Journal of Mathematical Control Science and Applications
Vol. 7 No. 2 (July-December, 2021)

Submitted: 17th August 2021 *Revised: 27th September 2021* *Accepted: 30th November 2021*

REDUCED ORDER OBSERVER BASED MODIFIED ENHANCED ADAPTIVE CNF CONTROLLER

ABHIJIT KULKARNI

Abstract. Alternative solutions for control problems are the motivation for many types of problems. Alternate solution for reduced order observer based CNF controller is proposed in this work. Two numerical problems are considered and reliable simulation results are outcomes of the proposed controller. Novelty of the work is adaptive updating of nonlinear function in control law, scaled exponential function and improvement in results because of this change. Twofold improvements are proposed in the existing modified enhanced CNF controller which comprises a linear control and a nonlinear control law. First one is the adaptively updated nonlinear function in control law. Second is inclusion of scaled exponential function in control law. These modifications reduce RMS errors while tracking a sinusoidal target reference with multiple frequencies in a faster way without large overshoots. Simulation results are presented with comparison between modified enhanced CNF and modified enhanced adaptive CNF controller.

1. Introduction

Many of the system are linear systems with actuators as an important part. Actuator does appear in terms of pneumatic, electric, hydraulic or micro-actuators using latest technology such as micro-actuators. Actuator saturation is an important virtue of safety aspect. Actuation saturation can be viewed in application contexts such as those used in aeroplanes, control valves, power control devices. Composite nonlinear feedback control is a proven technique for such systems. System control with reliable results is always expected including performance parameters such as settling time, overshoot and error variables. System tracking responses, however, can be very challenging in sinusoidal desired signals as system components may have their own characteristics like hysteresis, inertia, interaction. Motivation of this work is that many systems can be modelled by linear systems with actuator saturation. In this work an alternative reliable approach in modified enhanced composite nonlinear feedback (CNF) controller is proposed. Novelty of the work is in adaptively updating nonlinear control law component. Availability of such alternate adaptive functions motivates researchers to explore more control opportunities and to investigate further such functions' advantages. Given a linear system under actuator constraints there exists an adaptive nonlinear function

2000 *Mathematics Subject Classification.* Primary 93-10; Secondary 93D05.

Key words and phrases. Composite Non-linear Feedback Control, Adaptive Updating, Observer, Alternate Solution.

with which semi-global asymptotic tracking can be achieved for a desired reference tracking. Novelty of the work is adaptive updating of nonlinear function in control law, scaled exponential function and improvement in results because of this change. Twofold improvements are proposed in existing modified enhanced CNF controller which comprises a linear control and a nonlinear control law. First one is adaptively updated nonlinear function in control law. Second is inclusion of scaled exponential function in control law. These modifications reduce RMS errors while tracking a sinusoidal target reference with multiple frequencies in a faster way without large overshoots. Simulation results are presented with comparison between modified enhanced CNF and modified enhanced adaptive CNF controller. Alternate approach presented here definitely open a new area for further set of system solutions. Literature in this regard is [1] through [7]. References from [1] to [6] appear intermittently. In [7], observer gain is chosen using Riccati equation. Further authors in [7] proposed topology-induced containment controller and it is extended to the output feedback scenario.

This work is organized as follows. **Section 2** explains controller design. Controller design in context to state feedback and reduced order measurement feedback cases are in detail discussed here. This also includes Lyapunov stability analysis with state feedback. **Section 3** is about what modification is brought in existing control law in terms of scaled functions. **Section 4** talks about illustrative example 1. **Section 5** is about XY-table servo system example (i.e., illustrative example 2). Last **Section 6** concludes the work.

2. Controller design

This section describes system considered, the auxiliary system, reference generator and integrator augmentation followed by state feedback and reduced order measurement feedback which is fruitful in the form of proposed controller. Consider a linear system:

$$\left.\begin{array}{c} \dot{x} = Ax + Bsat(u) + Ew, x(0) = x_0 \\ y = C_1 x \\ h = C_2 x \end{array}\right\} \tag{2.1}$$

$$sat(u) = sgn(u) min\left\{u_{max}, |u|\right\} \tag{2.2}$$

where $x \in R^n, u \in R, y \in R^p, h \in R, w \in R$ represent state, control input, measurement output, controlled output and disturbance input to system.

A, B, E, C_1, C_2 are constant matrices of appropriate dimensions. Designs of the proposed controller have two major parts. This includes reference generator and proposed adaptive CNF controller. Controller design is like [1], [2] as far as linear control law design is considered. So, the steps are reconsidered from those two references for smooth flow of design. Objective is to design a reduced order observer based adaptive CNF controller to track a desired signal as fast as possible and without large overshoot. The auxiliary system described below from [2] generates a *reference signal* that is to be tracked:

$$\Sigma_{aux} : \begin{cases} \dot{x_e} = Ax_e + Bu_e, x_e(0) = x_{e0} \\ r = C_2 x_e \end{cases} \tag{2.3}$$

where $x_e \in R^n$, $u_e \in R$ and r are the state, input and output of the auxiliary system. Then linear control law is:

$$u_e = F_e x_e + r_s \tag{2.4}$$

where F_e is the feedback gain matrix and r_s is an external signal source. Combining 2.3 and 2.4 the reference generator becomes:

$$\Sigma_{REF} : \left\{ \begin{matrix} \dot{x_e} = (A + BF)x_e + Br_s, x_e(0) = x_{e0} \\ u_e = F_e x_e + r_s \\ r = C_2 x_e \end{matrix} \right\} \tag{2.5}$$

Then an integrator is augmented into given system which takes care of disturbances and steady state error [1]

$$\dot{x_i} := e := h - r = C_2 x - r \tag{2.6}$$

The integrator augmented with system then takes the form:

$$\left\{ \begin{matrix} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}sat(u) + \bar{B}_r r + \bar{E}w \\ \bar{y} = \bar{C}_1 \bar{x} \\ h = \bar{C}_2 \bar{x} \end{matrix} \right. \tag{2.7}$$

where

$$\bar{x} = \begin{pmatrix} x_i \\ x \end{pmatrix}, \bar{x_0} = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}, \bar{y} = \begin{pmatrix} x_i \\ y \end{pmatrix} \tag{2.8}$$

$$\bar{A} = \begin{bmatrix} 0 & C_2 \\ 0 & A \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, B_r = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{2.9}$$

and

$$\bar{E} = \begin{bmatrix} 0 \\ E \end{bmatrix}, \bar{C_1} = \begin{bmatrix} 1 & 0 \\ 0 & C_1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & C_2 \end{bmatrix} \tag{2.10}$$

In the proposed controller, the adaptive law associated with nonlinear function. Design of controller is carried in two cases: state feedback and reduced order measurement feedback case.

2.1. State feedback case. For the augmented system given by 2.7, the reference generator when all the state variables are measurable is given by:

$$\left\{ \begin{matrix} \dot{\bar{x}}_e = \bar{A}\bar{x_e} + \bar{B}u_e + \bar{B}_r r \\ u_e = \begin{bmatrix} 0 & F_e \end{bmatrix} \bar{x_e} + r_s \\ r = C_2 \bar{x_e} \end{matrix} \right. \tag{2.11}$$

with $\widetilde{x}_e \begin{bmatrix}0 & x_e\end{bmatrix}^T$, $\widetilde{x}_{e0} \begin{bmatrix}0 & x_{e0}\end{bmatrix}^T$

After defining $\widetilde{x} = \bar{x} - \bar{x_e}$ one gets

$$\dot{\widetilde{x}} = \bar{A}\bar{x} + \bar{B}\left\{sat\left(u\right) - u_e\right\} + \bar{E}w \tag{2.12}$$

The steps involved in proposed controller design are:

Step 1) Constructing linear law

$$u_L = F\widetilde{x} + u_e \tag{2.13}$$

Step 2) Then the nonlinear feedback portion of the proposed controller u_N is:

$$u_N = \rho\bar{B}^T P\widetilde{x} \tag{2.14}$$

where ρ is updated adaptively from

$$\left.\begin{array}{c}\dot{\rho} = k_1 \|\widetilde{x}\| \rho + \left(\rho^4\right) \times \widetilde{x}^T P \bar{B} \bar{B}^T P \widetilde{x}, k_1 > 0 \\ \rho(0) = \rho_0\end{array}\right\} \tag{2.15}$$

This nonlinear function yields better tracking performance.
Step 3) Combination of linear and nonlinear law:

$$u = u_L + u_N = F\widetilde{x} + u_e \bar{B}^T P \widetilde{x} \tag{2.16}$$

Theorem 2.1. *Consider the system in 2.1 with all the states measurable and bounded disturbance w. Then there exists a smooth nonlinear function ρ updated adaptively from 2.15 for a given initial condition $\rho(0) = \rho_0$, and the modified enhanced adaptive CNF control law comprising of 2.16, will take the output h to track a general reference r along with the system error $\widetilde{x} \to 0$ as time $t \to \infty$.*

Proof. Following the steps detailed in [2], the closed loop system comprising of the augmented plant in 2.7 and the modified enhanced adaptive CNF control law in 2.16 is given by $\dot{\widetilde{x}} = \left(\bar{A} + \bar{B}F\right)\widetilde{x} + \bar{B}v + \bar{E}w$ where $v := sat\,(u) - F\widetilde{x} - u_e$ and $u = F\widetilde{x} + u_e + \rho\bar{B}^T P\widetilde{x}$.

Note: c_δ is the largest positive scalar [3] such that the error $\widetilde{x} \in X\,(F, c_\delta)$, $\left\{\bar{x} : \bar{x}^T P \bar{x} \leq c_\delta\right\}$, $|F\widetilde{x} + u_e| \leq u_{max}$

Based on the maximum/minimum limit of u , the correlated v takes the form depending on the saturation regions as:

a) $\rho\bar{B}^T P\widetilde{x} < v < 0$ (Negative saturation region of u)
b) $\rho\bar{B}^T P\widetilde{x} = v$ (Linear region of u) and
c) $0 < v < \rho\bar{B}^T P\widetilde{x}$ (Positive saturation region of u).

To make the stability analysis easier further, for all these cases, v is expressed as $v = q\rho\bar{B}^T P\widetilde{x}$ [2] for $q \in [0, 1]$.Then

$$\dot{\widetilde{x}} = \left(\bar{A} + \bar{B}F + q\rho\bar{B}\bar{B}^T P\right)\widetilde{x} + \bar{E}w \tag{2.17}$$

□

2.2. Stability. Let the Lyapunov function be

$$V = \widetilde{x}^T P \widetilde{x} + \rho^- 2 \tag{2.18}$$

$$\dot{V} = \dot{\widetilde{x}}^T P \widetilde{x} + \widetilde{x}^T P \dot{\widetilde{x}} - 2\rho^{-3}\dot{\rho} \tag{2.19}$$

$$\begin{aligned} &= \left[\widetilde{x}^T \left(\bar{A} + \bar{B}F + q\rho\bar{B}\bar{B}^T P\right)^T + \left(\bar{E}w\right)^T\right] + \\ &\widetilde{x}^T P \left[\left(\bar{A} + \bar{B}F + q\rho\bar{B}\bar{B}^T P\right)\widetilde{x} + \left(\bar{E}w\right)\right] + \\ &-2\rho^{-3}\left[k_1 \|\widetilde{m}\| \rho + \left(\rho^4\right) \times \bar{x}^T P\bar{B}\bar{B}^T P\widetilde{x}\right]\end{aligned} \tag{2.20}$$

With substitution of $F = -R^{-1}\bar{B}^T P$

$$\begin{aligned}\dot{V} = &\left[\widetilde{x}^T \left(\bar{A} + \bar{B}\left(-R^{-1}\bar{B}^T P\right) + q\rho\bar{B}\bar{B}^T P\right)^T + \left(\bar{E}w\right)^T\right] P\widetilde{x} \\ &+\widetilde{x}^T P\left[\left(\bar{A} + \bar{B}\left(-R^{-1}\bar{B}^T P\right) + q\rho\bar{B}\bar{B}^T P\right)\widetilde{x} + \left(\bar{E}w\right)\right] \\ &-2\rho^{-3}\left[k_1 \|\widetilde{m}\| \rho + \left(\rho^4\right) \times \widetilde{x}^T P\bar{B}\bar{B}^T P\widetilde{x}\right]\end{aligned} \tag{2.21}$$

$$
\begin{aligned}
&= \widetilde{x}^T \left(\bar{A} + \bar{B}\left(-R^{-1}\bar{B}P\right)\right)^T P\widetilde{x} + \widetilde{x}^T \left(q\rho\bar{B}\bar{B}^T P\right)^T P\widetilde{x} + \left(\bar{E}w\right) P\widetilde{x} \\
&\quad + \widetilde{x}P\left(\bar{A} + \bar{B}\left(-R^{-1}\bar{B}P\right)\right)\widetilde{x} + \widetilde{x}^T P\left(q\rho\bar{B}\bar{B}^T P\right)\widetilde{x} + \widetilde{x}^T P\bar{E}w \\
&\quad -2\rho^{-3} \times \rho k_1 \left\|\widetilde{x}\right\| - 2\rho^{-3} \times \rho^4 \widetilde{x}^T P\bar{B}\bar{B}^T P\widetilde{x}
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
&= \widetilde{x}^T - \left(\bar{A}^T + P\bar{A} - P\bar{B}R^{-1}\bar{B}^T P\right)\widetilde{x} - \widetilde{x}^T P\bar{B}R^{-1}\bar{B}^T P\widetilde{x} + \\
&\quad 2q\rho\widetilde{x}^T P\bar{B}\bar{B}^T P\widetilde{x} + 2\widetilde{x}^T P\bar{E}w - 2\rho^{-3} \times \rho k_1 \left\|\widetilde{x}\right\| \\
&\quad -2\rho^{-3} \times \rho^4 \widetilde{x}^T P\bar{B}\bar{B}^T P\widetilde{x}
\end{aligned} \tag{2.23}
$$

$$
\begin{aligned}
&\leq \widetilde{x}^T Q\widetilde{x} - \widetilde{x}^T P\bar{B}R^{-1}\bar{B}^T P\widetilde{x} + 2q\rho\widetilde{x}^T P\bar{B}\bar{B}^T P\widetilde{x} + \\
&\quad 2\widetilde{x}^T P\bar{E}w - 2\rho^{-3} \times \rho k_1 \left\|\widetilde{x}\right\| - 2\rho^{-3} \times \rho^4 \widetilde{x}^T P\bar{B}\bar{B}^T P\widetilde{x}
\end{aligned} \tag{2.24}
$$

Next with following

$$
P\bar{B}\bar{B}^T P = N \tag{2.25}
$$

and

$$
P\bar{B}R^{-1}\bar{B}^T P = Z \tag{2.26}
$$

and taking terms $\bar{x}^T\bar{x}$ common from the two terms involving N and Z and letting

$$
Q + Z = T \tag{2.27}
$$

$$
\begin{aligned}
\dot{V} &\leq -\bar{x}^T T\widetilde{x}^T + 2q\rho\bar{x}^T N\widetilde{x} + 2\bar{x}^T P\bar{E}w \\
&\quad -2\rho^{-2} k_1 \left\|\widetilde{x}\right\| - 2\rho\widetilde{x}^T N\widetilde{x}
\end{aligned} \tag{2.28}
$$

Let $P = S^T S \Rightarrow S^T SP^{-1} = I$ Multiplying first term of $\dot{V}$ with

$$
S^T SP^{-1}P^{-1}S^T S = I
$$

and substitution of $S^T S = P$ in second term of $\dot{V}$, one gets

$$
\begin{aligned}
\dot{V} &\leq -\bar{x}^T S^T \left[\right] SP^{-1}TP^{-1}S^T S\widetilde{x} + 2\widetilde{x}^T S^T S\bar{E}w + \\
&\quad 2q\rho\bar{x}^T N\widetilde{x} - 2\rho^{-2} k_1 \left\|\widetilde{x}\right\| - 2\rho\widetilde{x}^T N\widetilde{x}
\end{aligned} \tag{2.29}
$$

with $|w| \leq \tau_w$.

Hence the second term in $\dot{V}$ i.e., $2\widetilde{x}^T S^T S\bar{E}w$ becomes $2\tau_w \left\|S\widetilde{x}\right\| \left\|S\widetilde{E}^T\right\|$. Further writing the term

$$
\begin{aligned}
\dot{V} &\leq -\lambda_{min}\left(S^T SP^{-1}TP^{-1}S^T\right) S\widetilde{x} + 2\tau_w \left\|S\widetilde{x}\right\| \left\|S\widetilde{E}^T\right\| + \\
&\quad 2q\rho\bar{x}^T N\widetilde{x} - 2\rho^{-2} k_1 \left\|\widetilde{x}\right\| - 2\rho\widetilde{x}^T N\widetilde{x}
\end{aligned} \tag{2.30}
$$

Now $\left\|S\widetilde{x}\right\| = \sqrt{\left(S\widetilde{x}\right)^T \left(S\widetilde{x}\right)} = \sqrt{\widetilde{x}S^T S\widetilde{x}} = \sqrt{\widetilde{x}P\widetilde{x}} = \left(\widetilde{x}P\widetilde{x}\right)^{\frac{1}{2}}$; similarly, $\left\|S\bar{E}^T\right\| = \left(\bar{E}P\bar{E}^T\right)^{\frac{1}{2}}$.

This implies that $2\tau_w \left\|S\widetilde{x}\right\| \left\|S\widetilde{E}^T\right\| = 2\tau_w \left(\widetilde{x}P\widetilde{x}\right)^{\frac{1}{2}} \left(\bar{E}P\bar{E}^T\right)^{\frac{1}{2}}$.

Also, in first bracketed term substitute $S^T S = P \& PP^{-1} = I \Rightarrow SP^{-1}TP^{-1}S^T = TP^{-1} = P^{-1}T$

$$
\begin{aligned}
\dot{V} &\leq -\lambda_{min}\left(S^T SP^{-1}TP^{-1}S^T\right) S\widetilde{x} + 2\tau_w \left(\widetilde{x}P\widetilde{x}\right)^{\frac{1}{2}} \left(\bar{E}P\bar{E}^T\right)^{\frac{1}{2}} + \\
&\quad 2q\rho\bar{x}^T N\widetilde{x} - 2\rho^{-2} k_1 \left\|\widetilde{x}\right\| - 2\rho\widetilde{x}^T N\widetilde{x}
\end{aligned} \tag{2.31}
$$

$\dot{V}$ is now partitioned as $\dot{V}_1$ and $\dot{V}_2$

Let

$$\left.\begin{array}{r}\dot{V}_1 \leq -\lambda_{min}\left(S^T S P^{-1} T P^{-1} S^T\right) S\widetilde{x} + 2\tau_w\left(\widetilde{x} P \widetilde{x}\right)^{\frac{1}{2}}\left(\bar{E} P \bar{E}^T\right)^{\frac{1}{2}} \\ \dot{V}_2 \leq 2q\rho\bar{x}^T N\widetilde{x} - 2\rho^{-2}k_1\left\|\widetilde{x}\right\| - 2\rho\widetilde{x}^T N\widetilde{x}\end{array}\right\} \tag{2.32}$$

Taking $\lambda_{min}\left(P^{-1}T\right)\left(\widetilde{x}^T\widetilde{x}\right)^{\frac{1}{2}}$ common from $\dot{V}_1$ one gets

$$\dot{V}_1 \leq -\lambda_{min}\left(P^{-1}T\right)\left(\widetilde{x}^T\widetilde{x}\right)^{\frac{1}{2}}\left[\left(\widetilde{x}^T\widetilde{x}\right)^{\frac{1}{2}} - 2\tau_w\lambda_{max}\left(PT^{-1}\right)\left(\bar{E}P\bar{E}^T\right)^{\frac{1}{2}}\right] \tag{2.33}$$

$$\dot{V}_1 \leq -\lambda_{min}\left(P^{-1}T\right)\left(\widetilde{x}^T\widetilde{x}\right)^{\frac{1}{2}}\left[\left(\widetilde{x}^T\widetilde{x}\right)^{\frac{1}{2}} - \gamma\right] \tag{2.34}$$

where

$$\gamma = 2\tau_w\lambda_{max}\left(PT^{-1}\right)\left(\bar{E}P\bar{E}^T\right)^{\frac{1}{2}} \tag{2.35}$$

It then follows that if $\left(\widetilde{x}^T P\widetilde{x}\right)^{\frac{1}{2}} > \gamma$ then

$$\dot{V}_1 \leq 0 \tag{2.36}$$

As the $\dot{V}_2$ involves terms associated with ρ, after taking $2\rho\widetilde{x}^T N\widetilde{x}$ common from first and third term, one gets

$$\dot{V}_2 \leq 2\rho\widetilde{x}^T N\widetilde{x}\left(q-1\right) - 2\rho^{-2}k_1\left\|\widetilde{x}\right\| \tag{2.37}$$

Now two cases are considered here; if ρ is positive or negative:

A) If ρ is positive

i) for $q = 1$

$$\dot{V}_2 \leq 0 - 2\rho^{-2}k_1\left\|\widetilde{x}\right\| < 0 \tag{2.38}$$

ii) for $0 \leq q < 1$ the term $(q-1)$ becomes negative, hence

$$\dot{V}_2 \leq -2\rho\widetilde{x}^T N\widetilde{x} - 2\rho^{-2}k_1\left\|\widetilde{x}\right\| < 0 \tag{2.39}$$

B) If ρ is negative

i) for $q = 1$

$$\dot{V}_2 \leq -2\rho^{-2}k_1\left\|\widetilde{x}\right\| \tag{2.40}$$

It then follows that $\dot{V}_2 < 0$ provided condition in 2.40 is satisfied.

ii) for $0 \leq q < 1$

$$\dot{V}_2 \leq +2\rho\widetilde{x}^T N\widetilde{x} - 2\rho^{-2}k_1\left\|\widetilde{x}\right\| \tag{2.41}$$

$$\dot{V}_2 \leq -2\rho\lambda_{min}\left(N\right)\left\|\widetilde{x}\right\|^2 - 2\rho^{-2}k_1\left\|\widetilde{x}\right\| \tag{2.42}$$

It then follows that

$$2\rho^{-2}k_1\left\|\widetilde{x}\right\| > 2\rho\lambda_{min}\left(N\right)\left\|\widetilde{x}\right\|^2 \tag{2.43}$$

$$\left\|\widetilde{x}\right\| < \frac{k_1}{\rho^3\lambda_{min}\left(N\right)} \tag{2.44}$$

This proves $\dot{V} < 0$ that subjected to fulfillment of 2.40. Hence the closed loop system in 2.13 is guaranteed to be semi-globally asymptotically stable.

2.3. Reduced order measurement feedback case. Noting that C_1 in the measurement output of the given system in 2.1 is in the form: $C_1 = \begin{bmatrix} I_P & 0 \end{bmatrix}$, the system from 2.7 is partitioned as:

$$\left\{ \begin{aligned} \begin{pmatrix} \dot{x}_i \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{bmatrix} 0 & C_{21} & C_{22} \\ 0 & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} x_i \\ x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ B_1 \\ B_2 \end{bmatrix} sat\left(u\right) + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ E_1 \\ E_2 \end{bmatrix} w \\ y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_P & 0 \end{bmatrix} \\ h &= \begin{bmatrix} 0 & C_{21} & C_{21} \end{bmatrix} \begin{pmatrix} x_i \\ x_1 \\ x_2 \end{pmatrix} \end{aligned} \right. \tag{2.45}$$

with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}, E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

and

$$\begin{pmatrix} x_i \\ x_1 \\ x_2 \end{pmatrix} = \bar{x}, \begin{pmatrix} x_i\left(0\right) \\ x_1\left(0\right) \\ x_2\left(0\right) \end{pmatrix} = \begin{pmatrix} 0 \\ x_{10} \\ x_{20} \end{pmatrix} = \bar{x}_0, \bar{y} = \begin{pmatrix} x_i \\ y \end{pmatrix} = \begin{pmatrix} x_i \\ x_1 \end{pmatrix}.$$

Objective is to design the reduced order observer based modified enhanced adaptive CNF control law where the variable x_2 is to be estimated. Reduced order observer takes the standard form [4]:

$$\begin{aligned} \dot{x}_c = &\left(A_{22} + K_R A_{12}\right) x_c + \left(B_2 + K_R B_1\right) sat\left(u\right) \\ &+ \left[A_{21} + K_R A_{11} - \left(A_{22} + K_R A_{12}\right) K_R\right] y \end{aligned} \tag{2.46}$$

with $x_c = \hat{x_c} + K_R y$, where K_R is observer gain, $\hat{x_c}$ is estimated state variable. Reduced order observer based modified enhanced adaptive CNF controller takes the form:

$$u = \left(F + \rho \bar{B} P\right) \left[\begin{pmatrix} x_i \\ x_1 \\ x_c - K_R y \end{pmatrix} - \bar{x}_e \right] + u_e \tag{2.47}$$

where $\bar{x}_e = \begin{pmatrix} 0 & x_{e1} & x_{e2} \end{pmatrix}^T$ with the nonlinear gain ρ is adaptively updated using:

$$\left. \begin{aligned} \dot{\rho} &= k_1 \left\| \widetilde{m} \right\| \rho + \left(\rho^4\right) \times \widetilde{m}^T P \bar{B} B^T P \widetilde{m} \\ \widetilde{m} &= \left[\begin{pmatrix} x_i \\ x_1 \\ x_c - K_R y \end{pmatrix} - \bar{x}_e \right], k_1 > 0 \\ \rho\left(0\right) &= \rho_0 \end{aligned} \right\} \tag{2.48}$$

Theorem 2.2. *For a given system in 2.1, there exists a scalar nonlinear gain ρ updated from adaptive law as in 2.48, the reduced order observer based modified enhanced adaptive CNF law given by 2.47 will drive the system-controlled output to track the desired signal semi-globally asymptotically provided* [2]*:*

$$1) \forall \bar{x} \in X\left(F, c_{R\delta}\right) := \left\{ \bar{x} : \bar{x}^T \begin{bmatrix} P & 0 \\ 0 & Q_R \end{bmatrix} \bar{x} \leq c_{R\delta} \right., \delta \in (0,1) \textit{ and } c_{R\delta} > \gamma^2 R.$$

$$2) \textit{Initial conditions: } \in X\left(F, c_{R,\delta}\right)$$

$$3) \left|u_e\right| \leq \delta u_{max}$$

The proof goes on the similar lines of [2] and [3]. Further as $\widetilde{m} \to 0 \Rightarrow x \to x_e$ in a finite time and controlled output $h \to r$.

3. Modification of the existing control law

To improve the response of the closed loop system, i.e., to make it faster, the linear control law is scaled by a smooth function $(\alpha - e^{-}at)$ where α, a are positive constants. Similarly, the nonlinear law is scaled by a function $e^{-}bt$ where b is a small positive constant. So, the control law in 2.47 becomes

$$u = \left(F \times K_{11} + \rho \bar{B} P K_{22}\right) \left[\begin{pmatrix} x_i \\ x_1 \\ x_c - K_r y \end{pmatrix} - \bar{x}_e \right] + u_e \tag{3.1}$$

where $K_{11} = \alpha - e^{-}at$, $K_{22} = e^{-}bt$

Remark: The nonlinear function adaptive law makes this modified enhanced adaptive CNF controller robust. The initial value of ρ i.e., $\rho(0)$ can be chosen to be positive, negative or zero. In the literature on enhanced CNF controller, ρ is a nonlinear gain function with $\rho < 0$ [[1], [5]]. Hence the proposed adaptive CNF controller is robust.

The steps for the design of proposed adaptive CNF controller are summarised: Augment integrator with plant as given in 2.7. Based on the desired signal r, gain matrix F, F_e and external signal generator r_s , design auxiliary reference generator in 2.3 and then form the reference generator in 2.5. Implement nonlinear adaptive law from 2.48 with the selection of reduced order observer gain K_R that locates the poles of $A_{22} + K_R A12$ suitably in the left-half plane. Choose constants related with exponential scaling functions α, a, b and implement the control law in 3.1.

Exponential scaling functions can be helpful in many situations. These functions can reduce control effort. For example, a vertical slope is difficult for a vehicle to climb; however, with a longer exponential road, the same travel is made easy. Another example is vertigo. While climbing down the steps of a staircase is problematic for vertigo patient. If these steps are climbed down with a reduced slope, this reduces dizziness definitely.

Two simulation examples are presented to validate the performance of the modified enhanced adaptive CNF control law. These are from [2]. First example is associated with tracking a multi-sinusoidal reference signal and the second example represents model of a practical high speed XY-table.

4. Illustrative Example 1

$$\left.\begin{array}{c} \dot{x} = \begin{bmatrix} 0 & 1 \\ -10 & 5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 100 \end{bmatrix} sat\,(u) + \begin{bmatrix} 0 \\ 100 \end{bmatrix} w \\ y = x_1 \\ u_{max} = 2 \\ w = -0.1 \end{array}\right\} \tag{4.1}$$

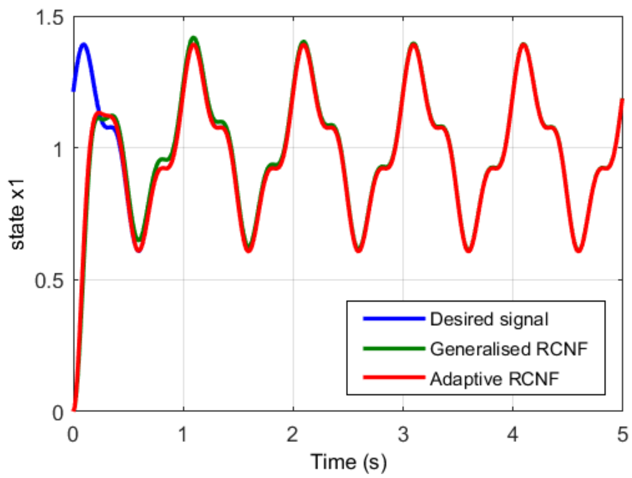


FIGURE 1. Comparison of output signal with multi-sinusoidal desired signal

Let

$$Q = diag\left\{1.5500 \quad 0.2500 \quad 0.0001\right\} \text{ and } R = 5.7500$$

$$P = \begin{bmatrix} 2.1539 & 0.2447 & 0.0124 \\ 0.2447 & 0.3231 & 0.0166 \\ 0.0124 & 0.0165 & 0.0012 \end{bmatrix}, F = -\begin{bmatrix} 0.5189 & 0.6578 & 0.1752 \end{bmatrix}$$

With desired $r = 1 + 0.3\sin\left(2\pi t + \frac{\pi}{4}\right) + 0.1\sin(6\pi t)$ and the feedback gain matrix $F_e = \begin{bmatrix} 0.0000 & -0.2948 & -0.0500 \end{bmatrix}$ the reference generator is formed as

$$\Sigma_{aux} : \begin{cases} \dot{x}_e = \begin{bmatrix} 0 & 1 \\ -{w_1}^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r_s \\ x_e\left(0\right) = \begin{pmatrix} 1.2121 \\ 3.2178 \end{pmatrix} \\ r = \begin{bmatrix} 1 & 0 \end{bmatrix} x_e \\ w_1 = 2\pi \end{cases} \tag{4.2}$$

$$r_s\left(t\right) = \frac{1}{100}\left[a_0 {w_1}^2 + a_2\left(w_1^2 - w_2^2\right)\sin\left(w_2 t\right)\right] \tag{4.3}$$

with $a_0 = 1, a_2 = 0.1, w_2 = 6\pi$. The nonlinear function adaptive law in 2.48 then becomes

$$\dot{\rho} = 0.0001\left\|\widetilde{x}_r\right\|\rho + \left(\rho^4\right)\widetilde{x}_r^T P\bar{B}\left(\bar{B}^T P\widetilde{x}_r\right) \tag{4.4}$$

Further with $a = 4.500, b = 0.0001$ and

$$K_{11} = 1.2500 - e^{-at}, K_{22} = e^{-bt} \tag{4.5}$$

and $F_n = \bar{B}^T P = \begin{bmatrix} 1.2371 & 1.6574 & 0.1224 \end{bmatrix}$ the control law in (1.53) becomes

$$v = \left(K_{11} \times F + K_{22} \times \rho \times \begin{bmatrix} 0.6344 & 1.3347 & 0.1159 \end{bmatrix}\right) \times \widetilde{x_r} + u_e \tag{4.6}$$

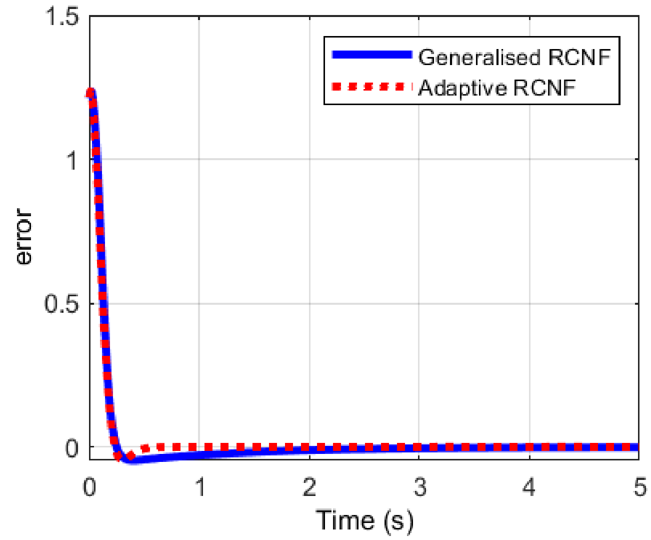


FIGURE 2. Error convergence of adaptive RCNF is faster and converges quickly

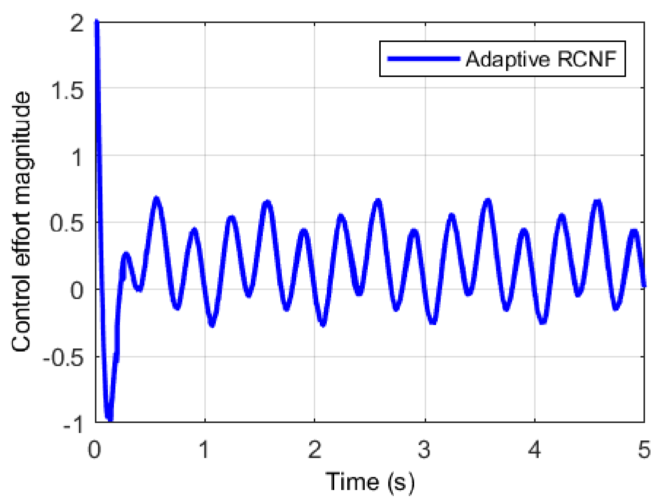


FIGURE 3. Control effort (Adaptive RCNF)

MATLAB simulation responses of example 1 are shown in figures 1-4. From figure 1, it is clear that, right from the first lower half cycle of , adaptive Reduced order modified enhanced CNF (adaptive RCNF) controller causes the output to start tracking and rapidly follows changing peaks, thus minimizing overshoots and undershoots. Figure 2 show that tracking error(difference of desired (r) and

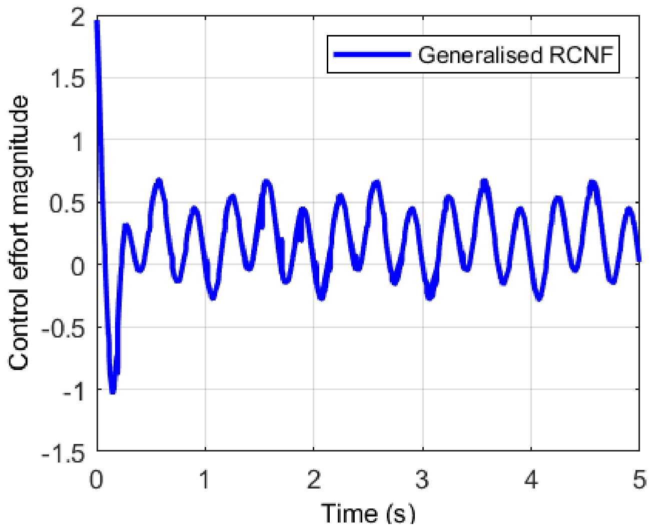


FIGURE 4. Control effort (Generalised RCNF)

TABLE 1. Comparing errors to check performance of controller

<table>
<tr><td>Example 1</td><td colspan="2">RMS error</td></tr>
<tr><td>GRCNF</td><td colspan="2">0.1753</td></tr>
<tr><td>RACNF</td><td colspan="2">0.1687</td></tr>
<tr><td>Example 2</td><td>RMS error
X axis (m)</td><td>RMS error
Y axis (m)</td></tr>
<tr><td>GRCNF</td><td>0.0127</td><td>0.0023</td></tr>
<tr><td>RACNF</td><td>0.0117</td><td>0.0008</td></tr>
</table>

controlled output (h)) converges towards zero a little faster in response to proposed controller. Error parameter is a standard performance evaluation parameter [6]. Further the error quickly settles near zero in 0.5 s, whereas it takes about 3 s for the same with adaptive RCNF and generalized RCNF controller respectively. The settling time can be considered in terms of the tracking error when it reaches zero. At this zero-error, the output equals desired output. So, the error reaches zero in 0.2367 s for proposed controller, whereas it takes 0.2525 s for generalized RCNF. Figure 3 and figure 4 shows controller outputs for adaptive and generalized RCNF. From Table 1, rms errors are compared. It is clear that proposed controller has improved performance in terms of this rms error criterion. The settling times are given in Table 2. Finally, from Table 3 the peak overshoot is also compared. So, in overall comparisons, proposed controller has improved performance in reducing rms errors, improving settling time and minimizing peak overshoot.

TABLE 2. Comparing settling times

Example 1	Example 1-Settling time (s)	
GRCNF	0.2525	
RACNF	0.2367	
Example 2	X axis Settling time (s)	Y axis Settling time (s)
GRCNF	0.4000	0.5500
RACNF	0.2541	0.5000

TABLE 3. Comparing peak overshoots

Example 1	Example 1-Peak overshoot (m)	
GRCNF	-0.0470	
RACNF	-0.0450	
Example 2	X axis Peak overshoot (m)	Y axis Peak overshoot (m)
GRCNF	7.5x10^{-3}	-14x10^{-3}
RACNF	11x10^{-3}	-6x10^{-3}

5. Illustrative Example 2

Now a problem of XY table trajectory tracking is considered. The X axis and Y axis models are taken from [2] as it is related with 2D trajectory design related. The maximum travel was 0.25 m in both directions. Here the associated control inputs represent electric current to the brush type dc servo-motor. The output displacement is in meters with system equations:

$$\Sigma_x : \begin{cases} \dot{x}_x = \begin{bmatrix} 0 & 1 \\ 0 & -2.825 \end{bmatrix} x_x + \begin{bmatrix} 0 \\ 8.034 \end{bmatrix} sat(u_x) \\ h_x = \begin{bmatrix} 1 & 0 \end{bmatrix} x_x \end{cases} \tag{5.1}$$

$$\Sigma_y : \begin{cases} \dot{x}_y = \begin{bmatrix} 0 & 1 \\ 0 & -3.226 \end{bmatrix} x_x + \begin{bmatrix} 0 \\ 6.774 \end{bmatrix} sat(u_y) \\ h_y = \begin{bmatrix} 1 & 0 \end{bmatrix} x_y \end{cases} \tag{5.2}$$

$$\Sigma_{rx} : \begin{cases} \dot{x}_{ex} = \begin{bmatrix} 0 & 1 \\ -w_1{}^2 & 0 \end{bmatrix} x_{ex} \\ x_{ex}(0) \begin{pmatrix} a_1 \sin(\phi) \\ a_1 w_1 \cos(\phi) \end{pmatrix} \\ r_x = \begin{bmatrix} 1 & 0 \end{bmatrix} x_{ex} \end{cases} \tag{5.3}$$

with ϕ a parameter used to create initial condition and desired signals

$$r_x(t) = 0.1\cos(0.4\pi t) \text{ and } r_y(t) = 0.1\sin(0.4\pi t) \tag{5.4}$$

5.1. Controller for X axis. Note that integrator $\dot{x}_{ix} = h_x - r_x$ is augmented in 4.5. Following are various matrices used

$$Q_x = diag\left\{0.2000 \quad 40.0000 \quad 0.0600\right\},$$
$$R_x = 0.0700 \text{ and}$$
$$P_x = \begin{bmatrix} 2.8443 & 0.2247 & 0.0061 \\ 0.2247 & 3.1866 & 0.0866 \\ 0.0061 & 0.0866 & 0.0050 \end{bmatrix}$$

The state feedback gain matrix for X axis is

$$F_x = -\begin{bmatrix} 1.6903 & 24.0906 & 2.2899 \end{bmatrix}$$

Next another feedback gain matrix (equivalent to F_e) is

$$F_{ex} = -\begin{bmatrix} 0.0448 & 0.2966 & 0.3516 \end{bmatrix}$$

The nonlinear function adaptive law for X axis becomes

$$\dot{\rho_x} = 0.0001 \times \|\widetilde{x}_{xr}\| \, \rho_x + \left(\rho_x{}^4\right) \times \widetilde{x}_{xr}^T P_x \bar{B}_x \bar{B}_x^T P_x \widetilde{x}_{xr} \tag{5.5}$$

with

$$\widetilde{x}_{xr} = \left[\begin{pmatrix} x_{ix} \\ h_x \\ x_{cx+12.175h_x} \end{pmatrix} - \begin{pmatrix} 0 \\ x_{ex} \end{pmatrix}\right] \text{ and } \rho_x(0) = -1.5000$$

With the constants $a_x = 11.4400$, $b_x = 0.0001$

$$K_{11x} = \gamma_1 \left(1 - \gamma_2 e^{-a_x t}\right), \gamma_1 = 0.8000, K_{22x} = e^{-b_x t} \tag{5.6}$$

Let the term equivalent to $\bar{B}P$ in 4.4 be represented by

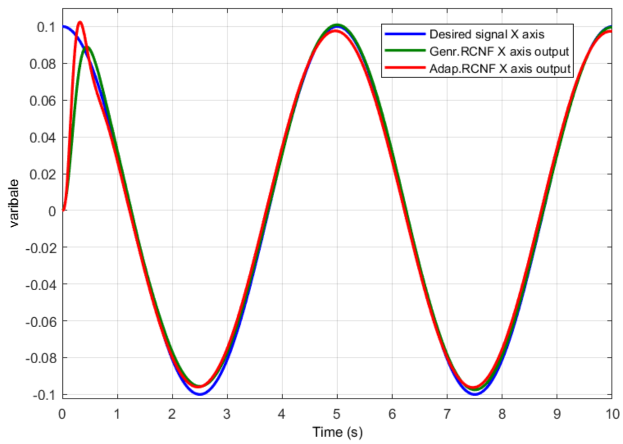
$$\begin{aligned} F_{nx} = \bar{B}_x P_x &= \begin{bmatrix} 0.0490 & 0.6959 & 0.0399 \end{bmatrix} \\ v_x = \left(K_{11x} \times F_x + K_{22x} \times \rho_y \times \begin{bmatrix} 0.0490 & 0.6959 & 0.0399 \end{bmatrix}\right) &\times (\widetilde{x}_{xr}) + u_{ex} \end{aligned} \tag{5.7}$$

5.2. Controller for Y axis.

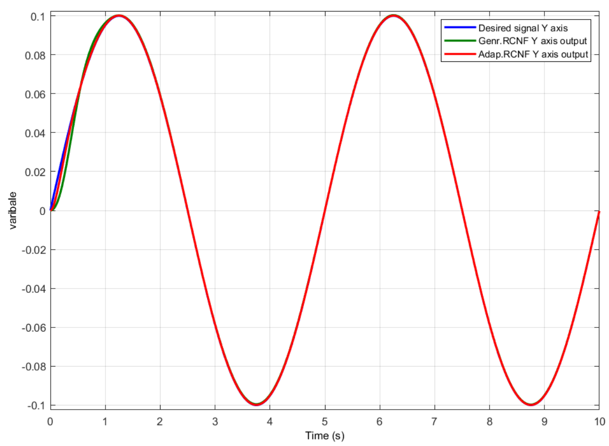
$$\Sigma_{ry} : \begin{cases} \dot{x}_{ey} = \begin{bmatrix} 0 & 1 \\ -w_1{}^2 & 0 \end{bmatrix} x_{ey} \\ x_{ey}(0) = \left(\begin{smallmatrix} 0 \\ a_1 w_1 \end{smallmatrix}\right) \\ r_y \begin{bmatrix} 1 & 0 \end{bmatrix} x_{ey} \end{cases} \tag{5.8}$$

$$u_{ey} = \begin{bmatrix} \frac{-w_1{}^2}{6.7740} & 0.4762 \end{bmatrix} x_{ey} \tag{5.9}$$

$$Q_y = diag\left\{0.3000 \quad 56.0000 \quad 0.0800\right\}, R_y = 0.5000 \text{ and}$$
$$P_y = \begin{bmatrix} 4.1383 & 0.5429 & 0.0286 \\ 0.5429 & 7.4526 & 0.3930 \\ 0.0286 & 0.3930 & 0.0339 \end{bmatrix}$$
$$F_y = -\begin{bmatrix} 0.7746 & 10.7201 & 1.4084 \end{bmatrix} \text{ and } F_{ey} = -\begin{bmatrix} 0.0531 & 0.2331 & 0.4762 \end{bmatrix}$$
$$\dot{\rho_y} = 0.0001 \times \|\widetilde{x}_{yr}\| \, \rho_y + \left(\rho_y{}^4\right) \times \widetilde{x}_{yr}^T P_y \bar{B}_y \bar{B}_y^T P_y \widetilde{x}_{yr} \tag{5.10}$$



(A) X axis Tracking



(B) Y axis Tracking

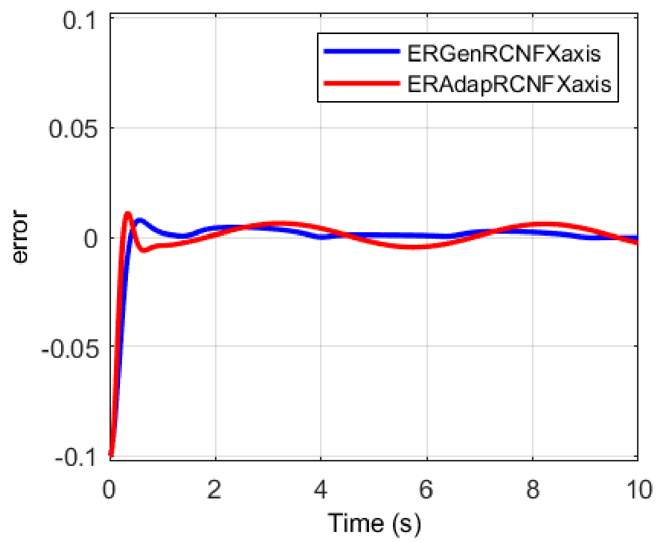
FIGURE 5. Comparison of generalized RCNF and Adaptive RCNF controller responses to 2-D desired signals

with

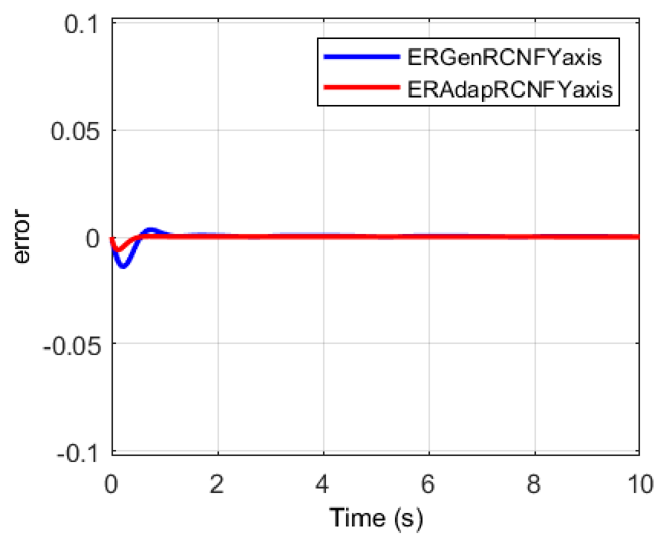
$$\widetilde{x}_{yr} = \left[\begin{pmatrix} x_{iy} \\ h_y \\ x_{cy+11.774h_y} \end{pmatrix} - \begin{pmatrix} 0 \\ x_{ey} \end{pmatrix} \right] \text{ and } \rho_y(0) = -0.1000 \tag{5.11}$$

$$v_y = \left(F_y + \rho_y \times \begin{bmatrix} 0.1936 & 2.6626 & 0.2297 \end{bmatrix}\right) \times (\widetilde{x}_{yr}) + u_{ey}$$

Remark: The control effort in both the simulation examples are bounded and there is no switching in the control effort. Major difficulties of the work include-



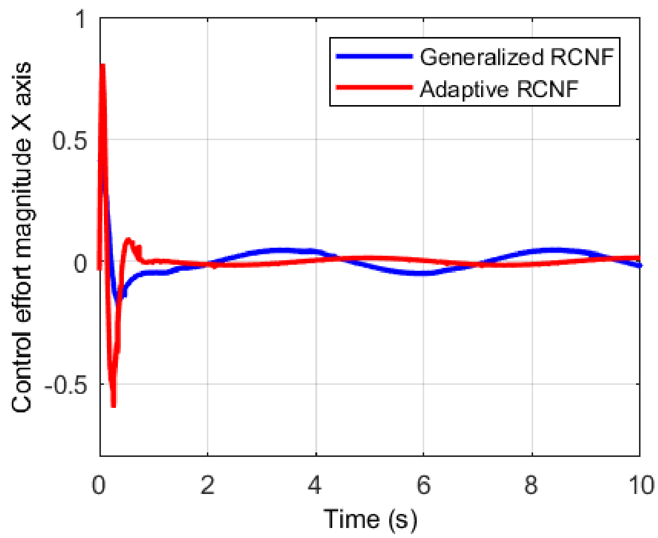
(A) X axis Tracking Errors



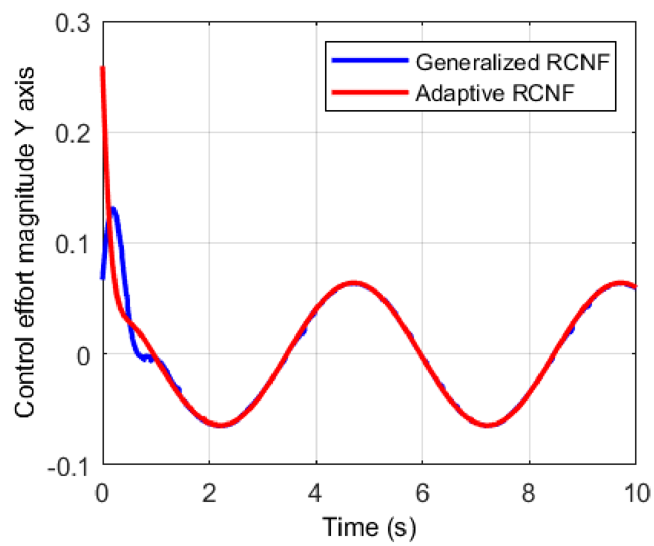
(B) Y axis Tracking Errors

FIGURE 6. X axis and Y axis tracking errors clearly shows the adaptive RCNF fast settling time responses

tuning of the control that will result in reduction in control effort, selection of state feedback gain matrix as they are specially concerned with frequency response of the system, disturbance value selection (it may be positive or it may be negative



(A) X axis Tracking Errors



(B) Y axis Tracking Errors

FIGURE 7. X axis and Y axis tracking errors clearly shows the adaptive RCNF fast settling time responses

or combination), selection of tracking input desired signal. Figures 5-1.9 are related to example 2. In figure 5, adaptive RCNF X axis output leads after about 0.25 s and gets tuned to the sinusoidal desired signal. Y axis output also quickly responds in a similar manner. The tracking error convergence in figure 1.6 again

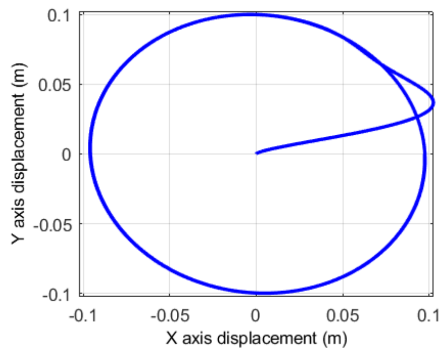


Figure 8. Circle formed by adaptive RCNF controller

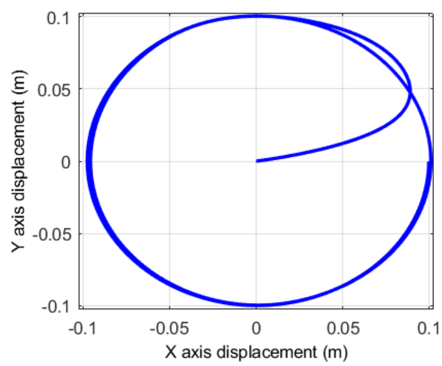


Figure 9. Circle formed by generalized RCNF controller

proves the faster response of the proposed controller (legend ERGenRCNFXaxis means tracking error for Generalized RCNF for X axis and legend ERAdapRCNFXaxis means tracking error for Adaptive RCNF for X axis; in legend: Y means Y axis). Figure 1.7 shows the control efforts associated to X and Y axis. Figure 1.8 and 1.9 compares circles drawn by adaptive RCNF controller and generalized CNF controller respectively. The circle drawn by the proposed controller has comparable response. The settling time for X axis response comes as 0.2541 s and 0.4000 s, whereas for Y axis response, the settling time is 0.5000 s and 0.55 s for adaptive RCNF and generalized CNF controller respectively. RMS errors which are compared in 1 are calculated with 'rms' function from MATLAB. In example 2, the rms error with proposed controller is slightly less for X axis and considerably less than generalized CNF for Y axis responses. Peak overshoots are slightly more however well within the acceptable limits. Hence in example 2 also the proposed controller has performed well in tracking of desired signals to a much more scalable (or can be called as 'reliable') level in all three aspects.

6. Conclusions

Alternative solutions to research problems are doors to novel work areas. In that direction a reduced order observer based modified enhanced adaptive RCNF controller technique is presented for general target references tracking. The proposed controller comprises of an adaptive nonlinear function along with an exponentially scaled control law. Under the application of this control law, the closed loop system remains semi-globally asymptotically stable. The simulation results of the proposed law are compared with generalized enhanced CNF control law for a set of two numerical examples. From the simulation results it is proven that proposed controller has performed effectively well in terms of improvement in settling time and reduction in rms error without large overshoot. The improvements in results show that there is scope of identification of set of problems in which a larger set of impartments are definitely possible. Hence this opens new opportunities for further research. Future scope involves making controller robust against variations in disturbances w.

Acknowledgment. (Authors are grateful to the reviewers' comments, suggestions and directions.)

References

1. Peng K., Chen B., Cheng G. and Lee T.H.: Modeling and compensation of nonlinearities and friction in a micro hard disk drive servo system with nonlinear feedback control, *IEEE Transactions on Control System Technology* **13-5** (September 2005) 708–720.
2. Cheng G., Peng K., Chen B. and Lee T.H.,: Improving transient performance in tracking general references using composite nonlinear feedback control and its application to high–speed XY–table positioning mechanism, *IEEE Transactions on Industrial Electronics* **54-2** (April 2007) 1039–1051.
3. Venkataramanan V., Peng K., Chen B., Lee T. H.: Discrete–time Composite Nonlinear Feedback control with an application in design of a hard disk drive servo system, *IEEE Transactions on Control System Technology* **11-1** (January 2003) 16–23.
4. Gopal, M.: *Digital Control and State Variable Methods Conventional and Intelligent Control Systems*, McGraw Hill, 2009.
5. Peng W. and Su J.,: Tracking controller for robot manipulators via composite nonlinear feedback law, *Journal of Systems Engineering and Electronics* **20-1** (February 2009) 129–135.
6. Mohammed Fahim Hassan and Hesham Talaat Attia: Stream Quality Control via a Constrained Nonlinear Time-delay Model, *Journal of Mathematical Control Science and Applications (JMCSA)* **3-2** (July-September 2017) 71–82.
7. Filippo Cacace, Mattia Mattioni, Salvatore Monaco, Lorenzo Ricciardi Celsi: Topology induced containment for general linear systems on weakly connected digraphs, *Automatica* **131** (2021) 109734.

Email address: kulkarni.abhijit@kbtcoe.org