eta}_{a+}u)(t) &= g(t,u(t)), \quad t \in [a,b] \\ ({}^\delta I^{1-\zeta}_{a+}u)(a) &= c_2, \qquad c_2 \in \mathbb{R}, \zeta = \mu + \eta(1-\mu) \end{cases} \tag{1.1}$$

for generalized Katugampola fractional differential equation by Schauder fixed point theorem and its equivalent integral equation is

$$u(t) = \frac{c_2}{\Gamma(\zeta)}\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{\zeta-1} + \frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1} g(p,u(p))dp \tag{1.2}$$

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* Corresponding author.

In the present article, we propose generalized fractional integrodifferential equation of the type

$$\begin{cases} ({}^{\delta}D^{\mu,\eta}_{a+}u)(t) & = g(t,u(t),\int_0^p k(p,\tau)u(\tau)d\tau), \quad 0<\mu<1, 0\le\eta\le 1, \\ ({}^{\delta}I^{1-\zeta}u)(a) & = \sum_{i=1}^{n}\lambda_i u(\xi_i), \qquad \mu\le\zeta=\mu+\eta(1-\mu), \end{cases} \tag{1.3}$$

where $t\in(a,T], \xi_i\in(a,T], \mu\in(0,1),\ \eta\in[0,1],\ \delta>0,\ \mu\le\zeta=\mu+\eta(1-\mu),\ \xi_i\in(a,T]$ and ${}^{\delta}D^{\mu,\eta}_{a+}$ and ${}^{\delta}I^{1-\zeta}$ denote generalized Katugampola fractional derivative of order μ and Katugampola fractional integral of order $1-\zeta$, function $g:(a,T]\times(a,T]\to\mathbb{R}$ is a given function, ξ_i are pre-fixed points satisfying $0<a<\xi_1\le\xi_2\le\cdots\le\xi_n<T$ and $\lambda_i=1,2,\cdots,n$ are real numbers.

2. Preliminaries

In this section, we see some important definitions and results that we use in the paper. Beta and Gamma functions are defined by

$$\Gamma(\mu)=\int_0^\infty t^{\mu-1}e^{-t}dt,\ B(\mu,\eta)=\int_0^1(1-t)^{\mu-1}t^{\eta-1}dt,\ \mu,\ \eta>0.$$

Definition 2.1. ([16]) The space $X^p_c(a,T), c\in\mathbb{R}, p\ge 1$ consists of all real valued Lebesgue measurable functions g on (a,T) for which $||g||_{X^p_c}<\infty$, where

$$||g||_{X^p_c}=\Big(\int_a^b |t^c g(t)|^p\frac{dt}{t}\Big)^{1/p},\ p\ge 1,\ \text{and}\ ||g||_{X^\infty_c}=\sup_{a\le t\le T}|t^c g(t)|.$$

In particular, when $c=\frac{1}{p}$ we get $X^c_{1/p}(a,T)=L_p(a,T)$.

Definition 2.2. ([6]) We denote by $C[a,T]$, a space of continuous functions g on $(a,T]$ with the norm

$$||g||_C=\max_{t\in[a,T]}|g(t)|.$$

The weighted space

$$C_{\zeta,\delta}[a,T]=\Big\{g:(a,T]\to\mathbb{R}:\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^\zeta g(t)\in C[a,T].\Big\} \tag{2.1}$$

with the norm

$$||g||_{C_{\zeta,\delta}}=\|\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^\zeta g(t)\|=\max_{t\in[a,T]}\Big|\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^\zeta g(t)\Big|\ \text{and}\ C_{0,\delta}[a,T]=C[a,T].$$

Definition 2.3. [6] Let $\Delta_\delta=\Big(t^{\delta-1}d/dt\Big),\ 0\le\zeta<1$. Also denote $C^n[a,T]$ the Banach space of functions g which are continuously differentiable, with Δ_δ on $[a,T]$ upto order $(n-1)$ and have derivative $\Delta^k_\delta g$ on $(a,T]$ such that $\Delta^n_\delta g\in C_{\zeta,\delta}[a,T]$.

$$C^n_{\Delta_\delta,\zeta}[a,T]=\Big\{\Delta^k_\delta g\in C[a,T], k=0,1,\cdots,n-1, \Delta^n_\delta g\in C_{\zeta,\delta}[a,T]\Big\}, n\in\mathbb{N}$$

with the norm given by

$$||g||_{C^n_{\Delta_\delta,\zeta}}=\sum_{k=0}^{n-1}||\Delta^k_\delta g||_C+||\Delta^n_\delta g||_{C_{\zeta,\delta}},\ ||g||_{C^n_{\Delta_\delta,\zeta}}=\sum_{k=0}^{n}\max_{t\in(a,T]}|\Delta^k_\delta g(t)|$$

In particular, for $n=0$, we get $C^0_{\Delta_\delta,\zeta}[a,T]=C_{\zeta,\delta}[a,T]$.

Definition 2.4. [7] Let $\mu > 0$ and $g \in X_c^p(a,T)$, where X_c^p is as defined in Definition (2.1). Then the left-sided Katugampola fractional integral ${}^{\delta}I_{a+}^{\mu}$ of order μ is defined as

$$ {}^{\delta}I_{a+}^{\mu}g(t) = \frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^{\delta}-a^{\delta}}{\delta}\Big)^{\mu-1}g(p)dp, t > a \tag{2.2}$$

Definition 2.5. [8] Let $\mu \in \mathbb{R}^+ - \mathbb{N}$ and $n = [\mu]+1$, where $[\mu]$ is the integer part of μ. The left sided Katugampola fractional derivative ${}^{\delta}D_{a+}^{\mu}$ is defined as

$$ {}^{\delta}D_{a+}^{\mu}g(t) = \Delta_{\delta}^{n}({}^{\delta}I_{a+}^{n-\mu}g(p))(t) \tag{2.3}$$

$$ = \Big(t^{\mu-1}\frac{d}{dt}\Big)^n\frac{1}{\Gamma(n-\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^{\delta}-a^{\delta}}{\delta}\Big)^{n-\mu-1}g(s)ds \tag{2.4}$$

Definition 2.6. [6] The left-sided generalized Katugampola fractional derivative ${}^{\delta}D_{a+}^{\mu,\eta}$ of order $0 < \mu < 1$ and type $0 \le \eta \le 1$ is defined as

$$ ({}^{\delta}D_{a+}^{\mu,\eta}g)(t) = ({}^{\delta}I_{a+}^{\eta(1-\mu)}\Delta_{\delta}^{\delta}I_{a+}^{(1-\eta)(1-\mu)}g)(t) \tag{2.5}$$

for the functions for which the right-hand side expression exists.

Lemma 2.7. *([17]) Suppose that* $\mu > 0,\ \eta > 0,\ q \ge 1$ *and* $\delta,\ c \in \mathbb{R}$ *such that* $\delta \ge c$. *Then for* $g \in X_c^q(a,T)$, *the semigroup property of Katugampola integral is valid. i.e.*

$$ ({}^{\delta}I_{a+}^{\mu})({}^{\delta}I_{a+}^{\eta})g(t) = {}^{\delta}I_{a+}^{\mu+\eta}g(t) \tag{2.6}$$

Lemma 2.8. *([8]) Suppose that* $\mu > 0,\ 0 \le \zeta < 1$ *and* $g \in C_{\zeta,\delta}[a,T]$. *Then for* $t \in (a,T]$,

$$ ({}^{\delta}D_{a+}^{\mu})({}^{\delta}I_{a+}^{\mu})g(t) = g(t) \tag{2.7}$$

Lemma 2.9. *([8]) Suppose that* $\mu > 0,\ 0 \le \zeta < 1$ *and* $g \in C_{\zeta,\delta}[a,T]$. *and* ${}^{\delta}I_{a+}^{1-\mu}g \in C_{\zeta,\delta}^{1}[a,T]$. *Then,*

$$ ({}^{\delta}I_{a+}^{\mu})({}^{\delta}D_{a+}^{\mu})g(t) = g(t) - \frac{{}^{\delta}I_{a+}^{1-\mu}g(a)}{\Gamma(\mu)}\Big(\frac{t^{\delta}-a^{\delta}}{\delta}\Big)^{\mu-1} \tag{2.8}$$

Lemma 2.10. *([1]) If* ${}^{\delta}I_{a+}^{\mu}$ *and* ${}^{\delta}D_{a+}^{\mu}$ *are defined as in Definition 2.4 and 2.5 respectively, then*

$$ {}^{\delta}I_{a+}^{\mu}\Big(\frac{t^{\delta}-a^{\delta}}{\delta}\Big)^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma+1)}\Big(\frac{t^{\delta}-a^{\delta}}{\delta}\Big)^{\sigma+\mu-1},\ \mu \ge 0,\ \sigma > 0,\ t > a$$

$$ {}^{\delta}D_{a+}^{\mu}\Big(\frac{t^{\delta}-a^{\delta}}{\delta}\Big)^{\mu-1} = 0,\ 0 < \mu < 1$$

Remark 2.11. ([1]) For $0 < \mu < 1,\ 0 \le \eta \le 1$, the generalized Katugampola fractional derivative ${}^{\delta}D_{a+}^{\mu,\eta}$ can be written in terms of Katugampola fractional derivative as

$$ {}^{\delta}D_{a+}^{\mu,\eta} = ({}^{\delta}I_{a+}^{\eta(1-\mu)})\Delta_{\delta}({}^{\delta}I_{a+}^{1-\mu}) = ({}^{\delta}I_{a+}^{\eta(1-\mu)})({}^{\delta}D_{a+}^{\zeta}),\ \zeta = \mu + \eta(1-\mu)$$

Lemma 2.12. *([6]) Let* $\mu > 0,\ 0 < \zeta \le 1$ *and* $g \in C_{1-\zeta,\delta}[a,b]$. *If* $\mu > \zeta$, *then*

$$ {}^{\delta}I_{a+}^{\mu}g(a) = \lim_{t\to a+}({}^{\delta}I_{a+}^{\mu}g)(t) = 0$$

To discuss the main results, we need following spaces.

$$C^{\mu,\eta}_{1-\zeta,\delta}[a,T] = \Big\{ g \in C_{1-\zeta,\delta}[a,T] :^{\delta} D^{\mu,\eta}_{a+} g \in C_{1-\zeta,\delta}[a,T] \Big\},\ 0 < \zeta \le 1 \tag{2.9}$$

and

$$C^{\zeta}_{1-\zeta,\delta}[a,T] = \Big\{ g \in C_{1-\zeta,\delta}[a,T] :^{\delta} D^{\zeta}_{a+} g \in C_{1-\zeta,\delta}[a,T] \Big\},\ 0 < \zeta \le 1$$

as ${}^{\delta}D^{\mu,\eta}_{a+} g = ({}^{\delta}I^{\eta(1-\mu)}_{a+})({}^{\delta}D^{\zeta}_{a+})g$, it is clear that $C^{\zeta}_{1-\zeta,\delta}[a,T] \subset C^{\mu,\eta}_{1-\zeta,\delta}[a,T]$.

Lemma 2.13. *([8]) Let $\mu > 0,\ \eta > 0$ and $\zeta = \mu + \eta - \mu\eta$. If $g \in C^{\zeta}_{1-\zeta,\delta}[a,T]$, then*

$$({}^{\delta}I^{\zeta}_{a+})(({}^{\delta}D^{\zeta}_{a+})g(t) = ({}^{\delta}I^{\mu}_{a+})({}^{\delta}D^{\mu,\eta}_{a+})g(t) =^{\delta} D^{\eta(1-\mu)}_{a+} g(t)$$

Lemma 2.14. *([6]) Let $0 < \mu < 1,\ 0 \le \eta \le 1,\ \zeta = \mu+\eta-\mu\eta$. If $g : (a,T]\times\mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot,\ u(\cdot)) \in C_{1-\zeta,\delta}[a,T]$ for any $u \in C_{1-\zeta,\delta}[a,T]$ for any $u \in C^{\zeta}_{1-\zeta,\delta}[a,T]$ satisfies initial value problem (1.1) iff u satisfies the nonlinear Volterra integral equation (1.2).*

Theorem 2.15. *([1]) Let $0 < \mu < 1,\ 0 \le \eta \le 1,\ \zeta = \mu + \eta - \mu\eta$. If $g : (a,T] \times (a,T] \to \mathbb{R}$ is a function such that $g(\cdot,\ u(\cdot)) \in C_{1-\zeta,\delta}[a,T]$ for any $u \in C_{1-\zeta,\delta}[a,T]$ for any $u \in C^{\zeta}_{1-\zeta,\delta}[a,T]$ satisfies initial value problem (1.3) iff u satisfies the mixed-type nonlinear Volterra integral equation*

$$\begin{aligned} u(t) = {} & \frac{K}{\Gamma(\mu)}\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{\zeta-1} \sum_{i=1}^{n} \lambda_i \int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} \\ & \times g\left(p, u(p), \int_0^p k(p,\tau)u(\tau)d\tau\right) dp + \frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1} \\ & \times g\left(p, u(p), \int_0^p k(p,\tau)u(\tau)d\tau\right) dp \end{aligned} \tag{2.10}$$

where

$$K = \left(\Gamma(\zeta) - \sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\right)^{-1} \tag{2.11}$$

3. Main Results

In this section, we prove the existence and uniqueness results for our problem. For this, we state some hypotheses:

(H_1) $g : (a,T] \times (a,T] \to \mathbb{R}$ is a function such that $g(\cdot, u(\cdot)) \in C^{\eta(1-\mu)}_{1-\zeta,\delta}[a,T]$ for any $u \in C_{1-\zeta,\delta}$ and there exists a positive constant $M, N > 0$ such that for all $u_1, v_1, u_2, v_2 \in (a,T]$,

$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \le M|u_1 - v_1| + N|u_2 - v_2|. \tag{3.1}$$

(H_2) The constants

$$\Omega_1 = \frac{\Gamma(\zeta)}{\Gamma(\zeta+\mu)}\left(|K|\sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu+\zeta-1} + \Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu}\right) \tag{3.2}$$

and

$$\Omega_2 = \frac{T^{\zeta} N k_T}{\Gamma(\mu+1)} \Big[\sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^{\delta} - a^{\delta}}{\delta} \Big)^{\mu} + \Big(\frac{T^{\delta} - a^{\delta}}{\delta} \Big)^{\mu-\zeta+1} \Big] \tag{3.3}$$

are such that

$$M\Omega_1 + \Omega_2 < 1 \tag{3.4}$$

Theorem 3.1. *If the hypotheses* $(H_1) - (H_2)$ *hold and if*

$$\Theta = |K| \Big[\frac{M\Gamma(\zeta)}{\Gamma(\mu+\zeta)} \sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^{\delta} - a^{\delta}}{\delta} \Big)^{\mu+\zeta-1} + \frac{T^{\zeta} N k_T}{\Gamma(\mu+1)} \sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^{\delta} - a^{\delta}}{\delta} \Big)^{\mu} \Big] < 1 \tag{3.5}$$

then the problem (1.3) has at least one solution in $C^{\zeta}_{1-\zeta,\delta}[a,T] \subset C^{\mu,\eta}_{1-\zeta,\delta}[a,T]$.

Proof. To prove this result, it is enough to prove that the integral equation (2.10) has a solution.
Define an operator $T : C_{1-\zeta,\delta}[a,T] \to C_{1-\zeta,\delta}[a,T]$ by

$$\begin{aligned}(Tu)(t) = {} & \frac{K}{\Gamma(\mu)} \Big(\frac{t^{\delta} - a^{\delta}}{\delta} \Big)^{\zeta-1} \sum_{i=1}^{n} \lambda_i \int_a^{\xi_i} p^{\delta-1} \Big(\frac{\xi_i^{\delta} - p^{\delta}}{\delta} \Big)^{\mu-1} \times \\ & g\Big(p, u(p), \int_0^p k(p,\tau) u(\tau) d\tau \Big) dp + \frac{1}{\Gamma(\mu)} \int_a^t p^{\delta-1} \Big(\frac{t^{\delta} - p^{\delta}}{\delta} \Big)^{\mu-1} \\ & g\Big(p, u(p), \int_0^p k(p,\tau) u(\tau) d\tau \Big) dp \end{aligned} \tag{3.6}$$

Then T is well defined. Define $G(p) = g(p,0,0)$ and consider a closed ball

$$B_r = \{ u \in C_{1-\zeta,\delta}[a,T] : ||u||_{C_{1-\zeta,\delta}} \le r \} \text{ where } r \ge \frac{\Omega_1}{1 - (M\Omega_1 + \Omega_2)}$$

Let us divide the operator T into two parts T_1, T_2 on B_r as follows.

$$\begin{aligned}(T_1 u)(t) = {} & \frac{K}{\Gamma(\mu)} \Big(\frac{t^{\delta} - a^{\delta}}{\delta} \Big)^{\zeta-1} \sum_{i=1}^{n} \lambda_i \int_a^{\xi_i} p^{\delta-1} \Big(\frac{\xi_i^{\delta} - p^{\delta}}{\delta} \Big)^{\mu-1} \\ & g\left(p, u(p), \int_0^p k(p,\tau) u(\tau) d\tau \right) dp \end{aligned} \tag{3.7}$$

and

$$(T_2 u)(t) = \frac{1}{\Gamma(\mu)} \int_a^t p^{\delta-1} \Big(\frac{t^{\delta} - p^{\delta}}{\delta} \Big)^{\mu-1} g(p, u(p), \int_0^p k(p,\tau) u(\tau) d\tau) dp \tag{3.8}$$

Claim 1: For any $u,\ v \in B_r,\ T_1u + T_2v \in B_r$. Consider

$$
\begin{aligned}
&\Big|(T_1u)(t)\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\Big| \\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1}|g(p,u(p),\int_0^p k(p,\tau)u(\tau)d\tau)|dp \\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} \\
&\Big(|g(p,u(p),\int_0^p k(p,\tau)u(\tau)d\tau) - g(p,0,0)| + |g(p,0,0)|\Big)dp \\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} \\
&\Big(M|u(p)| + N\Big|\int_0^p k(p,\tau)u(\tau)d\tau\Big| + |G(p)|\Big)dp \\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} \\
&\times\Big(M|u(p)| + Nk_T\Big|\int_0^p \Big(\frac{\tau^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\Big(\frac{\tau^\delta - a^\delta}{\delta}\Big)^{1-\zeta}u(\tau)d\tau\Big| + |G(p)|\Big)dp \\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(M\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{1-\zeta}|u(p)| \\
&+\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{1-\zeta}Nk_T||u||_{C_{1-\zeta,\delta}}T^\zeta + \Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{1-\zeta}|G(p)|\Big)dp \\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} \\
&\Big(M||u||_{C_{1-\zeta,\delta}} + ||G||_{C_{1-\zeta,\delta}}\Big)dp \\
&+\frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1}Nk_T||u||_{C_{1-\zeta,\delta}}T^\zeta dp \\
&\leq \frac{|K|\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu+\zeta-1}\Big(M||u||_{C_{1-\zeta,\delta}} + ||G||_{C_{1-\zeta,\delta}}\Big) \\
&+\frac{|K|}{\Gamma(\mu+1)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu}Nk_T||u||_{C_{1-\zeta,\delta}}T^\zeta
\end{aligned}
$$

$$||T_1u||_{C_{1-\zeta,\delta}} \leq |K|\Big[M\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu+\zeta-1} + \frac{1}{\Gamma(\mu+1)}$$
$$\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu}T^\zeta N k_T\Big]||u||_{C_{1-\zeta,\delta}} + \frac{|K|\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu+\zeta-1}||G||_{C_{1-\zeta,\delta}} \tag{3.9}$$

For operator T_2, consider

$$\Big|(T_2v)(t)\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\Big|$$
$$\leq \frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}|g(p,v(p),\int_a^p k(p,\tau)v(\tau)d\tau)|dp$$
$$\leq \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}\times$$
$$\Big(|g(p,v(p),\int_a^p k(p,\tau)v(\tau)d\tau) - g(p,0,0)| + |g(p,0,0)|\Big)dp$$
$$\leq \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1}$$
$$\times\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{1-\zeta}(M|v(p)| + \int_a^p |k(p,\tau)v(\tau)d\tau| + |G(p)|)dp$$
$$\leq \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}$$
$$\times\Big(M|v(p)| + Nk_T\Big|\int_0^p\Big(\frac{\tau^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\Big(\frac{\tau^\delta - a^\delta}{\delta}\Big)^{1-\zeta}v(\tau)d\tau\Big| + |G(p)|\Big)dp$$
$$\leq \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\mu-1}\Big(M\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{1-\zeta}|v(p)|$$
$$+\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{1-\zeta}Nk_T||v||_{C_{1-\zeta,\delta}}T^\zeta + \Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{1-\zeta}|G(p)|\Big)dp$$
$$\leq \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^a p^{\delta-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(M||v||_{C_{1-\zeta,\delta}}$$
$$+||G||_{C_{1-\zeta,\delta}}\Big)dp + \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}Nk_T||v||_{C_{1-\zeta,\delta}}T^\zeta dp$$
$$\leq \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{\mu+\zeta-1}\Big(M||v||_{C_{1-\zeta,\delta}} + ||G||_{C_{1-\zeta,\delta}}\Big)$$
$$+\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu+1)}\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{\mu}Nk_T||v||_{C_{1-\zeta,\delta}}T^\zeta$$
$$\leq \Big[M\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu} + Nk_T T^\zeta\frac{1}{\Gamma(\mu+1)}\Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu-\zeta+1}\Big]||v||_{C_{1-\zeta,\delta}}$$
$$+\Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu}\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}||G||_{C_{1-\zeta,\delta}}$$

$$
\begin{aligned}
||T_2v||_{C_{1-\zeta,\delta}} &\leq \Big[M\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}+Nk_TT^{\zeta}\frac{1}{\Gamma(\mu+1)}\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu-\zeta+1}\Big]\\
&\times||v||_{C_{1-\zeta,\delta}}+\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}||G||_{C_{1-\zeta,\delta}} \qquad (3.10)
\end{aligned}
$$

Then using (3.9), (3.10) in

$$
\begin{aligned}
&||T_1u+T_2v||_{C_{1-\zeta,\delta}}\\
&\leq ||T_1u||_{C_{1-\zeta,\delta}}+||T_2v||_{C_{1-\zeta,\delta}}\\
&\leq |K|\Big[M\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}+\frac{1}{\Gamma(\mu+1)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu}T^{\zeta}Nk_T\Big]\\
&\times||u||_{C_{1-\zeta,\delta}}+\frac{|K|\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}||G||_{C_{1-\zeta,\delta}}\\
&+\Big[M\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}+Nk_TT^{\zeta}\frac{1}{\Gamma(\mu+1)}\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu-\zeta+1}\Big]||v||_{C_{1-\zeta,\delta}}\\
&+\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}||G||_{C_{1-\zeta,\delta}}\\
&\leq \Big\{|K|\Big[M\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}+\frac{1}{\Gamma(\mu+1)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu}\\
&+\Big[\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}M\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}+\frac{1}{\Gamma(\mu+1)}Nk_TT^{\zeta}\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu-\zeta+1}\Big]\Big\}r\\
&+\Big\{\frac{|K|\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}+\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big\}||G||_{C_{1-\zeta,\delta}}\\
&\leq \Big\{M\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big[|K|\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}+\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}\Big]\\
&+\frac{T^{\zeta}Nk_T}{\Gamma(\mu+1)}\Big[\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu}+\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu-\zeta+1}\Big]\Big\}r\\
&+\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big\{|K|\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}+\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}\Big\}||G||_{C_{1-\zeta,\delta}} \qquad (3.11)\\
&\leq (M\Omega_1+\Omega_2)r+\Omega_1\leq r \qquad (3.12)
\end{aligned}
$$

Thus, $T_1u+T_2v\in B_r$.
Claim 2: T_1 is a contraction.

Let $u, v \in B_r$ and consider

$$\begin{aligned}
&\Big|((T_1u)-(T_1v))(t)\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\Big|\\
&\le \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big|g\Big(p,u(p),\int_0^p k(p,\tau)u(\tau)d\tau\Big)\\
&-g\Big(p,v(p),\int_0^p k(p,\tau)v(\tau)d\tau\Big)\Big|dp\\
&\le \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big[M|u(p)-v(p)|\\
&+N\Big|\int_0^p k(p,\tau)[u(\tau)-v(\tau)]d\tau\Big|\Big]dp\\
&\le \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\Big[M\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}|u(p)-v(p)|\\
&+Nk_T\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\Big|\int_0^p\Big(\frac{\tau^\delta-a^\delta}{\delta}\Big)^{\zeta-1}||u-v||_{C_{1-\zeta_{1-\zeta,\delta}}}d\tau\Big|\Big]dp\\
&\le \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\\
&\times\Big[M+Nk_TT^\zeta\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\Big]dp||u-v||_{C_{1-\zeta_{1-\zeta,\delta}}}\\
&\le |K|\Big[\frac{M\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}\\
&+\frac{T^\zeta Nk_T}{\Gamma(\mu+1)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu}\Big]||u-v||_{C_{1-\zeta_{1-\zeta,\delta}}}\\
&||T_1u-T_1v||_{C_{1-\zeta,\delta}}\le\Theta||u-v||_{C_{1-\zeta_{1-\zeta,\delta}}}
\end{aligned}$$

This shows that T_1 is a contraction.
Claim 3: T_2 is compact.

For $0 < a < t_1 < t_2 < T$, consider

$$\begin{aligned}
&|(T_2u)(t_1) - (T_2u)(t_2)| \\
&\leq \Big|\frac{1}{\Gamma(\mu)}\int_a^{t_1} p^{\delta-1}\Big(\frac{t_1^\delta - p^\delta}{\delta}\Big)^{\mu-1} g(p, u(p), \int_0^p k(p,\tau)u(\tau)d\tau)dp \\
&- \frac{1}{\Gamma(\mu)}\int_a^{t_2} p^{\delta-1}\Big(\frac{t_2^\delta - p^\delta}{\delta}\Big)^{\mu-1} g(p, u(p), \int_0^p k(p,\tau)u(\tau)d\tau)dp\Big| \\
&\leq \frac{||g||_{C_{1-\zeta,\delta}}}{\Gamma(\mu)}\Big|\int_a^{t_1} p^{\delta-1}\Big(\frac{t_1^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1}dp \\
&- \int_a^{t_2} p^{\delta-1}\Big(\frac{t_2^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1}dp\Big| \\
&\frac{\Gamma(\zeta)||g||_{C_{1-\zeta,\delta}}}{\Gamma(\mu+\zeta)}\Big|\Big(\frac{t_1^\delta - a^\delta}{\delta}\Big)^{\mu+\zeta-1} - \Big(\frac{t_2^\delta - a^\delta}{\delta}\Big)^{\mu+\zeta-1}\Big|
\end{aligned}$$

One can see that as $t_2 \to t_1$, RHS of the above inquality tends to zero in both the cases $\mu + \zeta < 1$ or $\mu + \zeta \geq 1$. Hence, T_2 is equicontinuous. Hence, by the Arzela-Ascoli theorem, T_2 is compact on B_r. As all the conditions of Kransnoselskii fixed point theorems are satisfied, we can say that the problem has at least one solution in $C_{1-\zeta,\delta}[a, T]$ and hence in $C^\delta_{1-\zeta,\delta}[a, T]$. This completes the proof. □

Now, we prove uniqueness theorem.

Theorem 3.2. *If the hypotheses $(H_1) - (H_2)$ hold, then the problem has unique solution.*

Proof. From the operator defined in the previous theorem, we have
$T : C_{1-\zeta,\delta}[a, T] \to C_{1-\zeta,\delta}[a, T]$ by

$$\begin{aligned}
(Tu)(t) = &\frac{K}{\Gamma(\mu)}\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\sum_{i=1}^n \lambda_i \int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} \\
&g\Big(p, u(p), \int_0^p k(p,\tau)u(\tau)d\tau\Big)dp + \frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1} \\
&g\Big(p, u(p), \int_0^p k(p,\tau)u(\tau)d\tau\Big)dp, \qquad (3.13)
\end{aligned}$$

from the Claim 1 of the previous theorem, we can see that for
$u \in B_r, ||Tu||_{C_{1-\zeta,\delta}} \leq r$.

Next, we prove that T is a contraction. Consider

$$
\begin{aligned}
&\Big|((Tu)-(Tv))(t)\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\Big|\\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big|g\Big(p,u(p),\int_0^t k(p,\tau)u(\tau)d\tau\Big)\\
&-g\Big(p,v(p),\int_0^t k(p,\tau)v(\tau)d\tau\Big)\Big|dp\\
&+\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big|g\Big(p,u(p),\int_0^t k(p,\tau)u(\tau)d\tau\Big)\\
&-g\Big(p,v(p),\int_0^t k(p,\tau)v(\tau)d\tau\Big)\Big|dp\\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big[M|u(p)-v(p)|\\
&+N\Big|\int_0^t k(p,\tau)[u(\tau)-v(\tau)]d\tau\Big|\Big]dp\\
&+\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big[M|u(p)-v(p)|\\
&+N\Big|\int_0^t k(p,\tau)[u(\tau)-v(\tau)]d\tau\Big|\Big]dp\\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\\
&\times\Big[M\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}|u(p)-v(p)|+Nk_T\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\\
&\times\Big|\int_0^p\Big(\frac{\tau^\delta-a^\delta}{\delta}\Big)^{\zeta-1}||u-v||_{C_{1-\zeta_{1-\zeta,\delta}}}d\tau\Big|\Big]dp\\
&+\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\\
&\times\Big[M\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}|u(p)-v(p)|+Nk_T\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\\
&\times\Big|\int_0^p\Big(\frac{\tau^\delta-a^\delta}{\delta}\Big)^{\zeta-1}||u-v||_{C_{1-\zeta_{1-\zeta,\delta}}}d\tau\Big|\Big]dp\\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\\
&\times\Big[M+Nk_TT^\zeta\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\Big]dp||u-v||_{C_{1-\zeta_{1-\zeta,\delta}}}\\
&+\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\\
&\times\Big[M+Nk_TT^\zeta\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\Big]dp||u-v||_{C_{1-\zeta_{1-\zeta,\delta}}}\\
&\leq |K|\Big[\frac{M\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}\\
&+\frac{T^\zeta Nk_T}{\Gamma(\mu+1)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu}\Big]||u-v||_{C_{1-\zeta_{1-\zeta,\delta}}}\\
&+\Big[M\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}+NT^\zeta k_T\frac{1}{\Gamma(\mu+1)}\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu-\zeta+1}\Big]||u-v||_{C_{1-\zeta_{1-\zeta,\delta}}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Big\{ M \frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)} \Big[|K| \sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^\delta - a^\delta}{\delta} \Big)^{\mu+\zeta-1} + \Big(\frac{T^\delta - a^\delta}{\delta} \Big)^{\mu} \Big] \\
&+ \frac{T^\zeta k_T N}{\Gamma(\mu+1)} \Big[\sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^\delta - a^\delta}{\delta} \Big)^{\mu} + \Big(\frac{T^\delta - a^\delta}{\delta} \Big)^{\mu-\zeta+1} \Big] \Big\} ||u-v||_{C_{1-\zeta,\delta}} \\
&\leq (M\Omega_1 + \Omega_2) ||u-v||_{C_{1-\zeta,\delta}}
\end{aligned}
$$

$$
||Tu - Tv||_{C_{1-\zeta,\delta}} \leq (M\Omega_1 + \Omega_2) ||u-v||_{C_{1-\zeta,\delta}}, \forall u, v \in B_r \tag{3.14}
$$

Thus, T is a contraction map and hence by the Banach fixed point theorem, the operator has unique solution in $C_{1-\zeta,\delta}[a,T]$.

□

4. Examples

In this section, we apply the result to example.

Example 4.1. Consider the nonlocal problem

$$
\begin{cases}
({}^\delta D_{a+}^{\mu,\eta} u)(t) &= g\left(t, u(t), \int_1^t k(t,p)u(p)dp\right), \quad t \in (1,2] \\
({}^\delta I_{a+}^{1-\zeta} u)(1+) &= u(\frac{5}{3}), \qquad \zeta = \mu + \eta(1-\mu)
\end{cases} \tag{4.1}
$$

Set $\mu = \frac{1}{2},\ \eta = \frac{1}{5}$ then $\zeta = \frac{3}{5}$. Also, let $\delta = \frac{1}{2}$ and

$$
g(t, u(t), \int_1^t k(t,p)u(p)dp) = \Big(\frac{t^\delta - 1}{\delta} \Big)^{-1/12} + \frac{1}{8} \Big(\frac{t^\delta - 1}{\delta} \Big)^{13/14} u(t) + \frac{1}{7} \int_1^t u(p)dp,
$$

with $k(t,p) = 1$, so that $k_T = 1$, then

$$
\begin{aligned}
\Big(\frac{t^\delta - 1}{\delta} \Big)^{2/5} g(t, u(u(t))) &= \Big(\frac{t^\delta - 1}{\delta} \Big)^{19/60} + \frac{1}{8} \Big(\frac{t^\delta - 1}{\delta} \Big)^{93/70} u(t) \\
&+ \frac{1}{7} \Big(\frac{t^\delta - 1}{\delta} \Big)^{2/5} \int_1^t u(p)dp \in C[1,2]
\end{aligned}
$$

Therefore $g(t, u(t), \int_1^t k(t,p)u(p)dp) \in C_{2/5,1/2}[1,2]$.

Also, $|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq \frac{1}{8}|u_1 - v_1| + \frac{1}{7}|u_2 - v_2|,\ \forall\ u_1,\ u_2,\ v_1,\ v_2$ so $M = \frac{1}{8},\ N = \frac{1}{7}$.

$$
|K| = \left| \left(\Gamma(\frac{3}{5}) - 1.1 \Big(\frac{(5/3)^{\frac{1}{2}} - 1}{\frac{1}{2}} \Big)^{-2/5} \right)^{-1} \right| \approx 0.9807811549 < 1.
$$

and

$$
\begin{aligned}
\Theta = 0.9807811549 \Big[&\frac{1}{8} \frac{\Gamma(3/5)}{\Gamma(11/10)} (1.1) \Big(\frac{(5/3)^{\frac{1}{2}} - 1}{\frac{1}{2}} \Big)^{\frac{1}{10}} \\
&+ \frac{2^{3/5} \frac{1}{7} 1}{\Gamma(3/2)} (1.1) \Big(\frac{(5/3)^{\frac{1}{2}} - 1}{\frac{1}{2}} \Big)^{\frac{1}{2}} \Big] \approx 0.58426248648 < 1
\end{aligned}
$$

Hence, by the Theorem (3.1), the problem (4.1) has at least one solution in $C_{2/5,1/2}[1,2]$.

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M. A. Kazi: Department of Mathematics, N. B. Navale Sinhgad College of Engg., Kegaon, Solapur-413255, (M.S.) India
Email address: mansooratmsc@gmail.com

V. V. Kharat: Department of Mathematics, N. B. Navale Sinhgad College of Engg., Kegaon, Solapur-413255, (M.S.) India
Email address: vvkvinod9@gmail.com

A. R. Reshimkar: Department of Mathematics, D. B. F. Dayanand College of Arts and Science, Solapur-413002, (M.S.) India
Email address: anand.reshimkar@gmail.com

M. T. Gophane: Department of Mathematics, Shivaji University, Kolhapur-416004, (M.S.) India
Email address: gmachchhindra@gmail.com

S. D. Thikane: Department of Mathematics, Jaysingpur College, Jaysingpur-416101, (M.S.) India
Email address: surendra.thikane@gmail.com