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RESULTS ON TOTALLY CONTACT UMBILICAL SCREEN SLANT LIGHTL IKE SUBMANI FOLDS OF INDEFINITE KENMOTSU MANIFOLDS

D. P. SEMWAL
Department of Mathematics, DAV College, Sector - 10, Chandigarh - 160011, India
E-mail: dpsemwal27@gmail.com

ABSTRACT: The present article delves into the investigation of the geometric properties of screen slant lightlike submanifolds within the context of indefinite Kenmotsu manifolds with providing a non-trivial example. The main focus of this article is on totally contact umbilical screen slant lightlike submanifolds followed by a non-trivial example. Our study establishes a significant result, demonstrating that every totally contact proper umbilical screen slant lightlike submanifold of an indefinite Kenmotsu manifold is totally contact geodesic.

Keywords: Indefinite Kenmotsu manifolds; screen slant lightlike submanifolds; totally contact umbilical lightlike submanifolds.

AMS Subject Classification: 53B30, 53B25, 53B35

1. Introduction

The theory of lightlike submanifolds is very importent due to its potential applications in mathematical physics, especially in the general relativity as lightlike submanifolds can be models of event horizons, Cauchy's horizons, Kruskal's horizons etc. Due to this relevancy, lightlike submanifolds is an importent topic of research in the field of differential geometry. The concept of lightlike submanifolds, which is characterized by a degenerate induced metric, is thoroughly examined by Duggal and Bejancu in their seminal work [1]. Various classifications of lightlike submanifolds in indefinite Kenmotsu manifolds have been established, taking into consideration the action of the $(1,1)$-tensor field φ on the tangent and normal bundles of the submanifolds. To introduce the angle other than 0 and $\frac{\pi}{2}$, between the vector fields of the associated distributions, Sahin further investigated the slant lightlike submanifolds in the context of indefinite Hermitian manifolds [8], and then other generalized classes of slant lightlike submanifolds came into existence and developed in [9, 10]. Due to the physical appearance of these submanifolds, geometry of slant and screen slant, hemi slant lightlike submanifolds within indefinite Kenmotsu manifolds was explored in [3, 4, 6]. However, the totally umbilical characteristic of the ambient spaces has its own significance, therefore geometers investigated this characteristic in the slant structure [7].

In this paper, the geometry of screen slant lightlike submanifolds of indefinite Kenmotsu manifolds is studied and the integrability conditions for the associated distributions of screen slant lightlike submanifolds are obtained along with a non-trivial example. Then the article focuses on totally contact umbilical screen slant lightlike submanifolds followed by a nontrivial example. Further, the study establishes a significant result, demonstrating that every totally contact proper umbilical screen slant lightlike submanifold of an indefinite Kenmotsu manifold is totally contact geodesic.

2. Preliminaries

Consider a submanifold (M, g) of dimension m embedded in a semi-Riemannian manifold of dimension $(m + n)$ with a constant index q. Here, we have $1 \leq q \leq m + n - 1$ and $m, n \geq 1$. If the metric g becomes degenerate on the tangent bundle TM of M, then M is referred to as a lightlike submanifold. As a consequence of the degenerate metric g on M, the subspaces $TyM^{\perp}$ and TyM become degenerate orthogonal subspaces but are not complementary to each other. This leads to the existence of a radical subspace denoted as $Rad(TM) = TyM \cap TyM^{\perp}$. Moreover, if the mapping $Rad(TM) : M \to TM$ is defined such that it assigns $y \in M$ to $Rad(TyM)$, this gives rise to a smooth distribution of rank $r > 0$ on M. In such cases, M is classified as an r-lightlike submanifold, and the distribution $Rad(TM)$ is referred to as the radical distribution on M. The complementary non-degenerate subbundles $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, respectively, are known as the screen distribution in TM and the screen transversal distribution in $TM^{\perp}$. This can be expressed as:

$$TM = Rad(TM) \perp S(TM) \text{ and } TM^{\perp} = Rad(TM) \perp S(TM^{\perp}).$$

Also, there exists a local null frame $\{Ni\}$ of null sections with values in the orthogonal complement of $S(TM^{\perp})$ in $S(TM^{\perp})^{\perp}$ such that $\tilde{g}(N_i, \xi_j) = \delta_{ij}$, $\tilde{g}(N_i, N_j) = 0$, for $i, j \in \{1,2, \dots, r\}$,

where $\{\xi_j\}$ is local basis of $Rad(TM)$. It implies that $tr(TM)$ and $ltr(TM)$, respectively, be vector bundles in $T\widetilde{M}|M$and $S(TM^{\perp})^{\perp}$ with the property $tr(TM) = ltr(TM) \perp S(TM^{\perp})$, and

(1) $$T\tilde{M}|M = TM \oplus tr(TM) = S(TM) \perp \big(Rad(TM) \oplus ltr(TM)\big) \perp S(TM^{\perp}).$$

Consider $\overline{\nabla}$ as the Levi-Civita connection on $\overline{M}$. For $Y_1, Y_2 \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and

$W \in \Gamma(S(TM^{\perp}))$, In view of the decomposition (1), the Gauss-Weingarten formulae are as follows

(2) $$\overline{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h^l(Y_1, Y_2) + h^s(Y_1, Y_2),$$

(3) $$\overline{\nabla}_{Y_1} N = -A_N Y_1 + \nabla^l_{Y_1} N + \mathrm{D}^s(Y_1, N),$$

(4) $$\overline{\nabla}_{Y_1} W = -A_W Y_1 + \nabla^s_{Y_1} W + \mathrm{D}^l(Y_1, W).$$

Using metric connection $\overline{\nabla}$ and Eqs. (2), (3), (4), we get the following equations

(5) $$\bar{g}(h^s(Y_1, Y_2), W) + \bar{g}\big(Y_2, \mathrm{D}^l(Y_1, W)\big) = g(A_W Y_1, Y_2).$$

In particular, the induced connection ∇ is not metric connection. For $Y_1, Y_2, Y_3 \in \Gamma(TM)$, the following formula represents the expression for the induced connection.

(6) $\nabla_{Y_1} g(Y_2, Y_3) = \bar{g}(h^l(Y_1, Y_2), Y_3) + \bar{g}(h^l(Y_1, Y_3), Y_2)$.

Further, the relation between the curvature tensor $\bar{R}$ and R of $\bar{M}$ and M, respectively, is given by [1].

(7) $\bar{R}(Y_1, Y_2)Y_3 = R(Y_1, Y_2)Y_3 + A_{h^{l(Y_1,Y_3)}}Y_2 - A_{h^{l(Y_2,Y_3)}}Y_1 + A_{h^{s(Y_1,Y_3)}}Y_2 - A_{h^{s(Y_2,Y_3)}}Y_1$

$$+ (\nabla_{Y_1} h^l)(Y_2, Y_3) - (\nabla_{Y_2} h^l)(Y_1, Y_3) + (\nabla_{Y_1} h^s)(Y_2, Y_3) - (\nabla_{Y_2} h^s)(Y_1, Y_3)$$
$$+ D^l(Y_1, h^s(Y_2, Y_3)) - D^l(Y_2, h^s(Y_1, Y_3)) + D^s(Y_1, h^l(Y_2, Y_3))$$
$$- D^s(Y_2, h^l(Y_1, Y_3)).$$

Definition 2.1 [2] An odd dimensional smooth semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called almost contact metric manifold with a structure (ϕ, V, η, ξ) if

(8) $\phi^2 Y_1 = -Y_1 + \eta(Y_1)V, \eta(V) = 1, \eta \circ \phi = 0, \phi V = 0$.

(9) $\bar{g}(\phi Y_1, \phi Y_2) = \bar{g}(Y_1, Y_2) - \eta(Y_1)\eta(Y_2)$ for all $Y_1, Y_2 \in \Gamma(T\bar{M})$

From (8) and (9), we get

(10) $\bar{g}(V, V) = 1$,

(11) $\bar{g}(Y_1, V) = \eta(Y_1)$,

(12) $\bar{g}(\phi Y_1, Y_2) = -\bar{g}(Y_1, \phi Y_2)$.

Definition 2.2 [5] An almost contact metric manifold with a structure (ϕ, V, η, ξ) is called an indefinite Kenmotsu structure if and only if

(13) $(\bar{\nabla}_{Y_1}\phi)Y_2 = -\bar{g}(Y_1, \phi Y_2)V + \eta(Y_2)\phi(Y_1)$, for $Y_1, Y_2 \in \Gamma(T\bar{M})$,

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{g}$.

From Eq. (13), for $Y_1, Y_2 \in \Gamma(T\bar{M})$, we obtain

(14) $\bar{\nabla}_{Y_1} V = -Y_1 + \eta(Y_1)V$.

3. Screen Slant Lightlike Submanifolds

Definition 3.1 [4] Let M be a $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ of index $2q$ such that $2q < dim(M)$ with structure vector field tangent to M. Then M is said to be a screen slant lightlike submanifold of $\bar{M}$ if following conditions are satisfied

(i) $Rad(TM)$ is invariant with respect to ϕ.

(ii) For any $x \in U \subset M$ and for any non-zero vector field $Y1$ tangent to $S(TM) = D \perp \{V\}$, $Y1$ and V linearly independent, the angle $\theta(Y1)$ (known as slant angle) between $\phi Y1$ and $S(TM)$ is constant, where D is the complementary distribution to V in screen distribution $S(TM)$. Therefore in view of above definition, the decomposition of tangent bundle is as:

$$TM = Rad(TM) \perp D \perp \{V\}.$$

Throughout the paper, $(M, g, S(TM))$ is considered as a lightlike submanifold of an indefinite Kenmotsu manifold with constant index $2q < dim(M)$ and structure vector field V is always tangent to M.

Consider a screen slant lightlike submanifold $(M, g, S(TM))$ of an indefinite Kenmotsu manifold $\bar{M}$ and P and Q are the projection morphisms on $Rad(TM)$ and $S(TM)$, respectively.

Then, for $Y_1 \in \Gamma(TM)$, we have

(15) $Y1 = PY1 + QY1 + \eta(Y1)V$

where $PY_1 \in (Rad(TM)), QY1 \in (S(TM))$ and $\bar{Q}Y1 = QY1 + \eta(Y1)V \in \Gamma(S(TM))$.

For any vector field $Y1 \in \Gamma(S(TM))$, we write

(16) $\phi(Y1) = TY1 + \omega Y1$

where $TY1 \in \Gamma(S(TM))$ and $\omega Y1 \in \Gamma(tr(TM))$.

Applying ϕ on (15) and using (16), we obtain

(17) $\phi(Y1) = TPY_1 + TQY_1 + \omega QY_1$,
where $TPY_1 \in \Gamma(Rad(TM)), TQY_1 \in \Gamma(S(TM))$ and $\omega QY_1 \in \Gamma(S(TM^{\perp}))$.
The screen transversal bundle $S(TM^{\perp})$ can be decomposed as
$S(TM^{\perp}) = \omega Q(S(TM)) \perp \mu$.
Then, for any $W \in \Gamma(S(TM^{\perp}))$, we have
(18) $\phi W = BW + CW,\ \phi N = CN$,
where $BW \in \Gamma(S(TM)), CW \in \Gamma(\mu)$ and $CN \in \Gamma(ltr(TM))$.
Next, following [4], a screen slant lightlike submanifold M of an indefinite Kenmotsu manifold $\bar{M}$ has the following properties:
(19) $T^2\bar{Q}Y_1 = -\cos^2\theta\,(\bar{Q}Y_1 - \eta(\bar{Q}Y_1)V)$,
(20) $g(T\bar{Q}Y_1, T\bar{Q}Y_2) = \cos^2\theta|STM[g(\bar{Q}Y_1, \bar{Q}Y_2) - \eta(\bar{Q}Y_1)\eta(\bar{Q}Y_2)]$,
(21) $\bar{g}(\omega\bar{Q}Y_1, \omega\bar{Q}Y_2) = \sin^2\theta|STM[g(\bar{Q}Y_1, \bar{Q}Y_2) - \eta(\bar{Q}Y_1)\eta(\bar{Q}Y_2)]$ for any $Y_1, Y_2 \in \Gamma(TM)$.
Example 1 Let $(\bar{M}, \bar{g}) = (R_2^9, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,-,+,+,+,+,+,+,+)$ with respect to canonical basis $\{\partial x_1, \partial y_1, \partial x_2, \partial y_2, \partial x_3, \partial y_3, \partial x_4, \partial y_4, \partial z\}$. Consider a submanifold M of R_2^9 defined by

$$x_1 = u_1, y_1 = -u_2, \quad x_2 = u_2, y_2 = u_1, x_3 = u_3, y_3 = -u_4,$$
$$x_4 = cos\theta\, u_3, \quad y4 = sin\theta\, u_3, \quad z = u_5$$

Then $Z_1 = e^{-z}(\partial x_1 + \partial y_2)$, $Z_2 = e^{-z}(\partial x_2 - \partial y_1)$,
$Z_3 = e^{-z}(\partial x_3 + \sin\theta\,\partial y_4 + \cos\theta\,\partial y_4)$, $Z_4 = e^{-z}(-\partial y_3)$, $Z_5 = V = \partial z$. $N_1 = \frac{e^{-z}}{2}(-\partial x_1 + \partial y_2)$, $N_2 = \frac{e^{-z}}{2}(\partial x_2 + \partial y_1)$
It implies that $\phi Z_1 = Z_2, \phi N_1 = N_2\ and\ D_\theta = Span\{Z_3, Z_4\}$.
D_θ is a slant distribution with a slant angle $\frac{\pi}{4}$. Therefore, M is a screen slant lightlike submanifold of an indefinite Kenmotsu manifold.
Theorem 3.2 Let M be a screen slant lightlike submanifold of an indefinite Kenmotsu manifold then, for $Y_1, Y_2 \in \Gamma(TM)$, the following equations hold
(22) $(\nabla_{Y_1}T)Y_2 = A_{\omega QY_2}Y_1 + Bh^s(Y_1, Y_2) - \bar{g}(\phi(Y_1), Y_2)V + \eta(Y_2)TPY_1 + \eta(Y_2)TQY_1$,
(23) $\phi h^l(Y_1, Y_2) = h^l(Y_1, TPY_2) + h^l(Y_1, TQY_2) + D^l(Y_1, \omega QY_2)$,
(24) $(\nabla_{Y_1}\omega)Y_2 = -h^s(Y_1, TPY_2) - h^s(Y_1, TQY_2) + Ch^s(Y_1, Y_2)(\eta(Y_2)\omega QY_1)$,
where $(\nabla_{Y_1}T)Y_2 = \nabla_{Y_1}TPY_2 + \nabla_{Y_1}TQY_2 - TP\nabla_{Y_1}Y_2 - TQ\nabla_{Y_1}Y_2$ and
$(\nabla_{Y_1}\omega)Y_2 = \nabla^s_{Y_1}\omega QY_2 - \omega Q\nabla_{Y_1}Y_2$.
Proof: For $Y_1, Y_2 \in \Gamma(TM)$, using Eq. (13), we obtain

$$\bar{\nabla}_{Y_1}\phi Y_2 = \phi\bar{\nabla}_{Y_1}Y_2 - \bar{g}(\phi(Y_1), Y_2)V + \eta(Y_2)\phi Y_1,$$

Using Eq. (17) in above equation, we get

$$\bar{\nabla}_{Y_1}(TPY_2 + TQY_2 + \omega QY_2) = \phi\bar{\nabla}_{Y_1}Y_2 - \bar{g}(\phi(Y_1), Y_2)V + \eta(Y_2)\phi Y_1.$$

which reduces to
(25) $\nabla_{Y_1}TPY_2 + h^l(Y_1, TPY_2) + h^s(Y_1, TPY_2) + \nabla_{Y_1}TQY_2 + h^l(Y_1, TQY_2) + h^s(Y_1, TQY_2)$

$$-A_{\omega QY_2}Y_1 + \nabla^s_{Y_1}\omega QY_2 + D^l(Y_1, \omega QY_2)$$
$$= TP\nabla_{Y_1}Y_2 + TQ\nabla_{Y_1}Y_2 + \omega Q\nabla_{Y_1}Y_2 + Ch^l(Y_1, Y_2)$$
$$+Bh^s(Y_1, Y_2) + Ch^s(Y_1, Y_2) - \bar{g}(\phi(Y_1), Y_2)V$$
$$+\eta(Y_2)(TPY_1 + TQY_1 + \omega QY_1).$$

Equating components of $TM, ltr(TM)$ and $S(TM^{\perp})$ in Eq. (25), we obtain

$(\nabla_{Y_1} T)Y2 = A_{\omega Q Y_2} Y_1 + Bh^s(Y_1, Y_2) - \bar{g}(\phi Y_1, Y_2)V + \eta(Y_2)(TPY_1 + TQY_1)$, $\phi h^l(Y_1, Y_2) = h^l(Y_1, TPY_2) + h^l(Y_1, TQY_2) + D^l(Y_1, \omega Q Y_2)$

and

$(\nabla_{Y_1} \omega)Y_2 = -h^s(Y_1, TPY_2) - h^s(Y_1, TQY_2) + Ch^s(Y_1, Y_2) + (\eta(Y_2)\omega Q Y_1)$.

This completes the proof.

Theorem 3.3 Let M be a screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then

(i) the radical distribution $Rad(TM)$ is integrable, if and only if, the screen transversal second fundamental form of M satisfies $h^s(Y_1, \bar{J}Y_2) = h^s(\bar{J}Y_1, Y_2)$, for $Y_1, Y_2 \in \Gamma(Rad(TM))$.

(ii) the screen distribution $S(TM)$ is integrable if and only if $Q(\nabla_{Y_1} TY_2 - \nabla_{Y_2} TY_1) = Q(A_{\omega Y_1} Y_2 - A_{\omega Y_2} Y_1)$, for all $Y_1, Y_2 \in \Gamma(S(TM))$.

Proof: From Eq. (24), we obtain $h^s(Y_1, \bar{J}Y_2) - Ch^s(Y_1, Y_2) = \omega \nabla_{Y_1} Y_2$ for $Y_1, Y_2 \in \Gamma(Rad(TM))$.

Thus we obtain $h^s(Y_1, \bar{J}Y_2) - h^s(\bar{J}Y_1, Y_2) = \omega P(Y_1, Y_2)$ which proves assertion (i).

On the other hand, from Eq. (22), we get

$\nabla_{Y_1} TY_2 - A_{\omega Y_2} Y_1 = T\nabla_{Y_1} Y_2 + Bh^s(Y_1, Y_2) - g(TY_1, Y_2)V + \eta(Y_2)TY_1$ for all $Y_1, Y_2 \in \Gamma(S(TM))$.

Hence, we get

$\nabla_{Y_1} TY_2 - \nabla_{Y_2} TY_1 + A_{\omega Y_1} Y_2 - A_{\omega Y_2} Y_1 = T(Y_1, Y_2) + 2g(Y_1, TY_2)V + \eta(Y_2)TY_1 - \eta(Y_1)TY_2$.

Thus, we obtain

$$Q(\nabla_{Y_1} TY_2 - \nabla_{Y_2} TY_1) + Q(A_{\omega Y_1} Y_2 - A_{\omega Y_2} Y_1) = QT[Y_1, Y_2],$$

which proves (ii).

Theorem 3.4 Let M be a screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. If $(\nabla_{Y_1} T)Y_2 = 0$ for $Y_1 \in \Gamma(TM)$ and $Y_2 \in \Gamma(\mathrm{Rad}(TM))$, then the induced connection ∇ is a metric connection.

Proof: Let, for any $Y_1 \in \Gamma(TM)$ and $Y_2 \in \Gamma(\mathrm{Rad}(TM))$, we have $(\nabla_{Y_1} T)Y_2 = 0$, then from Eq. (22), we obtain $Bh^s(Y_1, Y_2) = 0$, hence, for any $Z \in \Gamma(TM)$, $g(Bh^s(Y_1, Y_2), Z) = 0$ implies

(26) $\bar{g}(h^s(Y_1, Y_2), \omega Q Z) = 0.$

Expanding the expression $\bar{g}\left(\omega Q \nabla_{Y_1} Y_2, \varphi h^s(Y_1, Y_2)\right)$ by using Eq. (2), we obtain

$\bar{g}\left(\omega Q \nabla_{Y_1} Y_2, \varphi h^s(Y_1, Y_2)\right) = \bar{g}\left(\omega Q \nabla_{Y_1} Y_2, \varphi h^s \bar{\nabla}_{Y_1} Y_2\right) - \phi \nabla_{Y_1} Y_2 - \varphi h^l(Y_1, Y_2)).$

Since $ltr(TM)$ is invariant, using Eq. (13), we get

$\bar{g}\left(\omega Q \nabla_{Y_1} Y_2, \varphi h^s(Y_1, Y_2)\right) = \bar{g}\left(\omega Q \nabla_{Y_1} Y_2, h^s \bar{\nabla}_{Y_1} \varphi Y_2\right) - \bar{g}\left(\omega Q \nabla_{Y_1} Y_2, \omega Q \nabla_{Y_1} Y_2\right).$

which reduces to

$\bar{g}\left(\omega Q \nabla_{Y_1} Y_2, \varphi h^s(Y_1, Y_2)\right) = \bar{g}\left(\omega Q \nabla_{Y_1} Y_2, h^s(Y_1, \varphi Y_2)\right) - \bar{g}\left(\omega Q \nabla_{Y_1} Y_2, \omega Q \nabla_{Y_1} Y_2\right).$

Using Eq. (26) in above equation, we obtain

(27) $\bar{g}\left(\omega Q \nabla_{Y_1} Y_2, \varphi h^s(Y_1, Y_2)\right) = -\bar{g}\left(\omega Q \nabla_{Y_1} Y_2, \omega Q \nabla_{Y_1} Y_2\right).$

Using Eq. (21) in Eq. (27), for $Y_1 \in \Gamma(TM)$ and $Y_2 \in \Gamma(\mathrm{Rad}(TM))$, we obtain

(28) $\bar{g}\left(\omega Q\nabla_{Y_1}Y_2, \varphi h^s(Y_1, Y_2)\right) = -\sin^2\theta\, g\left(Q\nabla_{Y_1}Y_2, Q\nabla_{Y_1}Y_2\right).$

Also from Eq. (9), we obtain

$$\bar{g}\left(\omega Q\nabla_{Y_1}Y_2, \varphi h^s(Y_1, Y_2)\right) = \bar{g}\left(\nabla_{Y_1}Y_2, h^s(Y_1, Y_2)\right) - \eta(\nabla_{Y_1}Y_2)\eta(h^s(Y_1, Y_2)).$$

which reduces to

$$\bar{g}\left(\phi\nabla_{Y_1}Y_2, \varphi h^s(Y_1, Y_2)\right) = 0.$$

Using Eq. (17) in above equation, we obtain

(29) $\bar{g}\left(TQ\nabla_{Y_1}Y_2, \varphi h^s(Y_1, Y_2)\right) = -\bar{g}\left(\omega Q\nabla_{Y_1}Y_2, \varphi h^s(Y_1, Y_2)\right).$

Using Eq. (26) in Eq. (29), we obtain

(30) $\bar{g}\left(\omega Q\nabla_{Y_1}Y_2, \varphi h^s(Y_1, Y_2)\right) = 0.$

From Eqs. (28) and (30), we get

$$\sin^2\theta\, g\left(Q\nabla_{Y_1}Y_2, Q\nabla_{Y_1}Y_2\right) = 0.$$

Since M is a proper screen slant lightlike submanifold and $S(TM)$ is Riemannian, we obtain

$\nabla_{Y_1}Y_2 \in \Gamma(Rad(TM))$. This implies $Rad(TM)$ is parallel. This completes the proof.

4. Totally Contact Umbilical Lightlike Submanifolds

Definition 4.1 [11] Let M be a submanifold of an indefinite Kenmotsu manifold $\bar{M}$ and structure vector field V tangent to M. If the second fundamental form of a submanifold is of the form

(31) $h(Y_1, Y_2) = [g(Y_1, Y_2) - \eta(Y_1)\eta(Y_2)]\alpha + \eta(Y_1)h(Y_2, V) + \eta(Y_2)h(Y_1, V),$

for any vector fields Y_1 and Y_2 tangent to M, α being a vector field normal to M, then M is called totally contact umbilical submanifold.

From Eq. (31), we obtain

(32) $h^l(Y_1, Y_2) = [g(Y_1, Y_2) - \eta(Y_1)\eta(Y_2)]\alpha_L + \eta(Y_1)h^l(Y_2, V) + \eta(Y_2)h^l(Y_1, V)$

and

(33) $h^s(Y_1, Y_2) = [g(Y_1, Y_2) - \eta(Y_1)\eta(Y_2)]\alpha_S + \eta(Y_1)h^s(Y_2, V) + \eta(Y_2)h^s(Y_1, V),$

where $\alpha_L \in \Gamma(ltr(TM))$ and $\alpha_S \in \Gamma(S(TM^\perp))$.

Example 2 Let $(\bar{M}, \bar{g}) = (R_2^9, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,-,+,+,+,+,+,+,+)$ with respect to canonical basis $\{\partial x_1, \partial y_1, \partial x_2, \partial y_2, \partial x_3, \partial y_3, \partial x_4, \partial y_4, \partial z\}$. Consider a submanifold M of R_2^9 defined by

$$x_1 = u_1, y_1 = -u_2, \quad x_2 = u_2, y_2 = u_1, x_3 = u_3, y_3 = -u_4, \quad x_4 = cosu_4, \quad y4 = sinu_4, \quad z = u_5$$

Then $Z_1 = e^{-z}(\partial x_1 + \partial y_2)$, $Z_2 = e^{-z}(\partial x_2 - \partial y_1)$,

$Z_3 = e^{-z}(\partial x_3)$, $Z_4 = e^{-z}(-\partial y_3 + \sin u_4\, \partial x_4 + \cos u_4\, \partial y_4)$, $Z_5 = V = \partial z$. $N_1 = \frac{e^{-z}}{2}(-\partial x_1 + \partial y_2)$, $N_2 = \frac{e^{-z}}{2}(\partial x_2 + \partial y_1)$

It implies that $Rad(TM) = Span\{Z_1, Z_2\}$. Also, $\phi Z_1 = Z_2$, $\phi N_1 = N_2$ and $D_\theta = Span\{Z_3, Z_4\}$ is a slant distribution with a slant angle $\frac{\pi}{4}$. Therefore, M becomes a screen slant lightlike submanifold with screen transversal bundle $S(TM^\perp)$ spanned by

$W_1 = e^{-z}(\cos u_4 \partial x_4 + \sin u_4 \partial y_4)$ and $W_2 = e^{-z}(-\cos u_4 \partial x_4 - \partial y_3 + \sin u_4 \partial y_4)$.

Now, using Gauss formula and by direct computation, for every $Y \in \Gamma(TM)$, we have

$$\overline{\nabla}_Y Z_1 = \overline{\nabla}_Y Z_2 = \overline{\nabla}_Y Z_3 = 0.$$

Also, we can see that $\overline{\nabla}_Y Z_4$, for $Y \in \Gamma(TM)$ except $Y = Z_4$ as

$$\overline{\nabla}_{Z_4} Z_4 = -4e^{-z}(\cos u_4 \partial x_4 + \sin u_4 \partial y_4) = -4W_1.$$

Thus by using Gauss formula, we have $h^l(Y_1, Y_2) = 0$ for all $Y_1, Y_2 \in \Gamma(TM)$. Also $h^s(Y_1, Y_2) = 0$ for all $Y_1, Y_2 \in \Gamma(TM)$ except $h^s(Z_4, Z_4) = -2g(Z_4, Z_4)W_1$.

Therefore, M is a totally contact umbilical screen slant lightlike submanifold of R_2^9.

Theorem 4.2 Let M be a totally contact umbilical screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then, at least one of the following is true

(i) M is a screen real lightlike submanifold.

(ii) (ii) $D = \{0\}$.

(iii) (iii) If M is a proper screen slant lightlike submanifold then $\alpha_s \in \Gamma(\mu)$.

Proof: Since M is a totally contact umbilical screen slant lightlike submanifold, using Eq. (31), for $Y_1 \in \Gamma(D)$, we have $h(TQY_1, TQY_1) = g(TQY_1, TQY_1)\alpha$.

Using Eqs. (20) and (2) in above equation, we obtain

(34) $\overline{\nabla}_{TQY_1} TQY_1 - \nabla_{TQY_1} TQY_1 = \cos^2\theta\, g(QY_1, QY_1)\alpha.$

From Eq. (13), we get

(35) $\overline{\nabla}_{TQY_1} \phi Y_1 = \phi \overline{\nabla}_{TQY_1} QY_1 + g(TQY_1, TQY_1)V.$

Using Eq. (35) in Eq. (34), we obtain

$$\phi \overline{\nabla}_{TQY_1} QY_1 + g(TQY_1, TQY_1)V - \overline{\nabla}_{TQY_1} \omega QY_1 - \nabla_{TQY_1} TQY_1 = \cos^2\theta\, g(QY_1, QY_1)\alpha.$$

Using Eqs. (2), (4) and (20) in above equation, we obtain

(36) $\mathrm{T}\nabla_{TQY_1} QY_1 + \omega \nabla_{TQY_1} QY_1 + g(TQY_1, QY_1)\phi\alpha_L + g(TQY_1, QY_1)B\alpha_s$
$+ g(TQY_1, QY_1)C\alpha_s + A_{\omega QY_1} TQY_1 - \nabla^s_{TQY_1} \omega QY_1 - D^l(TQY_1, \omega QY_1)$
$-\nabla_{TQY_1} TQY_1 = \cos^2\theta\, g(QY_1, QY_1)(\alpha - V).$

Equating transversal components in Eq. (36), we obtain

(37) $\omega \nabla_{TQY_1} QY_1 + g(TQY_1, QY_1)\phi\alpha_L + g(TQY_1, QY_1)C\alpha_s - \nabla^s_{TQY_1} \omega QY_1$
$-D^l(TQY_1, \omega QY_1) = \cos^2\theta\, g(QY_1, QY_1)\alpha.$

Also, for any $Y_1, Y_2 \in \Gamma(D)$, taking covariant derivative of Eq. (21) with respect to TQY_1, we obtain

(38) $g(\nabla^s_{TQY_1} \omega QY_1, \omega QY_1) = \sin^2\theta\, g(\nabla_{TQY_1} QY_1, QY_1).$

From Eq. (21), we get

(39) $\bar{g}(\omega \nabla_{TQY_1} QY_1, \omega QY_1) = \sin^2\theta\, g(\nabla_{TQY_1} QY_1, QY_1).$

Now, taking the inner product of Eq. (37) with ωQY_1, we obtain

(40) $\bar{g}(\omega \nabla_{TQY_1} QY_1, \omega QY_1) - g(\nabla^s_{TQY_1} \omega QY_1, \omega QY_1) = \cos^2\theta\, g(QY_1, QY_1)\bar{g}(\alpha_s, \omega QY_1).$

Using Eqs. (38) and (39) in Eq. (40), we obtain

$$\cos^2\theta\, g(QY_1, QY_1)\bar{g}(\alpha_s, \omega QY_1) = 0.$$

Thus from above equation, we get either $\theta = \frac{\pi}{2}$ or $Y_1 = 0$ or $\alpha_s \in T(\mu)$.
This completes the proof.

Theorem 4.3 Every totally contact proper umbilical screen slant lightlike submnaifold of an indefinite Kenmotsu manifold is totally contact geodesic.

Proof: Since M is a totally contact umbilical screen slant lightlike submanifold, using Eq. (31), for $Y_1 = QY_1 \in \Gamma(D)$, we have $h(TQY_1, TQY_1) = g(TQY_1, TQY_1)\alpha$.

Using Eq. (20) in above equation, we obtain

$$h(TQY_1, TQY_1) = \cos^2\theta\, g(QY_1, QY_1)\alpha.$$

From Eq. (13), we get

$$\overline{\nabla}_{TQY_1}\phi Y_1 = \phi\overline{\nabla}_{TQY_1}Y_1 - g(\phi TQY_1, QY_1)V.$$

Using Eq. (12) in above equation, we obtain

$$\overline{\nabla}_{TQY_1}\phi Y_1 = \phi\overline{\nabla}_{TQY_1}Y_1 + g(TQY_1, TQY_1)V.$$

Using Eqs. (17) and (4) in above equation, we obtain

(41) $$\nabla_{TQY_1}TQY_1 + \mathrm{h}(TQY_1, TQY_1) - A_{\omega QY_1}TQY_1 + \nabla^s_{TQY_1}\omega QY_1 + D^l(TQY_1, \omega QY_1)$$
$$= T\nabla_{TQY_1}QY_1 + \omega\left(\nabla_{TQY_1}QY_1\right) + Ch^l(TQY_1, QY_1)\phi\alpha_L + Bh^s(TQY_1, QY_1) + Ch^s(TQY_1, QY_1)$$
$$+ g(TQY_1, TQY_1)V.$$

Equating the transversal part in Eq. (41), we get

(42) $$\mathrm{h}(TQY_1, TQY_1) = \omega(\nabla_{TQY_1}QY_1) - \nabla^s_{TQY_1}\omega QY_1 - D^l(TQY_1, \omega QY_1) + Ch^l(TQY_1, QY_1)$$
$$+ Ch^s(TQY_1, QY_1).$$

Since M is a totally contact umbilical screen slant lightlike submanifold and $g(TQY_1, QY_1) = 0$, therefore $Ch(TQY_1, QY_1) = g(TQY_1, QY_1)C(\alpha) = 0$.

Using Eq. (20) in Eq. (42), we obtain

$$\cos^2\theta\, g(QY_1, QY_1)\alpha = \omega\nabla_{TQY_1}QY_1 - \nabla^s_{TQY_1}\omega QY_1 - D^l(TQY_1, \omega QY_1).$$

Taking the inner product of above equation with ωQY_1, we obtain

(43) $$\cos^2\theta\, g(QY_1, QY_1)\bar{g}(\alpha_s, \omega QY_1) = -\bar{g}\left(\nabla^s_{TQY_1}\omega QY_1, \omega QY_1\right) + \bar{g}(\omega\nabla_{TQY_1}QY_1, \omega QY_1).$$

On the other hand, for any $Y_1 = QY_1 \in \Gamma(D)$, Eq. (21) implies that

(44) $$\bar{g}\left(\omega\nabla_{TQY_1}QY_1, \omega QY_1\right) = \sin^2\theta\, g(\nabla_{TQY_1}QY_1, QY_1)$$

Taking covariant derivative of Eq. (21) with respect to TQY1, we obtain

(45) $$\bar{g}\left(\nabla^s_{TQY_1}\omega QY_1, \omega QY_1\right) = \sin^2\theta\, g(\nabla_{TQY_1}QY_1, QY_1).$$

Using Eqs. (44) and (45) in Eq. (43), we obtain

$$\cos^2\theta\, g(QY_1, QY_1)\bar{g}(\alpha_s, \omega QY_1) = 0.$$

Since M is a proper screen slant lightlike submanifold and g is Riemannian metric on D, we have

$$\bar{g}(\alpha_s, \omega QY_1) = 0.$$

This implies $\alpha_s \in \Gamma(\mu)$. Now, for any $Y_1, Y_2 \in \Gamma(D)$, using the Kenmotsu property of $\overline{M}$, we have

$$\overline{\nabla}_{Y_1}\phi Y_2 = \phi\overline{\nabla}_{Y_1}Y_2 - \bar{g}(\phi(Y_1), Y_2)V.$$

Using Eq. (17) in above equation, we get

$$\overline{\nabla}_{Y_1}(TQY_2 + \omega QY_2) = \phi\overline{\nabla}_{Y_1}Y_2 - \bar{g}(\phi(Y_1), Y_2)V.$$

Using Eqs. (2) and (4), we obtain

(46) $$\nabla_{Y_1}TQY_2 + h(Y_1, TQY_2) - A_{\omega QY_2}Y_1 + \nabla^s_{Y_1}\omega QY_2 + D^l(Y_1, \omega QY_2)$$

$$= \phi\left(\nabla_{Y_1}Y_2 + h(Y_1, Y_2)\right) - \bar{g}(\phi(Y_1), Y_2)V.$$

Using Eq. (31) in above equation, we obtain

(47) $$\nabla_{Y_1}TQY_2 + g(Y_1, TQY_2)\alpha - A_{\omega QY_2}Y_1 + \nabla^s_{Y_1}\omega QY_2 + D^l(Y_1, \omega QY_2)$$
$$= T\nabla_{Y_1}Y_2 + \omega(\nabla_{Y_1}Y_2) + g(Y_1, Y_2)\phi\alpha - \bar{g}(\phi(Y_1), Y_2)V.$$

Now, taking the inner product of above equation with $\phi\alpha_s$ and using the fact that μ is invariant subbundle of $T\bar{M}$, we obtain

(48) $$\bar{g}(\nabla^s_{Y_1}\omega QY_2, \phi\alpha_s) = g(Y_1, Y_2)\bar{g}(\phi\alpha_s, \phi\alpha_s).$$

Again, using Eq. (13), for $Y_1 \in \Gamma(D)$ and $\alpha_s \in \Gamma(\mu)$, we obtain

$$\bar{\nabla}_{Y_1}\phi\alpha_s = \phi\bar{\nabla}_{Y_1}\alpha_s.$$

Using Eq. (4) in above equation, we get

(49) $-A_{\phi\alpha_s}Y_1 + \nabla^s_{Y_1}\phi\alpha_s + D^l(Y_1, \phi\alpha_s) = -TA_{\alpha_s}Y_1 - \omega A_{\alpha_s}Y_1 + B\nabla^s_{Y_1}\alpha_s + C\nabla^s_{Y_1}\alpha_s + \phi D^l(Y_1, \alpha_s).$

Taking the inner product of Eq. (49) with ωQY_2 and using $C\nabla^s_{Y_1}\phi\alpha_s \in \Gamma(\mu)$, we get

(50) $$\bar{g}(\nabla^s_{Y_1}\phi\alpha_s, \omega QY_2) = -\bar{g}(\omega A_{\alpha_s}Y_1, \omega QY_2) = -\sin^2\theta\, g(A_{\alpha_s}Y_1, Y_2).$$

Since $\bar{\nabla}$ is a metric connection, we have

(51) $$\bar{g}(\nabla^s_{Y_1}\omega QY_2, \phi\alpha_s) = -\bar{g}(\nabla^s_{Y_1}\phi\alpha_s, \omega QY_2,).$$

Using Eqs. (48) and (50) in Eq. (51), we get

(52) $$g(Y_1, Y_2)\bar{g}(\alpha_s, \alpha_s) = \sin^2\theta\, g(A_{\alpha_s}Y_1, Y_2),$$

and using (5), we obtain

(53) $$g(A_{\alpha_s}Y_1, Y_2) = \bar{g}(h^s(Y_1, Y_2), \alpha_s).$$

Using Eq. (53) in Eq. (52), we get

(54) $$\cos^2\theta\, g(Y_1, Y_2)\bar{g}(\alpha_s, \alpha_s) = 0.$$

Since M is a proper screen slant lightlike submanifold and g is Riemannian metric on D, we have $\alpha_s = 0$. Next, for $Y_1 \in \Gamma(D)$, from Eq. (13), we have

$$\bar{\nabla}_{Y_1}\phi Y_1 = \phi\bar{\nabla}_{Y_1}Y_1 - \bar{g}(\phi(Y_1), Y_1)V + \eta(Y_1)\phi(Y_1).$$

Using Eqs. (2), (4) and the fact $\bar{g}(\phi(Y_1), Y_1) = 0$, we obtain

(55) $$\nabla_{Y_1}TQY_1 + h(Y_1, TQY_1) - A_{\omega QY_1}Y_1 + \nabla^s_{Y_1}\omega QY_1 + D^l(Y_1, \omega QY_1)$$
$$= T\nabla_{Y_1}Y_1 + \omega(\nabla_{Y_1}Y_1) + Ch^l(Y_1, Y_1) + Bh^s(Y_1, Y_1) + Ch^s(Y_1, Y_1).$$

Since M is a totally contact umbilical lightlike submanifold, $h(Y_1, TQY_1) = 0$, equating the transversal components in above equation, we obtain

(56) $$\nabla^s_{Y_1}\omega QY_1 + D^l(Y_1, \omega QY_1) = \omega(\nabla_{Y_1}Y_1) + Ch^l(Y_1, Y_1) + Ch^s(Y_1, Y_1).$$

Taking the inner product of Eq. (56) with $\phi\xi$ where $\xi \in \Gamma(Rad(TM))$ and using Eq. (9), we obtain

(57) $\bar{g}(\phi h^l(Y_1, Y_1), \phi\xi) - \bar{g}(D^l(Y_1, \omega QY_1), \phi\xi) = \bar{g}(h^l(Y_1, Y_1), \xi) - \bar{g}(D^l(Y_1, \omega QY_1), \phi\xi) = 0.$

From Eq. (5), we get

$$\bar{g}(h^s(Y_1, \phi\xi),\ \omega QY_1) + \bar{g}(D^l(Y_1, \omega QY_1), \phi\xi) = g(A_{\omega QY_1}Y_1, \phi\xi).$$

Using Eq. (26) in above equation, we obtain

(58) $$\bar{g}(D^l(Y_1, \omega QY_1), \phi\xi) = 0.$$

Hence, using Eq. (58) and the property of totally contact umbilical lightlike submanifold in Eq. (57), we obtain

$\bar{g}(h^l(Y_1, Y_1), \xi) = g(Y_1, Y_1)g(\alpha_L, \xi) = 0.$

Since M is a proper screen slant lightlike submanifold and g is Riemannian metric on D, from above equation $\alpha_L = 0$. This completes the proof.

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D. P. SEMWAL
Department of Mathematics, DAV College, Sector - 10, Chandigarh - 160011, India
E-mail: dpsemwal27@gmail.com