

## ADIABATIC FILTRATION OF AN IDEAL GAS IN A HOMOGENEOUS AND ISOTROPIC POROUS MEDIUM

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ABSTRACT. Stationary adiabatic and isentropic filtration of ideal and van der Waals gases in porous media is considered. The explicit formulae for the corresponding boundary problems are given and critical phenomena in gas filtration are studied.

### 1. Introduction

In this paper we study the gas filtration through a porous medium. The first valuable results in this area were obtained by Leibenson L.S. ([6]) and Muskat, M. ([7]). They have proposed a generalization of the Navier-Stokes equations for this case where they substituted the Newton law by the Darcy one.

We consider 3-dimensional stationary gas filtration for the case of ideal and van der Waals gases. Condition for stationary not only simplifies the mathematical model but has practical reason because development of such processes take a long time and also for control of them we use the *cascade method*.

The case of ideal gases is closed to the Leibenson method but we've included more detail using of the thermodynamical properties of the medium. The case of van der Waals gases was considered to show critical phenomena in filtration processes and especially phase transitions.

In both cases we presented the explicit formulae for solutions of the corresponding Dirichlet problem.

### 2. Basic equations

Stationary gas filtration in homogeneous porous 3 dimensional media is described by the following system of differential equations [6, 7]:

- conservation of mass

$$\operatorname{div}(\rho \mathbf{U}) = 0, \quad (2.1)$$

- conservation of momentum, or Darcy law

$$\mathbf{U} = -\frac{k}{\mu} \nabla p, \quad (2.2)$$

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where  $\rho$  is the density of a gas,  $p$  – pressure,  $\mu$  – dynamic viscosity,  $k$  – permeability,  $\mathbf{U}$  – vector field of rate of the filtration, and  $\nabla_p$  – the gradient of pressure.

In addition to these equations, we'll assume that coefficients  $k, \mu$  are constants, the filtration is either adiabatic, i.e.

$$\mathbf{U}(\sigma) = 0, \quad (2.3)$$

or isenthalpic, i.e.

$$\mathbf{U}(\eta) = 0, \quad (2.4)$$

where  $\sigma$  is a specific entropy,  $\eta$  is a specific enthalpy and by  $\mathbf{U}(f)$  we'll denote the derivative of a function  $f$  along vector field  $\mathbf{U}$ .

It is worth to remark that the last conditions in the cases of sources or sinks lead us to local constancy entropy or enthalpy around them. Later on we'll extend this observation and shall propose some kind of "ergodicity hypothesis". Namely, we'll assume that entropy, or respectively enthalpy, are constant in the entire domain.

### 3. Filtration of ideal gases

**3.1. State equations.** The thermodynamic variables  $(p, \rho, \varepsilon, T, \sigma)$ , where  $\varepsilon$  is a specific inner energy and  $T$  is the temperature, should satisfy the state equations, which, for the case of ideal gases have the form:

- Claperon-Mendeleev equation

$$p = R\rho T, \quad (3.1)$$

and

- the inner energy equation

$$\varepsilon = \frac{n}{2}RT, \quad (3.2)$$

where  $n$  – degree of freedom and  $R$  is the universal gas constant.

It follows from these equations (see, for example, [2]) that

$$\sigma = R \ln \left( \frac{\varepsilon^{n/2}}{\rho} \right) + \text{const}, \quad (3.3)$$

and

$$\eta = \left( 1 + \frac{n}{2} \right) \varepsilon. \quad (3.4)$$

Due to these relations (3.3) adiabatic equation (2.3) takes the form

$$\frac{n+2}{n} \frac{\nabla_p(p)}{p} = \frac{\nabla_p(\rho)}{\rho}, \quad (3.5)$$

and enthalpic equation

$$\frac{\nabla_p(p)}{p} = \frac{\nabla_p(\rho)}{\rho}. \quad (3.6)$$

We'll write down both these equations in unified form

$$\lambda \frac{\nabla_p(p)}{p} = \frac{\nabla_p(\rho)}{\rho}, \quad (3.7)$$

where  $\lambda = \frac{n}{n+2}$ , for the adiabatic filtration and  $\lambda = 1$  for the isenthalpic one.

Remark that this equation shows us that the function

$$H = p^\lambda \rho^{-1}$$

is the first integral for the gradient vector field  $\nabla_p$ .

Going back to the mass conservation equation we get

$$\operatorname{div}(\rho \mathbf{U}) = 0 \implies \operatorname{div}(\rho \nabla_p) = 0 \implies \rho \Delta p + \nabla_p(\rho) = 0,$$

or

$$\Delta p + \frac{\nabla_p(\rho)}{\rho} = 0,$$

where  $\Delta$  is the Laplace operator.

Finally, by using the unified equation, we get for ideal gases the basic equation in the form

$$\Delta p + \lambda \frac{\nabla_p(p)}{p} = 0. \quad (3.8)$$

Straightforward computations show that the following relation is valid.

**Lemma 3.1.** *For any smooth function  $f(p)$  one has*

$$\Delta(f(p)) = f'(p) \Delta(p) + f''(p) \nabla_p(p).$$

**Corollary 3.2.** *Differential equation (3.8) is equivalent to the following*

$$\Delta(q) = 0, \quad (3.9)$$

where  $q = p^{\lambda+1}$ .

**3.2. The model of source or sink for ideal gases.** Consider rotation invariant solutions of equation (3.8) or (3.9). As we know function  $q = \frac{1}{4\pi r}$ , where  $r$  is the distance from a source, is the fundamental solution for the Laplace equation:  $\Delta(q) = -\delta_0$ .

Thus, let

$$p = \left( \frac{1}{4\pi r} \right)^{\frac{1}{1+\lambda}}, \quad (3.10)$$

be the corresponding pressure function.

Moreover, as we have seen, function  $H = p^\lambda \rho^{-1}$  is the 1-st integral of the gradient vector field  $\nabla_p$ .

Therefore, in the case of a source, we have that  $H = c^{-1}$  in a neighborhood of the source, where  $c$  is a constant.

Finally, we get

$$\rho = cp^\lambda$$

in a neighborhood of the source, or

$$\rho = c \left( \frac{1}{4\pi r} \right)^{\frac{\lambda}{1+\lambda}}. \quad (3.11)$$

Then, for the vector field  $\rho \nabla_p$  we get

$$\rho \nabla_p = \frac{c}{\lambda+1} \nabla_q,$$

and therefore

$$\operatorname{div} \left( \frac{k}{\mu} \rho \nabla p \right) = - \frac{k c}{(\lambda + 1) \mu} \delta_0, \quad (3.12)$$

for “source type” solution (3.10).

Summarizing, we get the following description of all thermodynamic variables in a neighborhood of a source.

**Theorem 3.3.** *In a neighborhood of a source the following functions*

$$p = \left( \frac{\gamma}{4\pi r} \right)^{\frac{1}{1+\lambda}}, \quad \rho = c \left( \frac{\gamma}{4\pi r} \right)^{\frac{\lambda}{1+\lambda}}, \quad T = \frac{1}{cR} \left( \frac{\gamma}{4\pi r} \right)^{\frac{1-\lambda}{1+\lambda}}$$

give us a model of source with intensity  $I$ , if

$$\gamma = \frac{(1 + \lambda) \mu}{k c} I.$$

*Proof.* It is easy to check that the scaling  $q \rightarrow \gamma q$ , gives us the following transformation

$$p \rightarrow \gamma^{\frac{1}{1+\lambda}} p, \quad \rho \rightarrow \gamma^{\frac{\lambda}{1+\lambda}},$$

and

$$\operatorname{div} (\rho \nabla p) \rightarrow \gamma \operatorname{div} (\rho \nabla p).$$

□

**3.3. The Dirichlet boundary problem for filtration of ideal gases.** Consider an open and connected domain  $\mathbf{D} \subset \mathbb{R}^3$  with a smooth boundary  $\partial\mathbf{D}$ , equipped with a set  $\mathbf{A} = \{a_i, i = 1, \dots, N\} \subset \mathbf{D}$  of points, with given intensities  $I_i$ .

We are looking for a smooth solution  $p$  of system (2.1,2.2) in domain  $\mathbf{D} \setminus \mathbf{A}$  for an ideal gas, having given intensities at point  $a_i$  and given values of pressure and temperature on the boundary:

$$p|_{\partial\mathbf{D}} = p_0, \quad T|_{\partial\mathbf{D}} = T_0. \quad (3.13)$$

First of all we define coefficients  $\gamma_i$  as follows

$$\gamma_i = \frac{(1 + \lambda) \mu}{k c} I_i, \quad (3.14)$$

where assumed that  $c$  is a common constant for all sources  $a_i$ , i.e. the filtration process is *adiabatic or isentropic in the strong sense*: specific entropy and specific enthalpy are constants in the entire domain  $\mathbf{D}$ , then on the boundary we have

$$c = \frac{p_0^{1-\lambda}}{RT_0}. \quad (3.15)$$

Now, define a function  $q$  as  $q(x) = q_0(x) + q_1(x)$ , where

$$q_0(x) = \frac{1}{4\pi} \sum_{i=1}^N \frac{\gamma_i}{|x - a_i|},$$

and  $q_1(x)$  is a smooth in domain  $\mathbf{D}$  harmonic function, having the following boundary values

$$q_1|_{\partial\mathbf{D}} = p_0^{1+\lambda} - q_0|_{\partial\mathbf{D}}. \quad (3.16)$$

Summarizing we get the following theorem.

**Theorem 3.4.** *A solution  $(p, T)$  of system differential equations (2.1, 2.2) for strong adiabatic or isentropic filtration of ideal gases in the domain  $\mathbf{D}$ , which is smooth in the domain except of points in  $\mathbf{A}$ , having at these points given intensities  $I_i$  and having given values  $p_0, T_0$  on the boundary of  $\mathbf{D}$ , has the form*

$$\begin{aligned} p &= \left( \frac{1}{4\pi} \sum_{i=1}^N \frac{\gamma_i}{|x - a_i|} + q_1 \right)^{\frac{1}{1+\lambda}}, \\ T &= \frac{1}{cR} \left( \frac{1}{4\pi} \sum_{i=1}^N \frac{\gamma_i}{|x - a_i|} + q_1 \right)^{\frac{1-\lambda}{1+\lambda}}, \end{aligned} \quad (3.17)$$

where  $q_1$  is a harmonic function in domain  $\mathbf{D}$  with boundary values (3.16), and constants  $\gamma_i, c$  given by relations (3.15), (3.14).

#### 4. Filtration of van der Waals gases

**4.1. State equations.** For the case van der Waals gases we have the following state equations (see, for example, [2]):

$$\begin{aligned} (p + a\rho^2)(\rho^{-1} - b) - RT &= 0, \\ \varepsilon - \frac{n}{2}RT + a\rho &= 0, \\ \sigma - R \ln \left( (\rho^{-1} - b)(\varepsilon + a\rho)^{n/2} \right) &= 0, \end{aligned}$$

where constants  $a, b$  are positive, constant  $a$  takes in account intermolecular forces and constant  $b$ — molecular volume, as above  $n$  is the degree of freedom.

This model describes gas or liquid (depending on the density) in domain where

$$p > a\rho^2(1 - 2b\rho), \quad b\rho < 1.$$

In domain where

$$p < a\rho^2(1 - 2b\rho), \quad b\rho < 1, \quad (4.1)$$

we have intermediate state or condensation.

The curve

$$p = a\rho^2(1 - 2b\rho), \quad b\rho < 1,$$

consist of the states where the phase transition occurs (see, [2] and [4], for more details).

The following contact transformation

$$(p, \rho, \varepsilon, T, \sigma) \rightarrow \left( \frac{p}{p_c}, \frac{\rho}{\rho_c}, \frac{\varepsilon}{\varepsilon_c}, \frac{T}{T_c}, \frac{\sigma}{\sigma_c} \right), \quad (4.2)$$

where

$$p_c = \frac{a}{27b^2}, \quad \rho_c = \frac{1}{3b}, \quad \varepsilon_c = \frac{a}{9b}, \quad T_c = \frac{8a}{27bR}, \quad \sigma_c = \frac{3R}{8}$$

are the critical values of the thermodynamic variables ([2]), maps the state equations to the reduced form:

$$\begin{aligned} 3p - (p + 8T)\rho + 9\rho^2 - 3\rho^3 &= 0, \\ \varepsilon - \frac{4n}{3}T + 3\rho &= 0, \\ \sigma - \frac{4n}{3}\ln(\varepsilon + 3\rho) - \frac{8}{3}\ln\left(\frac{3}{\rho} - 1\right) - \sigma_0 &= 0. \end{aligned} \quad (4.3)$$

Assuming that the entropy is a constant we get from equations (4.3) that

$$p = 3\rho^2 - c\left(\frac{3}{\rho} - 1\right)^{-\alpha}, \quad (4.4)$$

where

$$\alpha = 1 + \frac{2}{n}, \quad c = \frac{6}{n} \exp\left(\frac{3\sigma}{4}\right).$$

**Theorem 4.1.** *Basic equations (2.1,2.2) for adiabatic filtration of van der Waals gases are equivalent to equation*

$$\Delta(Q(\rho)) = 0,$$

where

$$Q(\rho) = 2\rho^3 + 3\alpha c\left(\frac{3}{\rho} - 1\right)^{-\alpha} \text{LerchPhi}\left(1 - \frac{3}{\rho}, 1, -\alpha\right) \quad (4.5)$$

is the Lerch function (see,[1]).and

The pressure and temperature functions given by (4.4) and

$$T = 8c\left(\frac{3}{\rho} - 1\right)^{1-\alpha}. \quad (4.6)$$

*Proof.* Vector field  $\rho\nabla_p$  is a gradient vector field  $\nabla_Q$ , where  $Q = \int \rho \frac{\partial p}{\partial \rho} d\rho$ , and  $p = p(\rho)$  given by equation (4.4).  $\square$

**4.2. Inversion of  $Q$ .** Function  $Q$  is defined by (4.5) and we found it by the requirement

$$\frac{dQ}{d\rho} = \rho \frac{dp}{d\rho},$$

where  $p$  satisfies (4.4).

Therefore, the condition

$$\frac{dQ}{d\rho} = 0 \iff \frac{dp}{d\rho} = 0,$$

can be written as follows

$$\rho^{2-\alpha} (3 - \rho)^{1+\alpha} = \frac{\alpha c}{2}.$$

Let us consider the function

$$w(\rho) = \rho^{2-\alpha} (3 - \rho)^{1+\alpha}$$

on the interval  $[0, 3]$ .

We have  $w(0) = w(3) = 0$  and  $w \geq 0$  on the interval.

Moreover,  $w' = 0$  in this interval at point  $\rho = 2 - \alpha$  only. Therefore,

$$\max_{\rho \in [0,3]} w(\rho) = w(2 - \alpha) = (2 - \alpha)^{2-\alpha} (1 + \alpha)^{1+\alpha},$$

and  $Q'(\rho) \neq 0$  on interval  $(0, 3)$  if

$$c > \frac{2}{\alpha} (2 - \alpha)^{2-\alpha} (1 + \alpha)^{1+\alpha}, \quad (4.7)$$

and has two roots in the opposites case.

**Proposition 4.2.** *The function  $Q$  is invertible if the specific entropy constant  $\sigma$  satisfies inequality (4.7), where*

$$c = \frac{6}{n} \exp\left(\frac{3\sigma}{4}\right).$$

**4.3. Phase transitions.** The phase transitions, condensation and gas-liquid areas could be found from the above theorem and formulae (4.1).

**Theorem 4.3.** *In adiabatic filtration of the van der Waals gases the condensation area and phase transition given by the following equations:*

- condensation area

$$\left(\frac{3}{\rho} - 1\right)^{\alpha+1} \rho^3 > 32c, \quad \rho < 3, \quad (4.8)$$

and

- curve of phase transitions

$$\left(\frac{3}{\rho} - 1\right)^{\alpha+1} \rho^3 - 32c = 0, \quad \rho < 3, \quad (4.9)$$

where  $Q(\rho)$  is a harmonic function.

It is easy to check that the function on left hand side of (4.8), similar to the above discussion, takes its maximum value at the point

$$\rho_c = 2 - \alpha,$$

and the corresponding value of constant  $c$  equals

$$c_c = \frac{1}{32} \left(2 + \frac{2}{n}\right)^{2+2/n} \left(1 - \frac{2}{n}\right)^{1-2/n}.$$

Therefore, if  $c > c_c$  we have no phase transitions, and if  $c < c_c$  the phase transitions occur at roots  $\rho_1, \rho_2$  of equation (4.9), where

$$\rho_1 < 1 - \frac{2}{n} < \rho_2.$$

**Example 4.4.** For monatomic gases we have  $n = 3$ , and therefore

$$c_c = 0,296\dots, \quad \rho_c = 0,33\dots$$

and

$$\rho_1 = 0,823\dots, \quad \rho_2 = 2,987\dots,$$

for  $c = 1/32$ .

**4.4. The model of source or sink for van der Waals gases.** Similar to the case of ideal gases we have  $\rho \nabla p = \nabla_{Q(\rho)}$ , for the case of van der Waals gases. Therefore, the source at a point  $x_0$  of given intensity  $I$  and fixed entropy  $\sigma$  has the form

$$\begin{aligned} \rho &= Q^{-1} \left( \frac{I}{4\pi |x - x_0|} \right), \\ T &= 8c \left( \frac{3}{\rho} - 1 \right)^{1-\alpha}, \\ p &= 3\rho^2 - c \left( \frac{3}{\rho} - 1 \right)^{-\alpha}, \end{aligned} \tag{4.10}$$

where as above

$$\alpha = 1 + \frac{2}{n}, \quad c = \frac{6}{n} \exp \left( \frac{3\sigma}{4} \right),$$

and  $Q$  given by (4.5).

Remark that  $Q^{-1}$  exists if condition (4.7) holds, in other case we need an additional separation of roots.

Also conditions (4.8) give us the areas of gas, liquid and condensation phases.

**4.5. The Dirichlet boundary problem for filtration of van der Waals gases.** As above, let's consider an open and connected domain  $\mathbf{D} \subset \mathbb{R}^3$  with a smooth boundary  $\partial\mathbf{D}$ , equipped with a set  $\mathbf{A} = \{a_i, i = 1, \dots, N\} \subset \mathbf{D}$  of points, with given intensities  $I_i$ .

We are looking for a smooth solution  $p$  of system (2.1,2.2) in domain  $\mathbf{D} \setminus \mathbf{A}$  for van der Waals gases, having given intensities at point  $a_i$ , given entropy constant  $c$ , and given values of pressure and temperature on the boundary:

$$\rho|_{\partial\mathbf{D}} = \rho_0.$$

Now, define functions  $Z_0$  and  $Z_1$  in such a way ,that

$$Z_0(x) = \frac{1}{4\pi} \sum_{i=1}^N \frac{I_i}{|x - a_i|},$$

and  $Z_1(x)$  is a smooth in domain  $\mathbf{D}$  harmonic function, having the following boundary values

$$Z_1|_{\partial\mathbf{D}} = Q(\rho_0) - Z_0|_{\partial\mathbf{D}}. \tag{4.11}$$

Then function  $Q(\rho) = Z_0 + Z_1$  gives us the solution of the Dirichlet boundary problem.

**Theorem 4.5.** *A solution  $(\rho, p, T)$  of system differential equations (2.1,2.2) for adiabatic filtration of van der Waals gases in the domain  $\mathbf{D}$ , which is smooth in the domain except of points in  $\mathbf{A}$ , having at these points given intensities  $I_i$  and the entropy constant  $c$  and having given value  $\rho_0$  on the boundary of  $\mathbf{D}$ , has the form*

$$\rho = Q^{-1} \left( \frac{1}{4\pi} \sum_{i=1}^N \frac{I_i}{|x - a_i|} + Z_1 \right),$$



where  $Z_1$  is a harmonic function in domain  $\mathbf{D}$  with boundary values (4.11). The corresponding temperature  $T$  and pressure  $p$  are given by equations (4.6) and (4.4).

*Remark 4.6.* In order to have well defined function  $Q^{-1}$  the constant  $c$  should satisfy condition (4.7) .

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