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NONLINEAR GENERALIZED FRACTIONAL MIXED INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITION

H. L. TIDKE, R. P. MAHAJAN, AND V. V. KHARAT

Abstract. The aim of this paper is to study a nonlocal Cauchy problem for nonlinear generalized mixed fractional integrodifferential equations with Katugampola derivative. Some new existence results of solutions for the given equations are obtained by using the fixed point theorems. Examples are presented to demonstrate the usefulness of our main results.

1. Introduction

The differential equations of fractional order are generalizations of classical differential equations of integer order. The history of fractional calculus dates back to the 17th century. So many mathematicians define the most used fractional derivatives, Riemann-Liouville in 1832, Hadamard in 1891 and Caputo in 1997 [15, 21]. Fractional calculus plays a very important role in several fields such as physics, chemical technology, economics, biology; see [6, 7, 8, 11, 12, 18, 20] and the basic theory of fractional calculus can be found in [1, 2, 5, 16, 19] references therein.

In 2011, Katugampola introduced a derivative that is a generalization of the Riemann-Liouville fractional operators and the fractional integral of Hadamard in a single form [13, 14]. The integrals are special cases when a parameter is defined at various values; when $\rho \to 0$, the Riemann-Liouville operators are obtained; when $\rho \to 1$, the Hadamard operators are obtained.

Further, in [3], the authors studied the fractional differential equations with Stieltjes and fractional integral boundary conditions using the generalized derivatives of the form

$$^{\rho}D^{\alpha_1}\mathrm{y(t)} = \mathrm{f}\Bigg(\mathrm{t,y(t)}\Bigg),\ t \in [0,T],$$

$$\mathrm{y(0){=}0},\ \int_0^T \mathrm{y(s)dH(s)} = \frac{\rho^{-\gamma}}{\Gamma(\gamma)}\int_0^{\zeta} \alpha_i \frac{s^{\rho_i - 10}x(s)}{(\zeta^{\rho} - s^{\rho})^{1-\gamma}},$$

where, $^{\rho}D^{\alpha_1}-$ generalized fractional derivative and $H-$continuous function.

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In [4], the authors discussed Caputo-type fractional differential equations with Katugampola type generalized fractional integral boundary conditions of the form

$$ {}^{\rho}{}_{c}D^{\alpha_1}_{0+}y(t) = f\Bigg(t, y(t)\Bigg),\ t \in [0,T], $$

$$ y(T) = \sum_{j=1}^{m} \sigma_i {}^{\rho}I^{\beta}_{0+}y(\eta_i) + k,\ \delta y(0) = 0\ \eta_i \in (0,T), $$

where ${}^{\rho}{}_{c}D^{\alpha_1}_{0+}$ denotes the Caputo fractional derivative and f is a continuous function.

Recently, the authors in [10] discussed the existence and stability of solution of the initial value problem (IVP):

$$ ({}^{\varrho}D^{\alpha,\beta}_{a+}x)(t) = f(t,x(t)),\ t \in J := (a,T], \tag{1.1} $$

$$ ({}^{\varrho}I^{1-\gamma}_{a+}x)(a) = c_2, \gamma = \alpha + \beta(1-\alpha), c_2 \in \mathbb{R}, \tag{1.2} $$

for generalized Katugampola fractional differential equation by using Schauder fixed point theorem and the equivalence between IVP (1.1)-(1.2) and the integral equation

$$ x(t) = \frac{c_2}{\Gamma(\gamma)}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} f(s,x(s))\mathrm{d}s. \tag{1.3} $$

In [9], using Krasnoselskii's fixed point theorem, Schauder fixed point theorem and Schaefer fixed point theorem, authors discussed the existence of solution of IVP with nonlocal initial condition:

$$ ({}^{\varrho}D^{\alpha,\beta}_{a+}x)(t) = f(t,x(t)),\ t \in J := (a,T], \tag{1.4} $$

$$ ({}^{\varrho}I^{1-\gamma}_{a+}x)(a+) = \sum_{j=1}^{m} \eta_j x(\xi_j), \alpha \le \gamma = \alpha + \beta(1-\alpha),\ \xi_j \in (a,T], \tag{1.5} $$

where ${}^{\varrho}D^{\alpha,\beta}_{a+}$ is the generalized Katugampola fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and ${}^{\varrho}I^{1-\gamma}_{a+}$ is the generalized Katugampola fractional integral with $\varrho > 0$. Authors also proved the equivalence between (1.4)-(1.5) and the integral equation

$$ \begin{aligned} x(t) =& \frac{K}{\Gamma(\alpha)}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1} \sum_{j=1}^{m}\eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} f(s,x(s))\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} f(s,x(s))\mathrm{d}s, \end{aligned} \tag{1.6} $$

where

$$ K = \left(\Gamma(\gamma) - \sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\right)^{-1}. \tag{1.7} $$

The above results motivate us and therefore, in this paper, we obtain the existence of solution of the following nonlinear generalized fractional integrodifferential equation (NGFIDE) of order α $(0 < \alpha < 1)$ and type $\beta \in [0,1]$:

$$({}^{\varrho}D^{\alpha,\beta}_{a+}x)(t) = f\Big(t, x(t), \int_a^t h(t,s)x(s)ds, \int_a^T k(t,s)x(s)ds\Big), \tag{1.8}$$

$$({}^{\varrho}I^{1-\gamma}_{a+}x)(t) = \sum_{j=1}^{m} \eta_j x(\xi_j), \alpha \leq \gamma = \alpha + \beta(1-\alpha), \tag{1.9}$$

for $t,\ \xi_j \in J := (a,T]$, where ${}^{\varrho}D^{\alpha,\beta}_{a+}$ is the generalized Katugampola fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and ${}^{\varrho}I^{1-\gamma}_{a+}$ is the generalized Katugampola fractional integral with $\varrho > 0$. The functions $f : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h,\ k : J \times J \to \mathbb{R}$ are continuous, and ξ_j are pre–fixed points satisfy $0 < a < \xi_1 \leq \ldots \leq \xi_m < T$ and η_j, $j = 1, 2, \ldots, m$ are real numbers.

First, we establish an equivalent mixed-type nonlinear Volterra Fredholm integral equation

$$\begin{aligned} x(t) = {} & \frac{K}{\Gamma(\alpha)} \left(\frac{t^{\varrho} - a^{\varrho}}{\varrho}\right)^{\gamma-1} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^{\varrho} - s^{\varrho}}{\varrho}\right)^{\alpha-1} \\ & \times f\Big(s, x(s), \int_a^s h(s,\tau)x(\tau)d\tau, \int_a^T k(s,\tau)x(\tau)d\tau\Big) \mathrm{d}s \\ & + \frac{1}{\Gamma(\alpha)} \int_a^t s^{\varrho-1} \left(\frac{t^{\varrho} - s^{\varrho}}{\varrho}\right)^{\alpha-1} \\ & \times f\Big(s, x(s), \int_a^t h(s,\tau)x(\tau)d\tau, \int_a^T k(s,\tau)x(\tau)d\tau\Big) \mathrm{d}s, \end{aligned} \tag{1.10}$$

where

$$K = \left(\Gamma(\gamma) - \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^{\varrho} - a^{\varrho}}{\varrho}\right)^{\gamma-1}\right)^{-1}, \tag{1.11}$$

for NGFIDE (1.8)-(1.9) in the weighted space of continuous functions $C_{1-\gamma,\varrho}[a,T]$ presented in the next section. We use the Krasnoselskii's fixed point theorem and Schauder fixed point theorem to prove the existence results for NGFIDE (1.8)-(1.9).

The rest of the paper is organized as follows. In Section 2, some definitions, notations and basic results are given. We prove the equivalent integral equation in Section 2 and the existence results proved in Section 3. Illustrative examples are given in the last section.

2. Preliminaries

Here we introduce some definitions and present preliminary results needed in our proofs later.

Let the Euler gamma and beta functions be defined, respectively, by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\mathrm{d}x, \quad \mathbf{B}(\alpha,\beta) = \int_0^1 (1-x)^{\alpha-1}x^{\beta-1}\mathrm{d}x,\ \alpha>0,\ \beta>0.$$

It is well known that $\mathbf{B}(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ for $\alpha > 0,\ \beta > 0$, see [15]. Throughout the paper, we consider $[a,T],\ 0 < a < T < \infty$ being a finite interval on $\mathbb{R}^+$ and $\varrho > 0$.

Definition 2.1 ([15])**.** The space $X_c^p(a,T),\ c \in \mathbb{R},\ p \geq 1$ consists of those real valued Lebesgue measurable functions g on (a,T) for which $\|g\|_{X_c^p} < \infty$, where

$$\|g\|_{X_c^p} = \left(\int_a^b |t^c g(t)|^p \frac{\mathrm{d}t}{t}\right)^{1/p}, \quad p \geq 1 \quad \text{and} \quad \|g\|_{X_c^\infty} = \operatorname*{ess\,sup}_{a\leq t\leq T} |t^c g(t)|$$

In particular, when $c = 1/p$, we see that $X_{1/p}^c(a,T) = L_p(a,T)$.

Definition 2.2 ([17])**.** We denote by $C[a,T]$ a space of continuous functions g on $(a,T]$ with the norm

$$\|g\|_C = \max_{t\in[a,T]} |g(t)|$$

The weighted space $C_{\gamma,\varrho}[a,T],\ 0 \leq \gamma < 1$ of functions g on $(a,T]$ is defined as

$$C_{\gamma,\varrho}[a,T] = \left\{ g : (a,T] \to \mathbb{R} : \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^\gamma g(t) \in C[a,T] \right\} \tag{2.1}$$

with the norm

$$\|g\|_{C_{\gamma,\varrho}} = \left\| \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^\gamma g(t) \right\|_C = \max_{t\in[a,t]} \left| \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^\gamma g(t) \right|,$$

and $C_{0,\varrho}[a,T] = C[a,T]$

Definition 2.3 ([17])**.** Let $\delta_\varrho = \left(t^{\varrho-1}\mathrm{d}/\mathrm{d}t\right),\ 0 \leq \gamma < 1$. Denote $C^n_{\delta_\varrho\gamma}[a,T]$ the Banach space of functions g which are continuously differentinble, with δ_ϱ, on $[a,T]$ upto order $(n-1)$ and have the derivative $\delta_\varrho^n g$ on $(a,T]$ such that $\delta_\varrho^n g \in C_{\gamma,\varrho}[a,T]$:

$$C^n_{\delta_{\varrho,\gamma}}[a,T] = \left\{ \delta_\varrho^k g \in C[a,T], k = 0,1,\ldots,n-1, \delta_\varrho^n g \in C_{\gamma,\varrho}[a,T] \right\}, \quad n \in \mathrm{N}$$

with the norm

$$\|g\|_{C^n_{\delta_{\varrho,\gamma}}} = \sum_{k=0}^{n-1} \left\|\delta_\varrho^k g\right\|_C + \left\|\delta_\varrho^n g\right\|_{C_{\gamma,\varrho}}, \quad \|g\|_{C^n_{\delta_\varrho}} = \sum_{k=0}^{n} \max_{t\in\Omega} \left|\delta_\varrho^k g(t)\right|.$$

In particular, for $n = 0$ we have $C^0_{\delta_\varrho\gamma}[a,T] = C_{\gamma,\varrho}[a,T]$.

Definition 2.4 ([13])**.** Let $\alpha > 0$ and $f \in X_c^p(a,T)$, where X_c^p is as in Definition 2.1. The left-sided Katugampola fractional integral ${}^\varrho I^\alpha_{a+}$ of order α is defined as

$${}^\varrho I^\alpha_{a+} f(t) = \int_a^t s^{\varrho-1} \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \frac{f(s)}{\Gamma(\alpha)}\mathrm{d}s,\ t > a. \tag{2.2}$$

Definition 2.5 ([14])**.** Let $\alpha \in \mathbb{R}^+\backslash N$ and $n = [\alpha] + 1$, where $[\alpha]$ is the integer part of α. The left-sided Katugampola fractional derivative ${}^{\varrho}D^{\alpha}_{a+}$ is defined as

$$\begin{aligned} {}^{\varrho}D^{\alpha}_{a+}f(t) &= \delta^n_{\varrho}\left({}^{\varrho}I^{n-\alpha}_{a+}f(s)\right)(t) \\ &= \left(t^{\varrho-1}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{n-\alpha-1}\frac{f(s)}{\Gamma(n-\alpha)}\mathrm{d}s. \end{aligned} \tag{2.3}$$

Definition 2.6 ([17])**.** The left-sided generalized Katugampola fractional derivative ${}^{\varrho}D^{\alpha,\beta}_{a+}$ of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ is defined as

$$\left({}^{\varrho}D^{\alpha,\beta}_{a+}f\right)(t) = \left({}^{\varrho}I^{\beta(1-\alpha)}_{a+}\delta_{\varrho}\,{}^{\varrho}I^{(1-\beta)(1-\alpha)}_{a+}f\right)(t), \tag{2.4}$$

for the functions for which the right-hand side expression exists.

Lemma 2.7 ([9])**.** *Suppose that $\alpha > 0,\ \beta > 0,\ p \geq 1$ and $\varrho, c \in \mathbb{R}$ such that $\varrho \geq c$. Then for $f \in X^p_c(a,T)$, the semigroup property of Katugampola integral is valid. This is*

$${}^{\varrho}I^{\alpha}_{a+}\,{}^{\varrho}I^{\beta}_{a+}f(t) = {}^{\varrho}I^{\alpha+\beta}_{a+}f(t). \tag{2.5}$$

Lemma 2.8 ([14])**.** *Suppose that $\alpha > 0,\ 0 \leq \gamma < 1$ and $f \in C_{\gamma,\varrho}[a,T]$. Then for all $t \in (a,T]$,*

$${}^{\varrho}D^{\alpha}_{a+}I^{\alpha}_{a+}f(t) = f(t). \tag{2.6}$$

Lemma 2.9 ([14])**.** *Suppose that $\alpha > 0,\ 0 \leq \gamma < 1,\ f \in C_{\gamma,\varrho}[a,T]$ and ${}^{\varrho}I^{1-\alpha}_{a+}f \in C^1_{\gamma,\varrho}[a,T]$. Then*

$${}^{\varrho}I^{\alpha}_{a+}\,{}^{\varrho}D^{\alpha}_{a+}f(t) = f(t) - \frac{{}^{\varrho}I^{1-\alpha}_{a+}f(a)}{\Gamma(\alpha)}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha-1}. \tag{2.7}$$

Lemma 2.10 ([9])**.** *Suppose ${}^{\varrho}I^{\alpha}_{a+}$ and ${}^{\varrho}D^{\alpha}_{a+}$ are defined as in Definitions 2.4 and 2.5, respectively. Then*

$${}^{\varrho}I^{\alpha}_{a+}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma+1)}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\sigma-1},\ \alpha \leq 0,\ \sigma > 0,\ t > a, \tag{2.8}$$

$${}^{\varrho}D^{\alpha}_{a+}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha-1} = 0,\ 0 < \alpha < 1. \tag{2.9}$$

Remark 2.11. For $0 < \alpha < 1,\ 0 \leq \beta \leq 1$, the generalized Katugampola fractional derivative ${}^{\varrho}D^{\alpha,\beta}_{a+}$ can be written in terms of Katugampola fractional derivative as

$${}^{\varrho}D^{\alpha,\beta}_{a+} = {}^{\varrho}I^{\beta(1-\alpha)}_{a+}\delta_{\varrho}{}^{\varrho}I^{1-\gamma}_{a+} = {}^{\varrho}I^{\beta(1-\alpha)}_{a+}{}^{\varrho}D^{\gamma}_{a+}, \quad \gamma = \alpha + \beta(1-\alpha).$$

Lemma 2.12 ([14])**.** *Let $\alpha > 0,\ 0 < \gamma \leq 1$ and $f \in C_{1-\gamma,\varrho}[a,b]$. If $\alpha > \gamma$, then*

$$\left({}^{\varrho}I^{\alpha}_{a+}f\right)(a) = \lim_{x\to a+}\left({}^{\varrho}I^{\alpha}_{a+}f\right)(t) = 0.$$

To discuss the existence of a solution of NGFIDE (1.8)-(1.9), we need the following spaces:

$$C^{\alpha,\beta}_{1-\gamma,\varrho}[a,T] = \left\{g \in C_{1-\gamma,\varrho}[a,T] : {}^{\varrho}D^{\alpha,\beta}_{a+}g \in C_{1-\gamma,\varrho}[a,T]\right\},\ 0 < \gamma \leq 1 \tag{2.10}$$

and

$$C^{\gamma}_{1-\gamma,\varrho}[a,T] = \left\{g \in C_{1-\gamma,\varrho}[a,T] : {}^{\varrho}D^{\gamma}_{a+}g \in C_{1-\gamma,\varrho}[a,T]\right\},\ 0 < \gamma \leq 1. \tag{2.11}$$

Since ${}^{\varrho}D^{\alpha,\beta}_{a+}g = {}^{\varrho}I^{\beta(1-\alpha)}_{a+}\,{}^{\varrho}D^{\gamma}_{a+}g$, it is obvious that $C^{\gamma}_{1-\gamma,\varrho}[a,T] \subset C^{\alpha,\beta}_{1-\gamma,\varrho}[a,T]$.

Lemma 2.13 ([9])**.** *Let $\alpha > 0$, $\beta > 0$ and $\gamma = \alpha + \beta - \alpha\beta$. If $g \in C^{\gamma}_{1-\gamma,\varrho}[a,T]$, then*

$$ {}^{\varrho}I^{\gamma}_{a+}\,{}^{\varrho}D^{\gamma}_{a+}g(t) = {}^{\varrho}I^{\alpha}_{a+}\,{}^{\varrho}D^{\alpha,\beta}_{a+}g(t) = {}^{\varrho}D^{\beta(1-\alpha)}_{a+}g(t). $$

To prove the equivalence between NGFIDE (1.8)-(1.9) with Volterra Fredholm integral equation (1.10), we note the following lemmas.

Lemma 2.14 ([14])**.** *Let $0 < \alpha < 1$, $0 \le \beta \le 1$, $\gamma = \alpha+\beta-\alpha\beta$. If $f : (a,T]\times\mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma,\varrho}[a,T]$ for any $x(\cdot) \in C_{1-\gamma,\varrho}[a,T]$, then $x(\cdot) \in C^{\gamma}_{1-\gamma,\varrho}[a,T]$ satisfies IVP* (1.1)-(1.2) *if and only if $x(\cdot)$ satisfies the nonlinear Volterra Fredholm integral equation.* (1.3)

Lemma 2.15 ([9])**.** *Let $0 < \alpha < 1$, $0 \le \beta \le 1$, $\gamma = \alpha+\beta-\alpha\beta$. If $f : (a,T]\times\mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma,\varrho}[a,T]$ for any $x(\cdot) \in C_{1-\gamma,\varrho}[a,T]$, then $x \in C^{\gamma}_{1-\gamma,\varrho}[a,T]$ satisfies IVP* (1.4)-(1.5) *if and only if x satisfies the nonlinear Volterra Fredholm integral equation.* (1.6)

The following lemma deals with equivalence of mixed-type integral equation (1.10) and the given NGFIDE (1.8)-(1.9).

Lemma 2.16. *Let $0 < \alpha < 1$, $0 \le \beta \le 1$ and $\gamma = \alpha + \beta - \alpha\beta$. Suppose that $\mathcal{F} : (a,b]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a function such that $\mathcal{F}(\cdot, y(\cdot), z(\cdot), w(\cdot)) \in C_{1-\gamma,\varrho}[a,b]$ for any $y(\cdot) \in C_{1-\gamma,\varrho}[a,b]$. Function $y(\cdot) \in C^{\gamma}_{1-\gamma,\varrho}[a,b]$ is a solution of NGFIDE* (1.8)–(1.9) *if and only if $y(\cdot)$ is a solution of the mixed-type nonlinear Volterra Fredholm integral equation* (1.10)

Proof. **Necessity Part:** By appling Lemma 2.14 and Lemma 2.15, a solution of NGFIDE (1.8)-(1.9) can be expressed as

$$ y(t) = \frac{{}^{\varrho}I^{1-\gamma}_{a+}y(a+)}{\Gamma(\gamma)}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1} + \int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\frac{\mathcal{F}\left(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\right)}{\Gamma(\alpha)}\mathrm{d}s \tag{2.12} $$

Hnce on setting $t = \xi_j$ in (2.12), we obtain

$$ \begin{aligned} &y(\xi_j) \\ &= \frac{{}^{\varrho}I^{1-\gamma}_{a+}y(a+)}{\Gamma(\gamma)}\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1} \\ &\quad + \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\frac{\mathcal{F}\left(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\right)}{\Gamma(\alpha)}\mathrm{d}s \end{aligned} \tag{2.13} $$

and by multiplying both sides of (2.13) by η_j, we get

$\eta_j y(\xi_j)$

$$= \frac{{}^{\varrho}I_{a+}^{1-\gamma}y(a+)}{\Gamma(\gamma)}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}$$
$$+\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\frac{\mathcal{F}\left(s,y(s),\int_a^s H(s,\tau)y(\tau)d\tau,\int_a^b K(s,\tau)y(\tau)d\tau\right)}{\Gamma(\alpha)}\mathrm{d}s \tag{2.14}$$

Using the given initial condition of NGFIDE (1.8)-(1.9), we have

$$\begin{aligned}&({}^{\varrho}I_{a+}^{1-\gamma}y)(a+)\\ &=\sum_{j=1}^{m}\eta_j y(\xi_j)\\ &=\frac{{}^{\varrho}I_{a+}^{1-\gamma}y(a+)}{\Gamma(\gamma)}\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\rho}-a^{\ell}}{\varrho}\right)^{\gamma-1}+\sum_{j-1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\ &\quad\times\frac{\mathcal{F}\left(s,y(s),\int_a^s H(s,\tau)y(\tau)d\tau,\int_a^b K(s,\tau)y(\tau)d\tau\right)}{\Gamma(\alpha)}\mathrm{d}s\end{aligned}$$

which implies

$$\begin{aligned}&({}^{\varrho}I_{a+}^{1-\gamma}y)(a+)\left(\Gamma(\gamma)-\sum_{j-1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\right)\\ &=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{j-1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho^{-1}}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\ &\quad\times\mathcal{F}\left(s,y(s),\int_a^s H(s,\tau)y(\tau)d\tau,\int_a^b K(s,\tau)y(\tau)d\tau\right)\mathrm{d}s\end{aligned} \tag{2.15}$$

i.e.

$$\begin{aligned}({}^{\varrho}I_{a+}^{1-\gamma}y)(a+)&=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\overline{\mathcal{K}}\sum_{j-1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho^{-1}}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\ &\quad\times\mathcal{F}\left(s,y(s),\int_a^s H(s,\tau)y(\tau)d\tau,\int_a^b K(s,\tau)y(\tau)d\tau\right)\mathrm{d}s\end{aligned} \tag{2.16}$$

where $\overline{\mathcal{K}}$ is as in (1.11). Substituting (2.16) into (2.12), we obtain the integral equation (1.10).

Sufficient Part: Applying ${}^{\varrho}I_{a+}^{1-\gamma}$ on both sides of the integral equation (1.10), we get

$${}^{\varrho}I_{a+}^{1-\gamma}y(t)=\frac{\overline{\mathcal{K}}}{\Gamma(\alpha)}{}^{\varrho}I_{a+}^{1-\gamma}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}$$

$$\times \mathcal{F}\left(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\right) \mathrm{d}s$$
$$+ {}^{\varrho}I_{a+}^{1-\gamma} {}^{\varrho}I_{a+}^{\alpha}\mathcal{F}\left(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\right) \mathrm{d}s,$$

using Lemmas 2.7 and 2.10, we have

$$\begin{aligned} {}^{\varrho}I_{a+}^{1-\gamma}y(t) = &\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\overline{\mathcal{K}}\sum_{j=1}^{m}\eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \\ &\times \mathcal{F}\Big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\Big)\mathrm{d}s \\ &+ {}^{\varrho}I_{a+}^{1-\beta(1-\alpha)}\mathcal{F}\Big(t, y(t), \int_a^t H(t,s)y(s)ds, \int_a^b K(t,s)y(s)ds\Big). \end{aligned} \tag{2.17}$$

Since $1-\gamma < 1-\beta(1-\alpha)$, Lemma 2.12 and limit $t \to a+$ gives

$$\begin{aligned} {}^{\varrho}I_{a+}^{1-\gamma}y(a) = &\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\overline{\mathcal{K}}\sum_{j=1}^{m}\eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \\ &\times \mathcal{F}\Big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\Big)\mathrm{d}s \end{aligned} \tag{2.18}$$

Setting $t = \xi_j$ in (1.10), we have

$$\begin{aligned} y(\xi_j) = &\frac{\overline{\mathcal{K}}}{\Gamma(\alpha)}\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\sum_{j=1}^{m}\eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \\ &\times \mathcal{F}\Big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\Big)\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \\ &\times \mathcal{F}\Big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\Big)\mathrm{d}s. \end{aligned}$$

Further, we have

$$\begin{aligned} &\sum_{j=1}^{m}\eta_j y(\xi_j) \\ &= \frac{\overline{\mathcal{K}}}{\Gamma(\alpha)}\sum_{j-1}^{m}\eta_j \int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1} \\ &\quad \times \mathcal{F}\left(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\right)\mathrm{d}s \sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\eta_j\frac{1}{\Gamma(\alpha)}\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\
&\quad\times\mathcal{F}\left(s,y(s),\int_a^s H(s,\tau)y(\tau)d\tau,\int_a^b K(s,\tau)y(\tau)d\tau\right)\mathrm{d}s\\
&=\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\frac{\mathcal{F}\left(s,y(s),\int_a^s H(s,\tau)y(\tau)d\tau,\int_a^b K(s,\tau)y(\tau)d\tau\right)}{\Gamma(\alpha)}\mathrm{d}s\\
&\quad\times\left(1+\overline{\mathcal{K}}\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\right)\\
&=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\overline{\mathcal{K}}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\
&\quad\times\mathcal{F}\left(s,y(s),\int_a^s H(s,\tau)y(\tau)d\tau,\int_a^b K(s,\tau)y(\tau)d\tau\right)\mathrm{d}s
\end{aligned}
\tag{2.19}
$$

Equations (2.18) and (2.19), imply that

$$
{}^{\varrho}I_{a+}^{1-\gamma}y(a+)=\sum_{j=1}^{m}\eta_j y\left(\xi_j\right)
$$

Operating ${}^{\varrho}D_{a+}^{\gamma}$ to both sides of (1.10), from Lemmas 2.10 and 2.14, it follows that

$$
{}^{\varrho}D_{a+}^{\gamma}y(t)={}^{\varrho}D_{a+}^{\beta(1-\alpha)}\mathcal{F}\Big(t,y(t),\int_a^t H(t,s)y(s)ds,\int_a^b K(t,s)y(s)ds\Big)
\tag{2.20}
$$

since $y\in C_{1-\gamma,\varrho}^{\gamma}[a,b]$, from the definition of $C_{1-\gamma,\varrho}^{\gamma}[a,b]$, we have ${}^{\varrho}D_{a+}^{\gamma}y\in C_{1-\gamma,\varrho}[a,b]$ then ${}^{\varrho}D_{a+}^{\beta(1-\alpha)}\mathcal{F}=\delta_{\varrho}{}^{\varrho}I_{a+}^{1-\beta(1-\alpha)}\mathcal{F}\in C_{1-\gamma,\varrho}[a,b]$.
For $\mathcal{F}\in C_{1-\gamma,\varrho}[a,b]$, obviously ${}^{\varrho}I_{a+}^{1-\beta(1-\alpha)}\mathcal{F}\in C_{1-\gamma,\varrho}[a,b]$, then ${}^{\varrho}I_{a+}^{1-\beta(1-\alpha)}\mathcal{F}\in C_{1-\gamma,\varrho}^{\delta_{\varrho}}[a,b]$. This means $\mathcal{F}$ and ${}^{\varrho}I_{a+}^{1-\beta(1-\alpha)}\mathcal{F}$ satisfy the conditions of Lemma 2.9. Lastly, applying ${}^{\varrho}I_{a+}^{1-\beta(1-\alpha)}$ to both sides of (2.20), Lemma 2.9 helps us to obtain

$$
\begin{aligned}
{}^{\varrho}D_{a+}^{a,\beta}y(t)=&\,\mathcal{F}\big(t,y(t),\int_a^t H(t,s)y(s)ds,\int_a^b K(t,s)y(s)ds\big)\\
&-\frac{{}^{\varrho}I_{a+}^{1-\beta(1-\alpha)}\mathcal{F}(a)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\beta(1-\alpha)-1}
\end{aligned}
$$

By Lemma 2.12, it is easy to observe that ${}^{\varrho}I_{a+}^{1-\beta(1-\alpha)}\mathcal{F}(a)=0$. Hence, it reduces to

$$
{}^{\varrho}D_{a+}^{\alpha,\beta}y(t)=\mathcal{F}\Big(t,y(t),\int_a^t H(t,s)y(s)ds,\int_a^b K(t,s)y(s)ds\Big).
$$

Hence, the sufficiency is proved. This completes the proof of the lemma. □

3. Existence of solutions

In this section, we state and prove the main results concerning the existence of a solution of NGFIDE (1.8)-(1.9) with help of using Krasnoselskii's fixed point theorem and Schauder fixed point theorem.
The first result proved by Krasnoselskii's fixed point theorem.

Theorem 3.1. *Suppose that:*

(H_1) $\mathcal{F} : (a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a function such that* $\mathcal{F}(\cdot, y(\cdot), z(\cdot), w(\cdot) \in C^{\beta(1-\alpha)}_{1-\gamma,\varrho}[a,b]$ *for any* $y \in C_{1-\gamma,\varrho}[a,b]$ *and there exists a positive constant* $\mathcal{L} > 0$ *such that for all* $y, z, w, \bar{y}, \bar{z}, \bar{w} \in \mathbb{R}$,

$$|\mathcal{F}(t,y,z,w) - \mathcal{F}(t,\bar{y},\bar{z},\bar{w})| \le \mathcal{L}(|y-\bar{y}| + |z-\bar{z}| + |w-\bar{w}|).$$

(H_2) *The constant*

$$\theta = \frac{\Gamma(\gamma)\mathcal{L}(1+(H_b+K_b)(b-a))}{\Gamma(\gamma+\alpha)} \times \left(|\overline{\mathcal{K}}| \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} + \left(\frac{b^\varrho - a^\varrho}{\varrho}\right)^{\alpha} \right) < 1,$$

where $\overline{\mathcal{K}}$ *is as in* (1.11), $H_b = \sup\{|H(t,s)| : a \le s \le t \le b\}$ *and* $K_b = \sup\{|K(t,s)| : a \le s \le t \le b\}$.

Then problem (1.8)-(1.9) *has at least one solution in* $C^{\gamma}_{1-\gamma,\varrho}[a,b] \subset C^{\alpha,\beta}_{1-\gamma,\varrho}[a,b]$.

Proof. In view of Lemma 2.16, it is sufficient to prove the existence of a solution for mixed-type integral equation (1.10). Then we define an operator $N : C_{1-\gamma,\varrho}[a,b] \to C_{1-\gamma,\varrho}[a,b]$ by

$$\begin{aligned}(\mathcal{N}y)(t) = {}& \frac{\overline{\mathcal{K}}}{\Gamma(\alpha)} \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} \sum_{j-1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \\ & \times \mathcal{F}\Big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\Big)ds \\ & + \frac{1}{\Gamma(\alpha)} \int_a^t s^{\varrho-1} \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \\ & \times \mathcal{F}\Big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b H(s,\tau)y(\tau)d\tau\Big)ds. \end{aligned} \tag{3.1}$$

It is clear form assumption that the operator $\mathcal{N}$ is well defined. Set $\overline{\mathcal{F}}(s) = \mathcal{F}(s,0,0,0)$ and

$$\varpi = \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \left(|\overline{\mathcal{K}}| \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{a+\gamma-1} + \left(\frac{b^\varrho - a^\varrho}{\varrho}\right)^{\alpha} \right) \|\overline{\mathcal{F}}\|_{C_{1-\gamma,\varrho}}.$$

Consider

$$B_r = \left\{ y \in C_{1-\gamma,\varrho}[a,b] : \|y\|_{C_{1-\gamma,\varrho}} \le r \right\}, \quad \text{where } r \ge \frac{\varpi}{1-\theta},\ \theta < 1.$$

Now, we write the operator $\mathcal{N}$ as combination of two operators $\mathcal{P}$ and $\mathcal{Q}$ on B_r as follows:

$$(\mathcal{P}y)(t) = \frac{\overline{\mathcal{K}}}{\Gamma(\alpha)}\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}$$
$$\times\, \mathcal{F}\big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\big)ds. \tag{3.2}$$

and

$$(\mathcal{Q}y)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}$$
$$\times\, \mathcal{F}\big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\big)\mathrm{d}s. \tag{3.3}$$

The proof is divided into three steps:

Step 1. For any $y, \bar{y} \in B_r$ we prove $\mathcal{P}y + \mathcal{Q}\bar{y} \in B_r$. For operator $\mathcal{P}$, multiplying both sides of (3.2) by $\left((t^\varrho - a^\varrho)/\varrho\right)^{1-\gamma}$, we have

$$(\mathcal{P}y)(t)\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma} = \frac{\overline{\mathcal{K}}}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}$$
$$\times\, \mathcal{F}\big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\big)\mathrm{d}s.$$

Then

$$\begin{aligned}
&\Big|(\mathcal{P}y)(t)\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma}\Big| \\
&\le \frac{|\overline{\mathcal{K}}|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} \\
&\quad\times \Big|\mathcal{F}\big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\big)\Big|\mathrm{d}s \\
&\le \frac{|\overline{\mathcal{K}}|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} \\
&\quad\times \Big(\Big|\mathcal{F}\big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\big) - \mathcal{F}(s,0,0,0)\Big| \\
&\qquad + \big|\mathcal{F}(s,0,0,0)\big|\Big)\mathrm{d}s \\
&\le \frac{|\overline{\mathcal{K}}|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1} \\
&\quad\times \Big(\mathcal{L}\big(|y(s)| + H_b\int_a^s |y(\tau)|d\tau + K_b\int_a^b |y(\tau)|d\tau\big) + |\overline{\mathcal{F}}(s)|\Big)\mathrm{d}s \\
&\le \frac{|\overline{\mathcal{K}}|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j} s^{\varrho-1}\Big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\Big)^{\alpha-1}\Big(\frac{s^\varrho - a^\varrho}{\varrho}\Big)^{\gamma-1}
\end{aligned}$$

$$\times \Big(\big(\frac{s^\varrho - a^\varrho}{\varrho}\big)^{1-\gamma} \mathcal{L}[1 + (H_b + K_b)(b-a)]|y(s)| + \big(\frac{s^\varrho - a^\varrho}{\varrho}\big)^{1-\gamma} |\overline{\mathcal{F}}(s)| \Big)$$

$$\leq \frac{|\overline{\mathcal{K}}|}{\Gamma(\alpha)} \sum_{j-1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \big(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\big)^{\alpha-1} \big(\frac{s^\varrho - a^\varrho}{\varrho}\big)^{\gamma-1}$$

$$\times \left[\mathcal{L}(1 + (H_b + H_b)(b-a))\|y\|_{C_{1-\gamma,\varrho}} + \|\overline{\mathcal{F}}\|_{C_{\gamma,\varrho}}\right]$$

$$\leq \frac{|\overline{\mathcal{K}}|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\big)^{\alpha+\gamma-1}$$

$$\times \mathbf{B}(\alpha,\gamma) \times \left[\mathcal{L}(1 + (H_b + K_b)(b-a))\|y\|_{C_{1-\gamma,\varrho}} + \|\overline{\mathcal{F}}\|_{C_{1-\gamma,\varrho}}\right],$$

which implies

$$\|\mathcal{P}y\|_{C_{1-\gamma,\varrho}} \leq \frac{\Gamma(\gamma)|\overline{\mathcal{K}}|}{\Gamma(\alpha+\gamma)} \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1}$$
$$\times \left[\mathcal{L}(1 + (H_b + K_b)(b-a))\|y\|_{C_{1-\gamma,\varrho}} + \|\overline{\mathcal{F}}\|_{C_{1-\gamma,\varrho}}\right]. \tag{3.4}$$

For operator $\mathcal{Q}$,

$$\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma} (\mathcal{Q}y)(t)$$
$$= \frac{1}{\Gamma(\alpha)} \int_a^t s^{\varrho-1} \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma}$$
$$\times \mathcal{F}\left(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\right) \mathrm{d}s, \tag{3.5}$$

using the same fact that we used in the case of operator $\mathcal{P}$ again, we obtain

$$\left|(\mathcal{Q}y)(t) \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma}\right|$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_a^t s^{\varrho-1} \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma}$$
$$\times \left|\mathcal{F}\left(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\right)\right| \mathrm{d}s$$
$$\leq \left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_a^t s^{\varrho-1} \left(\frac{t^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}$$
$$\times [\mathcal{L}(1 + (H_b + K_b)(b-a))|y(s)| + |\overline{\mathcal{F}}(s)|]\mathrm{d}s$$
$$\leq \frac{\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)} \left(\frac{b^\varrho - a^\varrho}{\varrho}\right)^{\alpha} \left[\mathcal{L}(1 + (H_b + K_b)(b-a))\|y(s)\|_{C_{1-\gamma,\varrho}} + \|\bar{\mathcal{F}(s)}\|_{C_{1-\gamma,\varrho}}\right].$$

This gives

$$\|\mathcal{Q}y\|_{C_{1-\gamma,\varrho}} \leq \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \left(\frac{b^\varrho - a^\varrho}{\varrho}\right)^{\alpha}$$

$$\times \left[\mathcal{L}(1+(H_b+K_b)(b-a))\|y(s)\|_{C_{1-\gamma,\varrho}} + \|\mathcal{F}\bar{(}s)\|_{C_{1-\gamma,\varrho}}\right]. \tag{3.6}$$

From equations (3.4) and (3.6) for every $y, \bar{y} \in B_r$ we obtain

$$\|\mathcal{P}y + \mathcal{Q}\bar{y}\|_{C_{1-\gamma,\varrho}} \leq \|\mathcal{P}y\|_{C_{1-\gamma,\varrho}} + \|\mathcal{Q}\bar{y}\|_{C_{1-\gamma,\varrho}} \leq \theta r + \varpi \leq r$$

which implies that $\mathcal{P}y + \mathcal{Q}\bar{y} \in B_r$.

Step 2. Now we prove that operator $\mathcal{P}$ is a contraction mapping.
Let $y, \bar{y} \in B_r$, for operator $\mathcal{P}$ we have,

$$\begin{aligned}
&|\big((\mathcal{P}y)(t) - (\mathcal{P}\bar{y})(t)\big)\big(\frac{t^\varrho - a^\varrho}{\varrho}\big)^{1-\gamma}| \\
&\leq \frac{|\overline{\mathcal{K}}|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_j} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \\
&\quad \times |\big[\mathcal{F}\big(s, y(s), \int_a^s H(s,\tau)y(\tau)d\tau, \int_a^b K(s,\tau)y(\tau)d\tau\big) \\
&\quad - \mathcal{F}\big(s, \bar{x}(s), \int_a^s H(s,\tau)\bar{y}(\tau)d\tau, \int_a^b K(s,\tau)\bar{y}(\tau)d\tau\big)\big]|\mathrm{d}s \\
&\leq \frac{|\overline{\mathcal{K}}|}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \int_a^{\xi_s} s^{\varrho-1} \left(\frac{\xi_j^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \mathcal{L}[1+(H_b+K_b)(b-a)]|y(s) - \tilde{y}(s)|\mathrm{d}s \\
&\leq \frac{\mathcal{L}[1+(H_b+K_b)(b-a)]|\overline{\mathcal{K}}|\mathbf{B}(\alpha,\gamma)}{\Gamma(\alpha)} \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} \|y - y\|_{C_{1-\gamma,\varrho}},
\end{aligned}$$

which is

$$\begin{aligned}
&\|\mathcal{P}y - \mathcal{P}\bar{y}\|_{C_{1-\gamma,\varrho}} \\
&\leq \frac{\mathcal{L}[1+(H_b+K_b)(b-a)]|\overline{\mathcal{K}}|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \sum_{j=1}^{m} \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} \|y - \bar{y}\|_{C_{1-\gamma,\varrho}} \\
&\leq \theta \|y - \bar{y}\|_{C_{1-\gamma,\varrho}}.
\end{aligned}$$

Hence, by assumption (H_2), the operator $\mathcal{P}$ is a contraction mapping.

Step 3. The operator $\mathcal{Q}$ is compact and continuous.
Since $\mathcal{F} \in C_{1-\gamma,\varrho}[a,b]$, by the definition of $C_{1-\gamma,\varrho}[a,b]$, it is obvious that $\mathcal{Q}$ is continuous. By Step 1, we have

$$\begin{aligned}
\|\mathcal{Q}y\|_{C_{1-\gamma,\varrho}} \leq \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \left(\frac{b^\varrho - a^\varrho}{\varrho}\right)^{\alpha} \\
\times \left[\mathcal{L}(1+(H_b+K_b)(b-a))\|y\|_{C_{1-\gamma,\varrho}} + \|\overline{\mathcal{F}}\|_{C_{1-\gamma,\varrho}}\right],
\end{aligned}$$

which means $\mathcal{Q}$ is uniformly bounded on B_r.
To prove the compactness of $\mathcal{Q}$, for any $0 < a < t_1 < t_2 \leq b$ we have

$$\begin{aligned}
&\left|(\mathcal{Q}y)(t_1) - (\mathcal{Q}y)(t_2)\right| \\
&= |\int_a^{t_1} s^{\varrho-1} \left(\frac{t_1^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \frac{\mathcal{F}(s, y(s), \int_a^s H(s,\tau)y(\tau, \int_a^b K(s,\tau)y(\tau)d\tau)}{\Gamma(\alpha)}\mathrm{d}s
\end{aligned}$$

$$-\int_a^{t_2} s^{\varrho-1}\left(\frac{t_2^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1} \frac{\mathcal{F}(s, y(s), \int_a^s H(s,\tau)y(\tau, \int_a^b K(s,\tau)y(\tau)d\tau)}{\Gamma(\alpha)} \mathrm{d}s|$$
$$\leq \frac{\|\mathcal{F}\|_{C_{1-\gamma,\varrho}}}{\Gamma(\alpha)} |\int_a^{t_1} s^{\varrho-1}\left(\frac{t_1^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}\left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} \mathrm{d}s$$
$$-\int_a^{t_2} s^{\varrho-1}\left(\frac{t_2^\varrho - s^\varrho}{\varrho}\right)^{\alpha-1}\left(\frac{s^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} \mathrm{d}s|$$
$$\leq \frac{\|\mathcal{F}\|_{C_{1-\gamma,\varrho}}\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \left|\left(\frac{t_1^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1} - \left(\frac{t_2^\varrho - a^\varrho}{\varrho}\right)^{\alpha+\gamma-1}\right|. \tag{3.7}$$

The term present on right-hand side of inequality (3.7) tends to zero as $t_2 \to t_1$ either $\alpha+\gamma < 1$ or $\alpha+\gamma \geq 1$. Therefore, $\mathcal{Q}$ is equicontinuous. Hence, recalling Arzelà-Ascoli theorem, the operator $\mathcal{Q}$ is compact on B_r.
In view of Krasnoselskii's fixed point theorem, NGFIDE (1.8)-(1.9) has at least one solution $y \in C_{1-\gamma,\varrho}[a,b]$. One can easily show that this solution is actually in $C^\gamma_{1-\gamma,\varrho}[a,b]$ by repeating the process from the proof of Lemma 2.16. This complete the proof. ☐

Finally, we will discuss the existence result by using Schauder fixed point theorem. For this, we need the following hypothesis:

(H_3) $\mathcal{F} : (a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $\mathcal{F}(\cdot, y(\cdot), z(\cdot), w(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma,\varrho}[a,b]$ for any $y,z,w \in C_{1-\gamma,\varrho}[a,b]$, and for all $y,z,w \in \mathbb{R}$ there exist $\mathcal{L} > 0$ and $\mathcal{M} \geq 0$ such that
$$|\mathcal{F}(t,y,z,w)| \leq \mathcal{L}(|y|+|z|+|w|) + \mathcal{M}.$$

Theorem 3.2. *Suppose that (H_2) and (H_3) hold. Then NGFIDE* (1.8)-(1.9) *has at least one solution in* $C^\gamma_{1-\gamma,\varrho}[a,b] \subset C^{\alpha,\beta}_{1-\gamma,\varrho}[a,b]$.

Proof. Let $B_\varepsilon = \left\{y \in C_{1-\gamma,\varrho}[a,b] : \|y\|_{C_{1-\gamma,\varrho}} \leq \varepsilon\right\}$ with $\varepsilon \geq \Omega/(1-\theta)$ for $\theta < 1$, where
$$\Omega = \frac{\mathcal{M}|\overline{\mathcal{K}}|}{\Gamma(\alpha+1)} \sum_{j=1}^m \eta_j \left(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\right)^\alpha + \frac{\mathcal{M}}{\Gamma(\alpha+1)}\left(\frac{b^\varrho - a^\varrho}{\varrho}\right)^{\alpha-\gamma+1}.$$

Consider the operator $\mathcal{N}$ on B_ε defined in (3.1). We conclude the theorem by considering the following three steps:
Step 1. First we prove that $\mathcal{N}(B_\varepsilon) \subset B_\varepsilon$. By hypotheses ($H_2$) and ($H_3$), for any $y \in C_{1-\gamma,\varrho}[a,b]$ and $\|y\|_{C_{1-\gamma,\varrho}}$ we have

$$\left|(\mathcal{N}y)(t)\left(\frac{t^\varrho - a^\varrho}{\varrho}\right)^{1-\gamma}\right|$$
$$\leq \Big[\frac{\mathcal{L}(1+(H_b+K_b)(b-a))\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \sum_{j=1}^m \eta_j \Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1}$$
$$+ \frac{\mathcal{L}[1+(H_b+K_b)(b-a)]\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\Big(\frac{b^\varrho - a^\varrho}{\varrho}\Big)^a\Big] \|y\|_{C_{1-\gamma,\varrho}}$$

$$+\frac{\mathcal{M}}{\Gamma(\alpha+1)}\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}+\frac{\mathcal{M}}{\Gamma(\alpha+1)}\left(\frac{b^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha-\gamma+1}.$$

This is

$$\|\mathcal{N}y\|_{C_{1-\gamma,\varrho}}\leq\theta\varepsilon+\Omega\leq\varepsilon,$$

which gives $\mathcal{N}\left(B_{\varepsilon}\right)\subset B_{\varepsilon}$.

We shall prove that $\mathcal{N}$ is completely continuous.

Step 2. $\mathcal{N}$ is continuous. Let $\{y_n\}$ be a sequence such that $y_n\to y$ in B_{ε}. Then for each $t\in(a,b]$, we have

$$\begin{aligned}
&\Big|\left((\mathcal{N}y)\left(y_n\right)-(\mathcal{N}y)(t)\right)\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{\gamma-1}\Big|\\
&\leq\frac{|\overline{\mathcal{K}}|}{\Gamma(\alpha)}\sum_{j=1}^{m}\eta_j\int_a^{\xi_j}s^{\varrho-1}\left(\frac{\xi_j^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\
&\quad\times\big|\mathcal{F}\big(s,y_n(s),\int_a^s H(s,\tau)y_n(\tau)d\tau,\int_a^b K(s,\tau)y_n(\tau)d\tau\big)\\
&\quad-\mathcal{F}\big(s,y(s),\int_a^s H(s,\tau)y(\tau)d\tau,\int_a^b K(s,\tau)y(\tau)d\tau\big)\big|\mathrm{d}s\\
&\quad+\left(\frac{t^{\varrho}-a^{\varrho}}{\varrho}\right)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_a^t s^{\varrho-1}\left(\frac{t^{\varrho}-s^{\varrho}}{\varrho}\right)^{\alpha-1}\\
&\quad\times\big|\mathcal{F}\big(s,y_n(s),\int_a^s H(s,\tau)y_n(\tau)d\tau,\int_a^b K(s,\tau)y_n(\tau)d\tau\big)\\
&\quad-\mathcal{F}\big(s,y(s),\int_a^s H(s,\tau)y(\tau)d\tau,\int_a^b K(s,\tau)y(\tau)d\tau\big)\big|\mathrm{d}s\\
&\leq\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\left(|\overline{\mathcal{K}}|\sum_{j=1}^{m}\eta_j\left(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha+\gamma-1}+\left(\frac{b^{\varrho}-a^{\varrho}}{\varrho}\right)^{\alpha}\right)\\
&\quad\times\big|\big|\mathcal{F}\big(\cdot,y_n(\cdot),\int_a^s H(s,\tau)y_n(\cdot)d\tau,\int_a^b K(s,\tau)y_n(\cdot)d\tau\big)\\
&\quad-\mathcal{F}\big(\cdot,y(\cdot),\int_a^s H(s,\tau)y(\cdot)d\tau,\int_a^b K(s,\tau)y(\cdot)d\tau\big)\big|\big|_{C_{1-\gamma,\varrho}},
\end{aligned}$$

this implies

$$\begin{aligned}
&\big|\big|\mathcal{N}y_n-\mathcal{N}y\big|\big|_{C_{1-\gamma,\varrho}}\\
&\leq\frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)}\big(|\overline{\mathcal{K}}|\sum_{j=1}^{m}\eta_j\big(\frac{\xi_j^{\varrho}-a^{\varrho}}{\varrho}\big)^{\alpha+\gamma-1}+\big(\frac{b^{\varrho}-a^{\varrho}}{\varrho}\big)^{\alpha}\big)\\
&\quad\times\big|\big|\mathcal{F}\big(\cdot,y_n(\cdot),\int_a^s H(s,\tau)y_n(\cdot)d\tau,\int_a^b K(s,\tau)y_n(\cdot)d\tau\big)\\
&\quad-\mathcal{F}\big(\cdot,y(\cdot),\int_a^s H(s,\tau)y(\cdot)d\tau,\int_a^b K(s,\tau)y(\cdot)d\tau\big)\big|\big|_{C_{1-\gamma,\alpha}}.
\end{aligned}$$

Thus, $\mathcal{N}$ is a continuous operator.
Step 3. Finally, we prove that $\mathcal{N}(B_\varepsilon)$ is relatively compact.
Since $\mathcal{N}(B_\varepsilon) \subset B_\varepsilon$, it follows that $\mathcal{N}(B_\varepsilon)$ is uniformly bounded. By following the procedure as we did in Step 3 in Theorem 3.1 , one can easily prove $\mathcal{N}$ is equicontinuous on B_ε.
As $\alpha \le \gamma < 1$ and noting (3.7), for any $0 < a < t_1 < t_2 \le b$ one has

$$\begin{aligned}
&|(\mathcal{N}y)(t_1) - (\mathcal{N}y)(t_2)| \\
&\le \frac{\|\mathcal{F}\|_{C_{1-\gamma,\varrho}}|\overline{\mathcal{K}}|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \sum_{j=1}^{m} \eta_j \Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1} \\
&\quad \times \left(\left(\frac{t_1^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} - \left(\frac{t_2^\varrho - a^\varrho}{\varrho}\right)^{\gamma-1} \right) + |(\mathcal{Q}y)(t_1) - (\mathcal{Q}y)(t_2)| \\
&\le \frac{\|\mathcal{F}\|_{C_{1-\gamma,\varrho}}|\overline{\mathcal{K}}|\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \sum_{j=1}^{m} \eta_j \Big(\frac{\xi_j^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1} \Big|\frac{t_2^\varrho - t_1^\varrho}{(t_1^\varrho - a^\varrho)\,(t_2^\varrho - a^\varrho)}\Big|^{1-\gamma} \\
&\quad + \frac{\|\mathcal{F}\|_{C_{1-\gamma,\varrho}}\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \Big|\Big(\frac{t_1^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1} - \Big(\frac{t_2^\varrho - a^\varrho}{\varrho}\Big)^{\alpha+\gamma-1}\Big| \to 0,
\end{aligned}$$

as $t_2 \to t_1$. Thus, $\mathcal{Q}$ is equicontinuous.
Hence, $\mathcal{N}(B_\varepsilon)$ is an equicontinuous set and therefore $\mathcal{N}(B_\varepsilon)$ is relatively compact.
As a consequence of Steps 1 to 3 together with Arzelà-Ascoli theorem, we can conclude that $\mathcal{N} : B_\varepsilon \to B_\varepsilon$ is completely continuous. By applying Schauder fixed point theorem, we complete the proof. ☐

4. Example

In this section, we will show the applications of our main results with two examples.

Example 4.1. Consider the nonlocal problem

$$\big({}^\varrho D_{a+}^{\alpha,\beta}\big)y(t) = \mathcal{F}(t, y(t), (\mathcal{H}y)(t), (\mathcal{K}y)(t)),\ t \in (1,2], \tag{4.1}$$

$$\big({}^\varrho I_{a+}^{1-\gamma}y\big)(1+) = 2y\left(\frac{5}{3}\right), \quad \gamma = \alpha + \beta(1-\alpha). \tag{4.2}$$

Denoting $\alpha = \frac{3}{4},\ \beta = \frac{1}{2}$ gives $\gamma = \frac{7}{8}$. Let $\varrho = \frac{1}{2} > 0$ and set

$$\begin{aligned}
\mathcal{F}\Big(t, y(t), (\mathcal{H}y)(t), (\mathcal{K}y)(t)\Big) &= \left(\frac{t^\varrho - 1}{\varrho}\right)^{-1/16} + \frac{1}{4}\left(\frac{t^\varrho - 1}{\varrho}\right)^{15/16} \sin y(t) \\
&\quad + \frac{1}{4}\mathcal{H}y(t) + \frac{1}{4}\mathcal{K}y(t),
\end{aligned}$$

where $(\mathcal{H}y)(t) = \int_1^t \frac{1}{(3+t)^2} y(s)ds$, and $(\mathcal{K}y)(t) = \int_1^2 \frac{1}{(4+t)^2}\frac{y(s)}{1+y(s)}ds$.
We can see that

$$\begin{aligned}
&\left(\frac{t^{1/2} - 1}{\frac{1}{2}}\right)^{1/8} \mathcal{F}\Big(t, y(t), (\mathcal{H}y)(t), (\mathcal{K}y)(t)\Big) \\
&= \left(\frac{t^{1/2} - 1}{\frac{1}{2}}\right)^{1/16} + \frac{1}{4}\left(\frac{t^{1/2} - 1}{\frac{1}{2}}\right)^{17/16} \sin y(t)
\end{aligned}$$

$$+\frac{1}{4}\left(\frac{t^{1/2}-1}{\frac{1}{2}}\right)^{1/8}(\mathcal{H}y)(t)+\frac{1}{4}\left(\frac{t^{1/2}-1}{\frac{1}{2}}\right)^{1/8}(\mathcal{K}y)(t)\in C[1,2] \tag{4.3}$$

i.e. $\mathcal{F}(t,y,(\mathcal{H}y)(t),(\mathcal{K}y)(t))\in C_{1/8,1/2}[1,2]$.
Moreover,

$$\begin{aligned}&|\mathcal{F}(t,y,(\mathcal{H}y)(t),(\mathcal{K}y)(t))-\mathcal{F}(t,\bar{y},(\mathcal{H}\bar{y})(t),(\mathcal{H}\bar{y})(t))|\\ &\le\frac{1}{4}\big(|y-\bar{y}|+|(\mathcal{H}y)(t)-(\mathcal{H}\bar{y})(t)+|(\mathcal{K}y)(t)-(\mathcal{K}\bar{y})(t)|\big).\end{aligned}$$

So, we have $\mathcal{L}=\frac{1}{4}$, $H_b=\frac{1}{16}$ and $K_b=\frac{1}{25}$.
Some elementary computations gives us

$$|\overline{\mathcal{K}}|=\left|\left(\Gamma(0.875)-2\Big(\frac{\left(\frac{5}{3}\right)^{1/2}-1}{\frac{1}{2}}\Big)^{-1/8}\right)^{-1}\right|\approx 0.9521<1$$

and

$$\begin{aligned}\theta&=\frac{\Gamma(0.875)\frac{1}{4}(1+\frac{1}{16}(2-1)+\frac{1}{25}(2-1))}{4\Gamma(1.625)}\\ &\quad\times\left(0.9521\times 2\Big(\frac{\left(\frac{5}{3}\right)^{1/2}-1}{\frac{1}{2}}\Big)^{5/8}+\left(\frac{2^{1/2}-1}{\frac{1}{2}}\right)^{3/4}\right)\\ &=0.0837\times 2.2260\\ &\approx 0.1864<1.\end{aligned}$$

All the assumptions of Theorem 3.1 are satisfied with $|\overline{\mathcal{K}}|\approx 0.9521$ and $\theta\approx 0.1864$. Therefore, problem (4.1)-(4.2) has at least one solution in $C_{1/8,1/2}[1,2]$.

Example 4.2. Consider the nonlocal problem

$$\big({}^{\varrho}D^{\alpha,\beta}_{a+}y\big)(t)=\mathcal{F}(t,y(t),(\mathcal{H}y)(t),(\mathcal{K}y)(t)),\ t\in(1,2], \tag{4.4}$$

$$\big({}^{\varrho}I^{1-\gamma}_{a+}y\big)(1+)=3y\Big(\frac{8}{7}\Big)+2y\Big(\frac{4}{3}\Big). \tag{4.5}$$

Denote $\alpha=\frac{1}{2}$, $\beta=\frac{3}{4}$ and $\varrho=\frac{1}{2}>0$. So $\gamma=\frac{7}{6}$ and $\big(\xi_1=\frac{8}{7}\big)\le\big(\xi_2=\frac{4}{3}\big)$. Set $\mathcal{F}(t,y(t),(\mathcal{H}y)(t),(\mathcal{K}y)(t))=\sin\big(\frac{1}{3}|y(t)|\big)+\frac{1}{3}\mathcal{H}y(t)++\frac{1}{3}\mathcal{K}y(t),\ t\in(1,2]$,
where

$$\mathcal{H}y(t)=\int_1^t\frac{1}{(3+t)^2}y(s)ds,$$

and

$$(\mathcal{K}y)(t)=\int_1^2\frac{1}{(4+t)^2}\frac{y(s)}{1+y(s)}ds.$$

It is easy to see that $\mathcal{F}(t,y(t),(\mathcal{H}y)(t),(\mathcal{K}y)(t))\in C_{1/8,1/2}[1,2]$ and

$$|\mathcal{F}(t,y,Hy(t))|\le\frac{1}{3}\big(|y|+|\mathcal{H}y(t)+|\mathcal{K}y(t)|\big).$$

So, we have $\mathcal{L}=\frac{1}{3}$, $\mathcal{M}=0$, $H_b=\frac{1}{16}$ and $H_b=\frac{1}{25}$.

Moreover,

$$
\begin{aligned}
|\overline{\mathcal{K}}| &= \left|\left(\Gamma(0.875) - \left(3\left(\frac{\left(\frac{8}{7}\right)^{1/2}-1}{\frac{1}{2}}\right)^{-1/8} + 2\left(\frac{\left(\frac{4}{3}\right)^{1/2}-1}{\frac{1}{2}}\right)^{-1/8}\right)\right)^{-1}\right| \\
&\approx 0.1973 < 1
\end{aligned}
$$

and

$$
\begin{aligned}
\theta &= \frac{\Gamma(0.875)\frac{1}{3}(1+\frac{1}{16}(2-1)+\frac{1}{25}(2-1))}{3\Gamma(1.375)} \\
&\quad \times \left(0.1973 \times 3\left(\frac{\left(\frac{8}{7}\right)^{1/2}-1}{\frac{1}{2}}\right)^{3/8} + 2\left(\frac{\left(\frac{4}{3}\right)^{1/2}-1}{\frac{1}{2}}\right)^{3/8}\right) \\
&= 0.1502 \times 1.5699 \\
&\approx 0.2358 < 1.
\end{aligned}
$$

With the values of $|\overline{\mathcal{K}}|$ and θ, problem (4.4)-(4.5) satisfies all the conditions of Theorem 3.2. Thus, problem (4.4)-(4.5) has at least one solution in $C_{1/8,1/2}[1,2]$.

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H. L. Tidke
Department of Mathematics,
School of Mathematical Sciences,
Kavayitri Bahinabai Chaudhari North Maharashtra University,
Jalgaon
E-mail address: tharibhau@gmail.com

R. P. Mahajan
Department of Mathematics,
R. C. Patel Arts, Commerce and Science College, Shirpur
E-mail address: rpmahajan9@gmail.com

V. V. Kharat
Department of Mathematics,
N. B. Navale Sinhgad College of Engineering, Kegaon, Solapur
E-mail address: vvkvinod9@gmail.com