Global and Stochastic Analysis
Vol. 6, No. 2, December (2019), 79-103

CLASSICAL SOLUTION TO A MULTIDIMENSIONAL STOCHASTIC BURGERS EQUATION VIA FORWARD-BACKWARD SDES

ALBERTO OHASHI AND EVELINA SHAMAROVA

Abstract. In this paper, we address the problem of existence and uniqueness of a global classical solution to a multidimensional stochastic Burgers equation without gradient-type assumptions on the force or the initial condition. The equation is first transformed to a random PDE, and then solved via the associated forward-backward SDE. Additionally, we obtain a new a priori gradient estimate valid for a large class of second-order quasilinear parabolic PDEs which becomes an important tool in our approach. Also, we study the stochastic Burgers equation in the vanishing viscosity limit.

1. Introduction

In this article, we obtain the existence and uniqueness of a global classical solution to the multidimensional stochastic Burgers equation

$$y(t,x) = h(x) + \int_0^t \big[\nu\Delta y(s,x) - (y,\nabla)y(s,x) + f(s,x,y)\big]ds + \eta(t,x) \tag{1.1}$$

on $[0,T]\times\mathbb{R}^n$, where h is a random initial data, f is a deterministic function representing force, and $\eta(t,x)$ is a noise smooth in x and rough in time. In particular, $\eta(t,x)$ can be a stochastic integral $\int_0^t g(s,x)dB_s$, assumed to be defined for each x, but this choice does not affect our analysis. Importantly, we do not assume that any of the functions f, η, or h are of gradient form.

In the past two decades many works have been dedicated to the problem of Burgers turbulence (see, e.g., [1, 3, 4, 6, 8, 11, 13, 22, 23]), that is, the study of solutions to a Burgers equation with a random initial condition or force. In the extensive survey on Burgers turbulence [2], Bec and Khanin refer the multidimensional extension of a stochastic Burgers equation in the non-potential case as an important open question. The authors illustrate that when the forcing and the initial data are potential (i.e., represented as gradients of other functions), the potential character of the velocity field is conserved by the dynamics, so the situation carry many similarities with the one-dimensional case [2]. Further, the authors in

---

Date: Date of Submission March 25, 2019; Date of Acceptance May 31, 2019, Communicated by Yuri E. Gliklikh.

1991 *Mathematics Subject Classification.* Primary 60H15; Secondary 35Q35, 35K59.

Key words and phrases. Stochastic Burgers equation, Forward-backward SDEs, Gradient estimate, Vanishing viscosity limit.

[2] explicitly pose the question of what happens when the potentiality assumption of the flow is dropped.

Our main motivation in studying the multidimensional viscous Burgers equation with smooth random forces is its application to the theory of hydrodynamical turbulence [2, 8, 24]. As such, equations of form (1.1) are frequently used as a model of randomly driven Navier-Stokes equations without pressure [7, 31].

In this work, we propose a method of obtaining a global classical solution to stochastic Burgers equation (1.1) based on a fixed point argument of the associated forward-backward SDE (FBSDE) and a gradient estimate. First, we transform (1.1) to a random PDE, and then introduce a sequence of stopping times making the noise globally bounded. This allows us to apply FBSDE techniques similar to the case of deterministic PDEs [15, 30], and also, to make use of our own result on a gradient estimate for PDEs by means of FBSDEs.

The interest in Burgers turbulence is motivated by its applications in cosmology [33], fluid dynamics [12], superconductors [5], etc. It is known that the Burgers equation arises as an asymptotic form of various nonlinear dissipative systems [2]. That is why a one-dimensional stochastic Burgers equation has been intensely studied over the last two decades in a variety of contexts and based on different techniques. The literature is vast, so we refer the reader to the series of works [6, 13, 14, 20], and references therein. The stochastic multidimensional potential case, i.e., when the force and the initial data are of the gradient form, has also been studied by some authors [1, 8, 10, 11, 25]. Since the potential Burgers equation can be reduced to a one-dimensional parabolic equation by a number of known approaches (see, e.g., [8, 10, 11]), the analysis is significantly simplified. We remark that in the present article, we consider the non-potential case for both, the random force and the initial condition, which does not allow us to apply any of the above techniques.

Further, we would like to mention article [9], where the authors prove the existence and uniqueness of a global strong solution to a non-potential multidimensional stochastic Burgers equation in the L_p-space with the number p bigger than the dimension of the equation. Although the stochastic Burgers equation in [9] has the form similar to (1.1), the approach of the aforementioned work completely differs from ours. Besides, from the hydrodynamical turbulence point of view, L_p-solutions do not appear suitable since they do not convey the meaning of the solution to (1.1) as the velocity of a fluid at a given point x in the space [29]. Also, our noise term is not assumed to take any specific form, unlike [9]. In fact, the choice of the forcing term $\eta(t,x)$ in physics literature is frequently made on the basis of the covariance of the form $\operatorname{cov}(\dot\eta^i(t,x),\dot\eta^j(t',x')) = \delta(t-t')\varphi_{ij}(x-x')$ (see, e.g., [7, 31]). However, the above relation is not satisfied by the stochastic-integral-type noise. Remark that in [9], the choice of the noise term as a stochastic integral plays a crucial role in the analysis. Another advantage of our method is the use of the associated FBSDE, which may allow the results of paper [17] on a forward-backward stochastic algorithm for PDEs to be applied to tackle equation (1.1) numerically.

Furthermore, we mention that in the deterministic case, the global existence and uniqueness of a classical solution to the multidimensional Burgers equation is known due to the results of Ladyzhenskaya et al [28], and follows as a particular case of a more general theory for systems of quasilinear parabolic PDEs. However, the results of [28] are not applicable to equation (1.1) since the noise is not differentiable in time.

As a byproduct of our approach, we obtain an a priori gradient estimate valid for a large class of quasilinear second order parabolic PDEs. Our bound is obtained exclusively by using the associated FBSDE. Previously, a gradient estimate by means of FBSDE techniques was obtained in [16]. However, the result of [16] cannot be applied to the present case. Indeed, in our work, the gradient estimate is used in the process of construction of the solution by glueing the solutions on short-time intervals, i.e., we deal with solutions defined on subintervals of $[0, T]$ but not on the entire interval. In this situation, the results of [16] do not guarantee that the gradient bound will be uniform over the length of the subinterval, while our result does guarantee that. Thus, our gradient estimate appears completely suitable for solving some class of PDEs by means of FBSDEs. Additionally, our approach to obtaining this bound is significantly simpler and shorter than in [16], although it is valid for a smaller class of PDEs.

Also, we remark that the classical book on quasilinear parabolic PDEs by Ladyzhenskaya et al [28] only provides an a priori gradient estimate for an initial-boundary value problem on a bounded domain.

Finally, we study the vanishing viscosity limit of equation (1.1). We investigate this problem only locally. Namely, we prove that on a small random time interval, there exists a unique classical solution to the inviscid stochastic Burgers equation and the solutions to viscous stochastic Burgers equations with the same force terms and the initial data converge to the inviscid solution uniformly in space and time. Note that even on a short time interval, many authors investigated the vanishing viscosity limit in hydrodynamics problems. As such, Ebin and Marsden [18] proved the convergence of local Sobolev-space-valued solutions of the Navier-Stokes equation to local solutions of the Euler equation. Golovkin [21] and Ladyzhenskaya [27] obtained the aforementioned convergence uniformly in space and time. Further, Ton [32] studied the local vanishing viscosity limit of a multidimensional deterministic Burgers equation in an L_2-space. Furthermore, Brzeźniak et al [9] proved that viscous solutions to a potential stochastic Burgers equation converge locally to an inviscid viscosity solution. It is known that even if the initial data and the force are smooth, a one-dimensional inviscid Burgers equation develops discontinuities (shocks) at a finite time, and, therefore, fails to have a global classical solution. Thus, one cannot expect a global uniform approximation of inviscid solutions by viscous. Finally, we remark that the inviscid multidimensional stochastic Burgers equations is also studied by means of the associated stochastic forward-backward system.

The organization of our paper is as follows. Section 2 is dedicated to the problem of existence and regularity of the solution to equation (1.1). In detail, in subsection 2.1 we discuss different forms of the noise that fit to our assumptions. Subsection 2.2 deals with the local existence, smoothness, and regularity of solutions to (1.1). Remark, that papers [15] and [30] seemingly deal with techniques similar to those in subsection 2.2. However, in [30], the FBSDEs under consideration are decoupled (unlike ours), while the assumptions of [15] (e.g., B.A2, Appendix B) are not satisfied by our FBSDE coefficients which fail to be Lipschitz together with their first and second order derivatives. Thus, we are not able to directly apply the existing results in this topic and have to perform the arguments under different assumptions. Further, subsection 2.3 deals with an a priori gradient estimate for some class of quasilinear parabolic PDEs. The estimate is uniform over subintervals of a given time interval. The authors are not aware if gradient estimates with the aforementioned property exist in the literature. In subsection 2.4, we obtain the main result of this work, which is existence and smoothness of solution to equation (1.1). Finally, section 3 is dedicated to the vanishing viscosity limit.

2. Existence and uniqueness of solution to equation (1.1)

In this section, we show that under assumptions (A1)–(A3) below, equation (1.1) possesses a unique global solution $y(t,x)$ which is C^2-smooth in x and continuous in t.

2.1. Assumptions and choice of the noise. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space satisfying the usual conditions.

Assume the following:

(A1) $f(t,x,y)$ is an $\mathbb{R}^n$-valued deterministic function of class $\mathrm{C}_b^{0,2}([0,T]\times\mathbb{R}^{2n})$.

(A2) $\eta(t,x)$ is an $\mathbb{R}^n$-valued stochastic process which is $\mathcal{F}_t$-adapted for each x; moreover, a.s., $\eta(t,x)$ is of class $\mathrm{C}_b^{0,4}([0,T]\times\mathbb{R}^n)$ and $\eta(0,x)=0$.

(A3) For each $x\in\mathbb{R}^n$, $h(x)$ is an $\mathcal{F}_0$-measurable random variable, which, moreover, is of class $\mathrm{C}_b^2(\mathbb{R}^n)$ a.s.

Below, we give a few examples of the noise process $\eta(t,x)$ satisfying (A2).

Example 1. $\eta(t,x)=\int_0^t g(s,x)dB_s=\sum_{i=1}^d g_i(s,x)dB_s^i$, where B_t^i are independent real-valued $\mathcal{F}_t$-Brownian motions, and the stochastic integral is defined for each $x\in\mathbb{R}^n$. Let us show that $\eta(t,x)$ verifies (A2) for some integrands $g(t,x)$. Namely, we assume:

(i) For each $x\in\mathbb{R}^n$, $g(t,x)$ is a progressively measurable stochastic process with values in $\mathbb{R}^{d\times n}$ which takes the form $g(t,x)=\tilde g(t,\phi(x))$ for some $\mathbb{R}^l$-valued random function $\phi(x)$ such that for each x it is a random variable independent of B_t, $t\in[0,T]$.

(ii) For each $t\in[0,T]$, $\tilde g(t,\,\cdot\,)$ is of class $\mathrm{C}_b^{4+\alpha}(\mathbb{R}^l)$ a.s., $\alpha\in(0,1)$; ϕ is of class $\mathrm{C}_b^4(\mathbb{R}^n)$ a.s., and, furthermore, $\mathbb{E}\int_0^T\|\tilde g(t,\,\cdot\,)\|^p_{\mathrm{C}_b^{4+\alpha}(\mathbb{R}^l)}dt<\infty$ for some $p>2+\theta+(4+\theta^2)^{\frac12}$, where $\theta=\frac12\,\alpha^{-1}(n+1)$.

Remark 2.1. Recall that the space $C_b^{k+\alpha}(\mathbb{R}^m)$, $\alpha\in(0,1)$, $k\in\mathbb{N}$, is defined as the (Banach) space of functions $\zeta(x)$ possessing the finite norm

$$\|\zeta\|_{C_b^{k+\alpha}(\mathbb{R}^m)}=\|\zeta\|_{C_b^k(\mathbb{R}^m)}+[\nabla_x^k\zeta]_\alpha^x,$$

where the Hölder constant $[\vartheta]_\alpha^x$ is defined as

$$[\vartheta]_\alpha^x=\sup_{\substack{x,x'\in\mathbb{R}^m,\\ 0<|x-x'|<1}}\frac{|\vartheta(x)-\vartheta(x')|}{|x-x'|^\alpha}.$$

Remark 2.2. Assumptions (i) and (ii) are satisfied, in particular, when the functions $g(t,\,\cdot\,)$ have a common compact support $D \subset \mathbb{R}^n$. Then, take $\tilde{g}(t,x) = g(t,x)$ and $\phi(x) = x\xi(x)$, where $\xi(x)$ is a C^∞-cutting function for D, i.e., $\xi(x) = 1$ if $x \in D$, $\xi(x) = 0$ if x is outside of D_δ, a small δ-neighborhood of D, and, moreover, $0 \leqslant \xi(x) \leqslant 1$. Furthermore, assume that $g(t,x)$ satisfies the regularity and integrability assumptions from (i) and (ii).

Lemma 2.3. *Under assumptions (i) and (ii), there is a version of the stochastic integral* $\int_0^t g(s,x)dB_s$ *which belongs to the space* $\mathrm{C}_b^{0,4}([0,T]\times\mathbb{R}^n)$.

For the proof of Lemma 2.3, we need the next lemma.

Lemma 2.4. *Assume that for each* $x \in \mathbb{R}^n$, $\zeta(t,x)$ *is a progressively measurable* $\mathbb{R}^{d\times n}$*-valued stochastic process such that for each* $t \in [0,T]$, $\zeta(t,x)$ *belongs to class* $\mathrm{C}^{1+\alpha}(\mathbb{R}^n)$ *and* $\mathbb{E}\int_0^T \|\zeta(s,\,\cdot\,)\|^p_{\mathrm{C}^{1+\alpha}(\mathbb{R}^n)}ds < \infty$ *for a number* p *as in (ii). Then, the stochastic integral* $\int_0^t \zeta(s,x)dB_s$ *possesses a* $\mathrm{C}^{0,1}([0,T]\times\mathbb{R}^n)$*-modification.*

Proof. Let, for any function $\vartheta(x)$, $\Delta_\varepsilon^k\vartheta(x) = \varepsilon^{-1}\big(\vartheta(x+\varepsilon e_k)-\vartheta(x)\big)$. It is immediate to verify that

$$\begin{aligned}
&\mathbb{E}\Big|\Delta_\varepsilon^k \int_0^t \zeta(s,x)dB_s - \Delta_{\varepsilon'}^k \int_0^{t'} \zeta(s,x')dB_s\Big|^p \\
&\qquad\leqslant \gamma(p,T)\,\mathbb{E}\int_0^T \|\zeta(s,\,\cdot\,)\|^p_{\mathrm{C}^{1+\alpha}(\mathbb{R}^n)}ds\,\big(|\varepsilon-\varepsilon'|^{\alpha p} + |x-x'|^{\alpha p} + |t-t'|^{\frac{p}{2}-1}\big)
\end{aligned}$$

for some constant $\gamma(p,T)$. The statement of the lemma holds by the choice of p (as in (ii)) and Kolmogorov's continuity theorem. □

Proof of Lemma 2.3. Lemma 2.4 implies that the stochastic integral $\int_0^t \tilde{g}(s,z)ds$ possesses a $\mathrm{C}^{0,4}$-modification. This immediately implies that $\int_0^t \tilde{g}(s,\phi(x))ds$ possesses a $\mathrm{C}_b^{0,4}$-modification, i.e., its derivatives in x are bounded. □

Example 2. Assume $g(t,\,\cdot\,)$ takes values in $\mathcal{L}(H, H^k(\mathbb{R}^n))$, where H is a Hilbert space and $H^k(\mathbb{R}^n)$ is a Sobolev space with sufficiently large k. Further, let B_t be an H-valued cylindrical Brownian motion. Then, $\eta(t,\,\cdot\,) = \int_0^t g(s,\,\cdot\,)dB_s$ can be understood as an $H^k(\mathbb{R}^n)$-valued stochastic integral. This implies that $\eta(t,x)$ is in $\mathrm{C}_b^{0,4}([0,T],\mathbb{R}^n)$ by Kolmogorov's continuity theorem and Sobolev's imbedding $H^k(\mathbb{R}^n) \hookrightarrow \mathrm{C}_b^4(\mathbb{R}^n)$.

Example 3. Let $\dot{W}^i(t,x)$, $i = 1,\dots,n$, be independent space-time white noises, and let $\dot{W}^i_\varepsilon(t,x)$ be a regularization in x of $\dot{W}^i(t,x)$, that is, $\dot{W}^i_\varepsilon(t,x) = (\dot{W}^i(t,\,\cdot\,) * \rho_\varepsilon)(x)$, where ρ_ε is a standard mollifier supported on the ball of radius ε. Alternatively, one can write $W^i_\varepsilon(t,x) = (W^i(t,\,\cdot\,) * \partial^n_{x_1\dots x_n}\rho_\varepsilon)(x)$, where $W^i(t,x)$ is an $(n+1)$-parameter Brownian sheet. The filtration $\mathcal{F}_t$ can be taken as follows $\sigma\{W^i(s,x), 0 \leqslant s \leqslant t, i = 1,\dots,n, x \in \mathbb{R}^n\} \vee \sigma\{h(x), x \in \mathbb{R}^n\} \vee \mathcal{N}$, where $\mathcal{N}$ is the collection of $\mathbb{P}$-null sets. Remark that $\operatorname{cov}(\dot{W}^i_\varepsilon(t,x), \dot{W}^j_\varepsilon(t',x')) = \delta(t-t')\varphi_{ij}(x-x')$, where $\varphi_{ij}(y) = \delta_{ij}\int_{\mathbb{R}^n}\rho_\varepsilon(z)\rho_\varepsilon(z+y)dz$. Since we are interested in noises of class $\mathrm{C}_b^{0,4}(\mathbb{R}^n)$, define $\dot{\eta}^i(t,x)$ as $\dot{W}^i_\varepsilon(t,x)\xi(x)$, where $\xi(x)$, $x \in \mathbb{R}^n$, is a C^∞-cutting function for a bounded domain $D \subset \mathbb{R}^n$ (see Remark 2.2).

Remark 2.5. Everywhere below, the set full $\mathbb{P}$-measure, where $\eta(t,x)$ and $h(x)$ belong to classes $\mathrm{C}_b^{0,4}([0,T]\times\mathbb{R}^n)$ and $\mathrm{C}_b^2(\mathbb{R}^n)$, respectively, and $\eta(0,x)=0$, will be denoted by Ω_0.

2.2. Local existence for stochastic Burgers-type equations. We start with the following lemma whose proof is straightforward.

Lemma 2.6. *The substitution*

$$\hat{y}(t,x) = y(t,x) - \eta(t,x) \tag{2.1}$$

transforms (1.1) *to the following Burgers-type equation with random coefficients:*

$$\begin{cases} \partial_t \hat{y}(t,x) = \nu\Delta\hat{y}(t,z) - (\eta(t,x)+\hat{y},\partial_x)\hat{y}(t,x) + F(t,x,\hat{y}), \\ \hat{y}(0,x) = h(x), \end{cases} \tag{2.2}$$

where

$$F(t,x,\hat{y}) = f(t,x,\hat{y}+\eta(t,x)) + \nu\Delta\eta(t,x) - (\hat{y}+\eta,\partial_x)\eta(t,x). \tag{2.3}$$

Everywhere below throughout this subsection, we assume that η, F, and h possess deterministic bounds in the spaces $C_b^{0,2}([0,T]\times\mathbb{R}^n)$, $C_b^{0,2}([0,T]\times\mathbb{R}^{2n})$, and $\mathrm{C}_b^2(\mathbb{R}^n)$, respectively. Moreover, the force term F is not assumed to necessarily take form (2.3).

In Theorem 2.8 below, we prove the existence and uniqueness of a local $\mathcal{F}_t$-adapted $\mathrm{C}_b^{1,2}$-solution to (2.2). First, by doing the time change $\bar{y}(t,x)=\hat{y}(T-t,x)$, we transform (2.2) to the backward equation

$$\bar{y}(t,x) = h(x) + \int_t^T \big[\nu\Delta\bar{y}(s,x) - (\bar{\eta}(t,x)+\bar{y},\nabla)\bar{y}(s,x) + \bar{F}(s,x,y)\big]ds \tag{2.4}$$

with $\bar{F}(t,x,y) = F(T-t,x,y)$ and $\bar{\eta}(t,x)=\eta(T-t,x)$.

The following lemma will be useful.

Lemma 2.7. *Let W_t be a one-dimensional Brownian motion and $\mathcal{B}$ be a σ-algebra independent of the (augmented) natural filtration $\mathcal{F}_t^W$ of W_t. Assume that Φ_t is $\mathcal{F}_t^W\vee\mathcal{B}$-adapted and $\mathbb{E}\int_0^t|\Phi_s|^2ds<\infty$, $t>0$. Then, $\mathbb{E}\Big[\int_0^t\Phi_s dW_s|\mathcal{B}\Big]=0$ a.s.*

Proof. Let $0=s_1<\ldots<s_n=t$ be a partition. Note that for a simple $\mathcal{F}_t^W\vee\mathcal{B}$-adapted integrand $\Phi=\sum_i\Phi_i\,\mathbb{I}_{[s_i,s_{i+1})}$, it holds that

$$\begin{aligned}\mathbb{E}\Big[\int_0^t\Phi_s dW_s|\,\mathcal{B}\Big] &= \mathbb{E}\Big[\sum_i\Phi_i(W_{s_{i+1}}-W_{s_i})|\,\mathcal{B}\Big] \\ &= \sum_i\mathbb{E}\Big[\Phi_i\,\mathbb{E}\big[(W_{s_{i+1}}-W_{s_i})|\mathcal{F}_{s_i}^W\vee\mathcal{B}\big]\,|\,\mathcal{B}\Big]=0.\end{aligned}$$

Further, we note that if a sequence $\{\Phi_t^{(n)}\}$ of simple $\mathcal{F}_t^W\vee\mathcal{B}$-adapted integrands is such that $\mathbb{E}\int_0^t(\Phi_s^{(n)}-\Phi_s)^2ds\to 0$, then by the conditional Jensen's inequality and Itô's isometry, $\mathbb{E}\big(\mathbb{E}\big[\int_0^t\big(\Phi_s^{(n)}-\Phi_s\big)dW_s|\,\mathcal{B}\big]\big)^2\to 0$. □

Everywhere below, the symbol $\mathbb{E}_\tau$ will denote the conditional expectation with respect to $\mathcal{F}_{T-\tau}$.

Theorem 2.8. *Let, the functions $\bar{\eta}(t,x)$, $\bar{F}(t,x,y)$, $h(x)$ satisfy the assumptions:*

1) *$\bar{F}(t,x,y)$ and $\bar{\eta}(t,x)$ are $\mathcal{F}_{T-t}$-adapted for each $x, y \in \mathbb{R}^n$.*
2) *$\bar{\eta}(t,x)$ and $h(x)$ a.s. belong to spaces $\mathrm{C}_b^{0,2}([0,T]\times\mathbb{R}^n)$ and $\mathrm{C}_b^2(\mathbb{R}^n)$, respectively, and possess a deterministic bound K with respect to the norms of the spaces.*
3) *$\bar{F}(t,x,y)$ is of class $\mathrm{C}^{0,2}([0,T]\times\mathbb{R}^{2n})$ and satisfies the estimate $|\bar{F}(t,x,y)| + |\nabla_{(x,y)}\bar{F}(t,x,y)| + |\nabla^2_{(x,y)}\bar{F}(t,x,y)| \leqslant K(1+|y|)$ a.s.*

Then, there exists a constant γ_K, depending only on K, such that on $[T-\gamma_K, T]$, there exists an $\mathcal{F}_{T-t}$-adapted $\mathrm{C}_b^{1,2}$-solution $\bar{y}(t,x)$ to equation (2.4).

Proof. In what follows, γ_i, μ_i, $i = 1, 2, \ldots$, are positive deterministic constants that may depend only on p and K; in particular, they do not depend on ν. We will track the dependence of some constants on ν because it is important for the next section. Furthermore, the constants $\tilde{\gamma}_K$, $\dot{\gamma}_K$, $\hat{\gamma}_K$, $\bar{\gamma}_K$, γ_K are positive and deterministic, that depend only on K; they determine the length of the interval. Without loss of generality, these γ_K-type constants are assumed to be smaller than 1.

We prove the existence of an $\mathcal{F}_{T-t}$-adapted $C_b^{1,2}$-solution to (2.4) by means of the associated FBSDEs (see [15], [30]):

$$\begin{cases} X_t^{\tau,x} = x - \int_\tau^t \big(\bar{\eta}(s, X_s^{\tau,x}) + Y_s^{\tau,x}\big)ds + \sqrt{2\nu}(W_t - W_\tau) \\ Y_t^{\tau,x} = h(X_T^{\tau,x}) + \int_t^T \bar{F}(s, X_s^{\tau,x}, Y_s^{\tau,x})\,ds - \int_t^T Z_s^{\tau,x} dW_s, \end{cases} \tag{2.5}$$

where W_t is an n-dimensional Brownian motion independent of the filtration $\mathcal{F}_{T-t}$, and the upper index τ, x means that the process $X_t^{\tau,x}$ starts at x at time $\tau > 0$. For each $\tau \in (0,T)$, define the filtration

$$(\mathcal{G}_t^\tau)_{\tau \leqslant t \leqslant T} = \sigma\{W_s - W_\tau, s \in [\tau, t]\} \vee \mathcal{F}_{T-\tau}. \tag{2.6}$$

In what follows, when it does not lead to misunderstanding, we will often skip the upper index τ, x in $(X_t^{\tau,x}, Y_t^{\tau,x}, Z_t^{\tau,x})$ and similar processes to simplify notation.

Step1. Boundedness of $\mathbb{E}_\tau |Y_t^{\tau,x}|^p$ and modified FBSDE. Consider the backward SDE in (2.5). From the assumptions of the theorem and Itô's formula, it follows that $\mathbb{E}_\tau |Y_t^{\tau,x}|^p$ is bounded, a.s., for any solution $Y_t^{\tau,x}$ to this BSDE and for any $\mathcal{G}_t^\tau$-adapted process $X_t^{\tau,x}$. Indeed, since

$$\begin{aligned} &\big(|g|^p\big)' h = p|g|^{p-2}(g,h);\\ &\big(|g|^p\big)'' h_1 h_2 = p(p-2)|g|^{p-4}(g,h_1)(g,h_2) + p|g|^{p-2}(h_1,h_2) \end{aligned}$$

for $p \geqslant 2$, then, a.s.,

$$\begin{aligned} \mathbb{E}_\tau |Y_t|^p + p(p-2)\int_t^T \mathbb{E}_\tau \Big[|Y_s|^{p-4}\sum_{i=1}^n |(Z_s^i, Y_s)|^2\Big]\,ds + p\int_t^T \mathbb{E}_\tau \big[|Y_s|^{p-2}|Z_s|^2\big]\,ds \\ = \mathbb{E}_\tau |h(X_T)|^p + 2p\int_t^T \mathbb{E}_\tau \big[|Y_s|^{p-2}(\bar{F}(s, X_s, Y_s), Y_s)\big]\,ds. \end{aligned} \tag{2.7}$$

Since $|\bar{F}(t,x,y)| \leqslant K(1+|y|)$, then Young's inequality and Gronwall's lemma imply that for every (τ, x),

$$\mathbb{E}_\tau |Y_t|^p \leqslant \gamma_1 \quad \text{and} \quad |Y_\tau^{\tau,x}| \leqslant (\gamma_1)^{\frac{1}{p}} \quad \text{a.s.} \tag{2.8}$$

Moreover, γ_1 is the same for all $(\tau, x) \in [0,T] \times \mathbb{R}^n$.

Now let $\delta = (\gamma_1)^{\frac{1}{p}}$ for some fixed p, and let $\zeta_\delta(y) = \xi_\delta(y)y$, where $\xi_\delta(y)$ is a C^∞-cutting function for the ball B_δ of radius δ centered at the origin (see Remark 2.2). We modify $\bar{F}$ by introducing $\zeta_\delta(y)$ instead of y as follows:

$$\bar{F}_\delta(t,x,y) = \bar{F}(t,x,\zeta_\delta(y)). \tag{2.9}$$

Together with Assumption 3), this implies that $|\bar{F}_\delta|$ is uniformly bounded by $K(1+\delta)$. Further, consider the modified FBSDE

$$\begin{cases} X_t^{\tau,x} = x - \int_\tau^t \big(\bar{\eta}(s, X_t^{\tau,x}) + Y_s^{\tau,x})\big) ds + \sqrt{2\nu}(W_t - W_\tau) \\ Y_t^{\tau,x} = h(X_T^{\tau,x}) + \int_t^T \bar{F}_\delta(s, X_s^{\tau,x}, Y_s^{\tau,x})\, ds - \int_t^T Z_s^{\tau,x} dW_s. \end{cases} \tag{2.10}$$

According to the results of [15] (Theorem A.1), there exists a constant $\tilde{\gamma}_K$, depending only on K (remark that δ also depends only on K), such that whenever $T-\tau \leqslant \tilde{\gamma}_K$, system (2.10) possesses a unique $\mathcal{G}_t^\tau$-adapted solution $(X_t^{\tau,x}, Y_t^{\tau,x}, Z_t^{\tau,x})$ on $[\tau, T]$ such that $X_t^{\tau,x}$ and $Y_t^{\tau,x}$ have continuous paths a.s.

Step 2. Continuity of the map $(\tau, x) \mapsto Y_\tau^{\tau,x}$ and solution to the original FBSDE. First, we prove that the map $[T-\dot{\gamma}_K, T] \times \mathbb{R}^n \to \mathrm{C}([T-\dot{\gamma}_K, T])$, $(\tau, x) \mapsto (X^{\tau,x}, Y^{\tau,x})$ has an a.s. continuous version for some constant $0 < \dot{\gamma}_K < \tilde{\gamma}_K$. This continuity will be required, in particular, for the proof of differentiability of $(X_t^{\tau,x}, Y_t^{\tau,x})$ with respect to x. Extend $X_s^{\tau,x}$ to $[T-\tilde{\gamma}_K, \tau]$ by x, and $Y_s^{\tau,x}$ by $Y_\tau^{\tau,x}$. By Corollary A.6 from [15], there exists a constant $\dot{\gamma}_K < \tilde{\gamma}_K$ such that for any $x, x' \in \mathbb{R}^n$, $\tau, \tau' \in [T-\dot{\gamma}_K, T]$,

$$\begin{aligned} \mathbb{E} \sup_{t\in[T-\dot{\gamma}_K,T]} |X_t^{\tau,x} - X_t^{\tau',x'}|^p + \mathbb{E} \sup_{t\in[T-\dot{\gamma}_K,T]} |Y_t^{\tau,x} - Y_t^{\tau',x'}|^p \\ \leqslant \gamma_2\big(|x-x'|^p + (1+|x|^p)|\tau-\tau'|^{\frac{p}{2}}\big), \end{aligned} \tag{2.11}$$

where $p \geqslant 2$. Pick $p > n$. Then, by Kolmogorov's continuity criterion in Banach spaces (see, e.g., [26]), there exists a continuous modification of the map $[T - \dot{\gamma}_K, T] \times \mathbb{R}^n \to \mathrm{C}([T-\dot{\gamma}_K, T])$, $(\tau, x) \mapsto (X^{\tau,x}, Y^{\tau,x})$. In particular, the map $(\tau, x) \mapsto Y_\tau^{\tau,x}$ is continuous a.s. This and (2.8) imply that $\sup_{\tau,x} |Y_\tau^{\tau,x}| < \delta$ a.s.

Further, according to Corollary A.4 of [15] and by the continuity in (τ, x) obtained above, a.s.,

$$Y_t^{\tau,x} = Y_\tau^{\tau, X_t^{\tau,x}} \qquad \text{for each } \tau \in (T-\tilde{\gamma}_K, T],\ t \in [\tau, T],\ x \in \mathbb{R}^n. \tag{2.12}$$

Therefore, $(X_t^{\tau,x}, Y_t^{\tau,x}, Z_t^{\tau,x})$ is also a solution to original FBSDE (2.5) on $[\tau, T]$.

Step 3. Differentiability of the FBSDEs solution in x. Boundedness of $\mathbb{E}_\tau |\partial X_t^{\tau,x}|^p$ and $\mathbb{E}_\tau |\partial Y_t^{\tau,x}|^p$. Now we proceed with the proof of differentiability. In Steps 3 and 4, we will write $\bar{F}$ instead of $\bar{F}_\delta$ (defined by (2.9)) to simplify notation, and thus assuming (without loss of generality) that $\bar{F}$ is bounded together with its spatial derivatives up to the second order.

For any function $\alpha(x)$, define $\Delta_\varepsilon^k \alpha(x) = \varepsilon^{-1}\big(\alpha(x+\varepsilon e_k) - \alpha(x)\big)$, $k = 1, \dots, n$, where $\{e_k\}_{k=1}^n$ is the orthonormal basis in $\mathbb{R}^n$. In particular, $\Delta_\varepsilon^k X_t =$

$\varepsilon^{-1}(X_t^{\tau,x+\varepsilon e_k} - X_s^{\tau,x})$, $k = 1,\dots,n$, and $\Delta_\varepsilon^k Y_t$, $\Delta_\varepsilon^k Z_t$ are defined similarly. Further, for a function Φ (which can be any of the functions $\bar F$, h, $\bar\eta$, or their gradients with respect to the spatial variables), we define $\nabla_2\Phi(t,u,v) = \partial_u\Phi(t,u,v)$, $\nabla_3\Phi(t,u,v) = \partial_v\Phi(t,u,v)$. Furthermore, we define

$$\begin{aligned}\nabla_2^{\varepsilon,k}\Phi_t &= \textstyle\int_0^1 \nabla_2\Phi(t, X_t + \lambda\varepsilon\Delta_\varepsilon^k X_t, Y_t)d\lambda,\\ \nabla_3^{\varepsilon,k}\Phi_t &= \textstyle\int_0^1 \nabla_3\Phi(t, X_t, Y_t + \lambda\varepsilon\Delta_\varepsilon^k Y_t)d\lambda,\end{aligned} \tag{2.13}$$

and note that

$$\nabla_2^{\varepsilon,k}\Phi_t = \int_0^1 \nabla_2\Phi(t,(1-\lambda)X_t^{\tau,x} + \lambda X_t^{\tau,x+\varepsilon e_k}, Y_t)d\lambda, \tag{2.14}$$

and similar for $\nabla_3^{\varepsilon,k}\Phi_t$. In case of just one spatial variable (like in h or η), we write ∇ instead of ∇_2 and $\nabla^{\varepsilon,k}$ instead of $\nabla_2^{\varepsilon,k}$. Note that

$$\Delta_\varepsilon^k\Phi_t = \nabla_2^{\varepsilon,k}\Phi_t\Delta_\varepsilon^k X_t + \nabla_3^{\varepsilon,k}\Phi_t\Delta_\varepsilon^k Y_t. \tag{2.15}$$

It is immediate to verify that the triple $(\Delta_\varepsilon^k X_t, \Delta_\varepsilon^k Y_t, \Delta_\varepsilon^k Z_t)$ solves the FBSDE

$$\begin{cases}\Delta_\varepsilon^k X_t = e_k - \int_\tau^t \left(\Delta_\varepsilon^k Y_s + \nabla^{\varepsilon,k}\bar\eta_s\Delta_\varepsilon^k X_s\right)ds,\\ \Delta_\varepsilon^k Y_t = \nabla^{\varepsilon,k}h_T\,\Delta_\varepsilon^k X_T + \int_t^T \left(\nabla_2^{\varepsilon,k}\bar F_s\Delta_\varepsilon^k X_s + \nabla_3^{\varepsilon,k}\bar F_s\Delta_\varepsilon^k Y_s\right)ds - \int_t^T \Delta_\varepsilon^k Z_s dW_s\end{cases} \tag{2.16}$$

on the same time interval $[\tau,T]$, where we proved the existence and uniqueness of solution to (2.5). Additionally, we define $(\Delta_0^k X_t, \Delta_0^k Y_t, \Delta_0^k Z_t)$ as the unique solution to FBSDE (2.16) whose coefficients are taken at $\varepsilon = 0$. Remark that setting $\varepsilon = 0$ in (2.14), we obtain $\nabla_2\Phi(t,X_t,Y_t)$ on the right-hand side. The existence and uniqueness of the triple $(\Delta_0^k X_t, \Delta_0^k Y_t, \Delta_0^k Z_t)$ follows from Theorem A.1 in [15].

Let us show that for $p \geqslant 2$, a.s.,

$$\max\left\{\mathbb{E}_\tau|\Delta_\varepsilon^k X_t|^p;\ \mathbb{E}_\tau|\Delta_\varepsilon^k Y_t|^p\right\} \leqslant \gamma_3 \quad \text{for all } \varepsilon \geqslant 0,\ t\in[\tau,T]. \tag{2.17}$$

Itô's formula and the BSDE in (2.16) imply

$$\begin{aligned}\mathbb{E}_\tau|\Delta_\varepsilon^k Y_t|^p &+ p(p-2)\int_t^T \mathbb{E}_\tau\Big[|\Delta_\varepsilon^k Y_s|^{p-4}\sum_{j=1}^n |(\Delta_\varepsilon^k Z_s^j, \Delta_\varepsilon^k Y_s)|^2\Big]ds\\ &+ p\int_t^T \mathbb{E}_\tau\big[|\Delta_\varepsilon^k Y_s|^{p-2}|\Delta_\varepsilon^k Z_s|^2\big]ds = \mathbb{E}_\tau\big[|\nabla^{\varepsilon,k}h_T\Delta_\varepsilon^k X_T|^p\big]\\ &+ 2p\int_t^T \mathbb{E}_\tau\Big[|\Delta_\varepsilon^k Y_s|^{p-2}(\nabla_2^{\varepsilon,k}\bar F_s\Delta_\varepsilon^k X_s + \nabla_3^{\varepsilon,k}\bar F_s\Delta_\varepsilon^k Y_s, \Delta_\varepsilon^k Y_s)\Big]ds.\end{aligned}$$

From here, by the forward SDE in (2.16) and Young's inequality, it follows that a.s. $\mathbb{E}_\tau|\Delta_\varepsilon^k Y_t|^p \leqslant \gamma_4\big(1 + \int_\tau^T \mathbb{E}_\tau|\Delta_\varepsilon^k Y_t|^p\,ds\big)$ for all $t\in[\tau,T]$ and $\varepsilon \geqslant 0$, which, together with the forward SDE in (2.16), implies (2.17).

Now let $\zeta_X(t) = \Delta_\varepsilon^k X_t - \Delta_{\varepsilon'}^k X_t$. Similarly, we define $\zeta_Y(t)$ and $\zeta_Z(t)$. The FBSDE for the triple $(\zeta_X(t), \zeta_Y(t), \zeta_Z(t))$ takes the form

$$\begin{cases}\zeta_X(t) = \int_\tau^t \left(\zeta_Y(s) + \nabla^{\varepsilon,k}\bar\eta_s\,\zeta_X(s) + \xi_s^X\right)ds,\\ \zeta_Y(t) = \nabla^{\varepsilon,k}h_T\,\zeta_X(T) + \varsigma_T + \int_t^T \big(\nabla_2^{\varepsilon,k}\bar F_s\,\zeta_X(s)\\ \qquad +\nabla_3^{\varepsilon,k}\bar F_s\,\zeta_Y(s) + \xi_s^Y\big)ds - \int_t^T \zeta_Z(s)dW_s,\end{cases} \tag{2.18}$$

where $\xi^X_s = (\nabla^{\varepsilon,k}\bar\eta_s - \nabla^{\varepsilon',k}\bar\eta_s)\Delta^k_{\varepsilon'}X_s$, $\xi^Y_s = (\nabla^{\varepsilon,k}_2\bar F_s - \nabla^{\varepsilon',k}_2\bar F_s)\Delta^k_{\varepsilon'}X_s + (\nabla^{\varepsilon,k}_3\bar F_s - \nabla^{\varepsilon',k}_3\bar F_s)\Delta^k_{\varepsilon'}Y_s$, and $\varsigma_T = (\nabla^{\varepsilon,k}h_T - \nabla^{\varepsilon',k}h_T)\Delta^k_{\varepsilon'}X_T$. Note that $\nabla^{\varepsilon,k}\bar\eta_s$ and $\nabla^{\varepsilon,k}h_T$ are bounded by K, and $\nabla^{\varepsilon,k}_i\bar F_s$, $i=2,3$, are bounded by $K(1+\delta)$, which follows from (2.13). Then, by standard arguments (which include an application of Itô's formula to $|\zeta_Y|^2$, elevating the both parts to the power $\frac{p}{2}$, and making use of the estimate $\mathbb{E}\big|\int_t^T(\zeta_Y(s),\zeta_Z(s)dW_s)\big|^{\frac{p}{2}} \leqslant \gamma_5(T-\tau)^{\frac{p}{4}}\mathbb{E}\sup_{[\tau,T]}|\zeta_Y|^p + \varepsilon\mathbb{E}\big(\int_t^T|\zeta_Z(s)|^2ds\big)^{\frac{p}{2}}$), there exists a constant $\check\gamma_k < \dot\gamma_K$ such that on the interval $[\tau,T]$ whose length is smaller than $\check\gamma_K$, for $p\geqslant 2$,

$$\mathbb{E}\sup_{[\tau,T]}|\zeta_X(t)|^p + \mathbb{E}\sup_{[\tau,T]}|\zeta_Y(t)|^p + \mathbb{E}\Big(\int_t^T|\zeta_Z(s)|^2ds\Big)^{\frac{p}{2}} \leqslant \gamma_6\Big(\mathbb{E}\,|\varsigma_T|^p + \mathbb{E}\int_\tau^T[|\xi^X_s|^p + |\xi^Y_s|^p]ds\Big) \leqslant \gamma_7\,|\varepsilon-\varepsilon'|^p. \quad (2.19)$$

The last inequality holds by the definition of ς_T, ξ^X_s, ξ^Y_s, and by virtue of (2.14) and (2.11). Combining (2.19) with Corollary A.6 from [15], we obtain that there exists a positive constant $\hat\gamma_K < \check\gamma_K$ such that for all $x,x'\in\mathbb{R}^n$, $\tau,\tau'\in[T-\hat\gamma_K,T]$, and $t\in[\tau,T]$,

$$\mathbb{E}\sup_{t\in[\tau,T]}|\Delta^k_\varepsilon X^{\tau,x}_t - \Delta^k_{\varepsilon'}X^{\tau',x'}_t|^p + \mathbb{E}\sup_{t\in[\tau,T]}|\Delta^k_\varepsilon Y^{\tau,x}_t - \Delta^k_{\varepsilon'}Y^{\tau',x'}_t|^p \leqslant \gamma_8(|\varepsilon-\varepsilon'|^p + |x-x'|^p + |\tau-\tau'|^{\frac{p}{2}}). \quad (2.20)$$

By Kolmogorov's continuity criterium, there exists a continuous version of the map $[0,+\infty)\times[T-\hat\gamma_K,T]\times[T-\hat\gamma_K,T]\times\mathbb{R}^n\to\mathbb{R}^{2n}$, $(\varepsilon,\tau,t,x)\mapsto(\Delta^k_\varepsilon X^{\tau,x}_t,\Delta^k_\varepsilon Y^{\tau,x}_t)$. This means that the map $[T-\hat\gamma_K,T]\times[T-\hat\gamma_K,T]\times\mathbb{R}^n\to\mathbb{R}^{2n}$, $(\tau,t,x)\mapsto(X^{\tau,t,x}_t,Y^{\tau,t,x}_t)$ is differentiable in x_k, and the derivative is continuous in (τ,t,x) a.s. In particular, there exists an a.s. continuous derivative $\partial_k Y^{\tau,x}_\tau$, and, by (2.17), a.s.,

$$|\partial_k Y^{\tau,x}_\tau| < \big(\gamma_3\big)^{\frac{1}{p}} \quad \text{for all } (\tau,x)\in[T-\hat\gamma_K,T]\times\mathbb{R}^n, \quad (2.21)$$

where $\partial_k = \partial_{x_k}$. This holds for all $k\in\{1,\dots,n\}$. Moreover, γ_3 does not depend on ν.

Step 4. Second order differentiability of the FBSDE solution in x. Boundedness of $\mathbb{E}_\tau|\partial^2_{ik}Y^{\tau,x}_t|^2$. Below, we use the symbol ∂_k for ∂_{x_k} and ∂^2_{ik} for $\partial^2_{x_ix_k}$. As in Step 3, we write F instead of F_δ to simplify notation.

Remark that (2.19) implies the differentiability in x of $Z^{\tau,x}_t$ with respect to the norm $(\mathbb{E}\int_\tau^T|\varphi(s)|^2ds)^{\frac12}$. Further, the FBSDE for $(\partial_kX_t,\partial_kY_t,\partial_kZ_t)$ takes the form:

$$\begin{cases}\partial_kX_t = e_k + \int_\tau^t\big(\partial_kY_s + \nabla\bar\eta_s\partial_kX_s\big)ds\\ \partial_kY_t = \nabla h(X_T)\partial_kX_T + \int_t^T\big(\nabla_2\bar F_s\partial_kX_s + \nabla_3\bar F_s\partial_kY_s\big)ds - \int_t^T\partial_kZ_s\,dW_s,\end{cases} \quad (2.22)$$

where $\nabla\bar\eta_s = \nabla\bar\eta(s,X_s)$, $\nabla h_T = \nabla h(X_T)$, $\nabla_i\bar F_s = \nabla_i\bar F(s,X_s,Y_s)$, $i=2,3$.

As in the previous step, define $\Delta^i_\varepsilon\partial_kX_t = \varepsilon^{-1}(\partial_kX^{\tau,x+\varepsilon e_i}_t - \partial_kX^{\tau,x}_s)$, $i=1,\dots,n$, and, similarly, $\Delta^i_\varepsilon\partial_kY_t$, $\Delta^i_\varepsilon\partial_kZ_t$. Applying the operation Δ^i_ε to FBSDE (2.22), using formula (2.15), and noticing that for any functions $\alpha_1(x)$ and $\alpha_2(x)$,

$\Delta^i_\varepsilon\big[\alpha_1(x)\alpha_2(x)\big] = \alpha_1(x)\Delta^i_\varepsilon\alpha_2(x) + \Delta^i_\varepsilon\alpha_1(x)\alpha_2(x+\varepsilon e_i)$, we obtain the FBSDE for the triple $(\Delta^i_\varepsilon\partial_k X_t, \Delta^i_\varepsilon\partial_k Y_t, \Delta^i_\varepsilon\partial_k Z_t)$

$$\begin{cases} \Delta^i_\varepsilon\partial_k X_t = -\int_\tau^t \big(\Delta^i_\varepsilon\partial_k Y_s + \nabla\bar\eta_s\,\Delta^i_\varepsilon\partial_k X_s + \vartheta^X_{s,\varepsilon}\big)ds,\\ \Delta^i_\varepsilon\partial_k Y_t = \nabla h_T\Delta^i_\varepsilon\partial_k X_T + \eta_{T,\varepsilon} + \int_t^T\big(\nabla_2\bar F_s\Delta^i_\varepsilon\partial_k X_s\\ \qquad +\nabla_3\bar F_s\Delta^i_\varepsilon\partial_k Y_s + \vartheta^Y_{s,\varepsilon}\big)ds - \int_t^T\Delta^i_\varepsilon\partial_k Z_s\,dW_s, \end{cases} \tag{2.23}$$

where

$$\begin{aligned} &\vartheta^X_{s,\varepsilon} = \nabla^{\varepsilon,i}\nabla\bar\eta_s\,\Delta^i_\varepsilon X_s\,\partial_k X_s^{\tau,x+\varepsilon e_i}; \quad \eta_{T,\varepsilon} = \nabla^{\varepsilon,i}\nabla h_T\,\Delta^i_\varepsilon X_T\,\partial_k X_T^{\tau,x+\varepsilon e_i};\\ &\vartheta^Y_{s,\varepsilon} = \nabla_2^{\varepsilon,i}\nabla_2\bar F_s\Delta^i_\varepsilon X_s\partial_k X_s^{\tau,x+\varepsilon e_i} + \nabla_3^{\varepsilon,i}\nabla_3\bar F_s\Delta^i_\varepsilon Y_s\partial_k Y_s^{\tau,x+\varepsilon e_i}\\ &\quad +\nabla_3^{\varepsilon,i}\nabla_2\bar F_s\Delta^i_\varepsilon Y_s\partial_k X_s^{\tau,x+\varepsilon e_i} + \nabla_2^{\varepsilon,i}\nabla_3\bar F_s\Delta^i_\varepsilon X_s\partial_k Y_s^{\tau,x+\varepsilon e_i}. \end{aligned} \tag{2.24}$$

Further, the triple $(\Delta^i_0\partial_k X_t, \Delta^i_0\partial_k Y_t, \Delta^i_0\partial_k Z_t)$ will denote the unique solution to FBSDE (2.23) whose coefficients $\vartheta^X_{s,\varepsilon}$, $\eta_{T,\varepsilon}$, and $\vartheta^Y_{s,\varepsilon}$ are taken at $\varepsilon = 0$. The existence and uniqueness of the above triple follows from Theorem A.1 in [15]. Let us show that, a.s.,

$$\max\{\mathbb{E}_\tau|\Delta^i_\varepsilon\partial_k X_t|^2, \mathbb{E}_\tau|\Delta^i_\varepsilon\partial_k Y_t|^2\} \leqslant \mu_1 \quad \text{for all } \varepsilon \geqslant 0,\ t\in[\tau,T]. \tag{2.25}$$

Itô's formula implies

$$|\Delta^i_\varepsilon\partial_k Y_t|^2 + \int_t^T |\Delta^i_\varepsilon\partial_k Z_s|^2 ds = |\nabla h(X_T)\Delta^i_\varepsilon\partial_k X_T + \eta_{T,\varepsilon}|^2 + 2\int_t^T (\nabla_2\bar F_s\,\Delta^i_\varepsilon\partial_k X_s$$
$$+ \nabla_3\bar F_s\,\Delta^i_\varepsilon\partial_k Y_s + \vartheta^Y_{s,\varepsilon}, \Delta^i_\varepsilon\partial_k Y_s)\,ds + \int_t^T (\Delta^i_\varepsilon\partial_k Y_s, \Delta^i_\varepsilon\partial_k Z_s dW_s).$$

From here, by using the forward SDE in (2.23), we conclude that there exists a constant $\bar\gamma_K < \hat\gamma_K$, depending only on K, such that for $\tau\in[T-\bar\gamma_K, T]$,

$$\mathbb{E}_\tau|\Delta^i_\varepsilon\partial_k Y_t|^2 \leqslant \mu_2\big(1 + \int_\tau^T \mathbb{E}_\tau\big(|\vartheta^X_{s,\varepsilon}|^2 + |\vartheta^Y_{s,\varepsilon}|^2\big)ds + \mathbb{E}_\tau|\eta_{T,\varepsilon}|^2\big) \quad \text{a.s.}$$

By the assumptions of the theorem and (2.17), the right-hand side of the above inequality is bounded a.s. This implies (2.25).

Now let us prove the existence of a continuous second derivative of the map $Y_\tau^{\tau,x}$. Let $\zeta_X(t) = \Delta^i_\varepsilon\partial_k X_t - \Delta^i_{\varepsilon'}\partial_k X_t$, $\zeta_Y(t) = \Delta^i_\varepsilon\partial_k Y_t - \Delta^i_{\varepsilon'}\partial_k Y_t$, $\zeta_Z(\varepsilon,t) = \Delta^i_\varepsilon\partial_k Z_t - \Delta^i_{\varepsilon'}\partial_k Z_t$. The FBSDE for the triple $(\zeta_X(t),\zeta_Y(t),\zeta_Z(t))$ takes the form:

$$\begin{cases} \zeta_X(t) = -\int_\tau^t \big(\zeta_Y(s) + \nabla\bar\eta_s\zeta_X(s) + \vartheta^X_{s,\varepsilon} - \vartheta^X_{s,\varepsilon'}\big)\,ds,\\ \zeta_Y(t) = \nabla h_T\zeta_X(T) + \eta_{T,\varepsilon} - \eta_{T,\varepsilon'} + \int_t^T\big(\nabla_2\bar F_s\zeta_X(s)\\ \qquad +\nabla_3\bar F_s\zeta_Y(s) + \vartheta^Y_{s,\varepsilon} - \vartheta^Y_{s,\varepsilon'}\big)ds - \int_t^T\zeta_Z(s)\,dW_s. \end{cases} \tag{2.26}$$

Note that FBSDE (2.26) has a similar structure with FBSDE (2.18). Thus, similar to (2.19), we conclude that there exists a constant $\mathring\gamma_K < \bar\gamma_K$ such that for $\tau\in[T-\mathring\gamma_K, T]$,

$$\mathbb{E}|\zeta_X(t)|^p + \mathbb{E}|\zeta_Y(t)|^p + \mathbb{E}\Big(\int_t^T |\zeta_Z(s)|^2 ds\Big)^{\frac{p}{2}} \leqslant \mu_3\big(\mathbb{E}\,|\eta_{T,\varepsilon} - \eta_{T,\varepsilon'}|^p +$$
$$\mathbb{E}\int_\tau^T [|\vartheta^X_{s,\varepsilon} - \vartheta^X_{s,\varepsilon'}|^p + |\vartheta^Y_{s,\varepsilon} - \vartheta^Y_{s,\varepsilon'}|^p]ds\big) \leqslant \mu_4\,|\varepsilon-\varepsilon'|^p \tag{2.27}$$

on $[\tau, T]$. The last inequality holds by (2.11) and (2.20). Combining (2.27) with Corollary A.6 from [15] (similar to the previous step), we obtain that there exists a positive constant $\gamma_K < \mathring{\gamma}_K$ such that for all $x, x' \in \mathbb{R}^n$, $\tau, \tau' \in [T-\gamma_K, T]$, and $t \in [\tau, T]$,

$$\mathbb{E}|\Delta^i_\varepsilon \partial_k Y^{\tau,x}_t - \Delta^i_{\varepsilon'} \partial_k Y^{\tau',x'}_t|^p \leqslant \mu_5(|\varepsilon - \varepsilon'|^p + |x - x'|^p + |\tau - \tau'|^{\frac{p}{2}}).$$

By Kolmogorov's continuity criterium, there exists a continuous version of the map $[0, +\infty) \times [T - \gamma_K, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $(\varepsilon, \tau, x) \mapsto \Delta^i_\varepsilon \partial_k Y^{\tau,x}_\tau$. This means that the map $[T - \gamma_K, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $(\tau, x) \mapsto \partial_k Y^{\tau,x}_\tau$ is differentiable in x_i and the derivative in continuous in (τ, x) a.s. Further, (2.25) implies that

$$|\partial^2_{ik} Y^{\tau,x}_\tau| \leqslant \sqrt{\mu_2} \quad \text{a.s.} \tag{2.28}$$

We remark that μ_2 depends only on K and does not depend on ν. Moreover, (2.28) holds uniformly in $(\tau, x) \in [T - \gamma_K, T] \times \mathbb{R}^n$ by continuity. This implies that there exists a set $\tilde{\Omega}$ of full $\mathbb{P}$-measure such that for all $\omega \in \tilde{\Omega}$, $Y^{\tau,x}_\tau$ is twice continuously differentiable in x, and, moreover, the derivatives of $Y^{\tau,x}_\tau$ up to the second order are bounded.

Step 5. Solution to random PDE (2.4). Define $\bar{y}(\tau, x, \omega) = Y^{\tau,x}_\tau(\omega)$ for each $\omega \in \tilde{\Omega}$. Note that $\bar{y}(\tau, x)$ is $\mathcal{F}_{T-\tau}$-measurable and by (2.12), a.s.,

$$Y^{\tau,x}_t = \bar{y}(t, X^{\tau,x}_t) \quad \text{for all } \tau, t \in [T - \gamma_K, T],\ x \in \mathbb{R}^n. \tag{2.29}$$

Let us prove that $\bar{y}(t, x)$ is a solution to (2.4). The idea of the proof is similar to that of Theorem 3.2 in [30]. However, we deal with the random coefficient case. Define $\mathcal{L}u = \nu\Delta u + (u + \bar{\eta}, \nabla)u$. We have

$$\bar{y}(t + h, x) - \bar{y}(t, x) = [\bar{y}(t + h, x) - \bar{y}(t + h, X^{t,x}_{t+h})] + [\bar{y}(t + h, X^{t,x}_{t+h}) - \bar{y}(t, x)].$$

Since $\bar{y}$ is of class $\mathrm{C}^{0,2}_b$, we can apply Itô's formula to the first term. Further, by (2.10) and (2.29), we substitute the second term with $-\int_t^{t+h} \bar{F}_\delta(s, X^{t,x}_s, \bar{y}(s, X^{t,x}_s)ds + \int_t^{t+h} Z^{t,x}_s dW_s$. Remark that, by (2.8), $\bar{F}_\delta(s, X^{t,x}_s, \bar{y}(s, X^{t,x}_s)) = \bar{F}(s, X^{t,x}_s, \bar{y}(s, X^{t,x}_s))$ so we can skip the index δ. Thus, we obtain that, a.s.,

$$\begin{aligned} \bar{y}(t+h, x) - \bar{y}(t, x) = -\int_t^{t+h} \mathcal{L}\bar{y}(t+h, X^{t,x}_s)ds - \sqrt{2\nu}\int_t^{t+h} \nabla\bar{y}(t+h, X^{t,x}_s)dW_s \\ - \int_t^{t+h} \bar{F}(s, X^{t,x}_s, \bar{y}(s, X^{t,x}_s))ds + \int_t^{t+h} Z^{t,x}_s dW_s \end{aligned}$$

for all (t, x, h). Fix a partition $\mathcal{P} = \{\tau = t_0 < t_1 < \cdots < t_n = T\}$. Taking the conditional expectation $\mathbb{E}_\tau$ and summing up, we obtain that, a.s.,

$$\bar{y}(\tau, x) - h(x) = \mathbb{E}_\tau \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \big(\mathcal{L}\bar{y}(t_{i+1}, X^{t_i,x}_s) + \bar{F}(s, X^{t_i,x}_s, \bar{y}(s, X^{t_i,x}_s))\big)ds. \tag{2.30}$$

Indeed, the conditional expectation of the stochastic integrals is zero by Lemma 2.7. Note that the expression under the integral sign is bounded, a.s., since $\mathcal{L}\bar{y}(t, x)$ is bounded by what was proved in the previous steps.

Further, $\mathcal{L}\bar{y}(t,x)$ and $\bar{F}(s, X_s^{t,x}, \bar{y}(s, X_s^{t,x}))$ are a.s. continuous in (t,x). Letting the mesh of $\mathcal{P}$ in (2.30) go to zero, by the conditional bounded convergence theorem, we obtain that $\bar{y}(t,x)$ solves (2.4) on $[T-\gamma_K, T]\times\mathbb{R}^n$. Further, by (2.8), (2.21), (2.28), and by equation (2.4) itself, we conclude that, a.s., $\bar{y}\in \mathrm{C}_b^{1,2}$. Finally, as we have already mentioned in Step 1, $\bar{y}$ is $\mathcal{F}_{T-\tau}$-adapted for each $x\in\mathbb{R}^n$. The theorem is proved. ☐

2.3. Gradient estimate. In this section, we present an FBSDE stochastic method to obtain a uniform in r bound for the gradient $\partial_x y(t,x)$ of the solution $y(t,x)$ to the following final value problem:

$$\begin{cases}\partial_t y(t,x) + \frac{1}{2}\mathrm{tr}(\partial^2_{xx} y(t,x)\sigma(t,x)\sigma(t,x)^\top) \\ +(\varphi(t,x,y(t,x)),\partial_x)y(t,x) + f(t,x,y(t,x),\partial_x y(t,x)\sigma(t,x,y)) = 0, \\ y(T,x) = h(x), \quad x\in\mathbb{R}^n,\ t\in[r,T],\ r\geqslant 0.\end{cases} \tag{2.31}$$

Here $\sigma(t,x)^\top$ is the transpose to the matrix σ, $\mathrm{tr}(\partial^2_{xx} y(t,x)\sigma(t,x)\sigma(t,x)^\top)$ is the vector whose l-th component is the trace of the matrix $\partial^2_{xx} y_l(t,x)\sigma(t,x)\sigma(t,x)^\top$, where $y_l(t,x)$ is the l-th component of $y(t,x)$, and $(\varphi(t,x,y(t,x)),\partial_x)$ is the formal scalar product of φ and the vector ∂_x with the coordinates $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$. Equation (2.31) is assumed to be $\mathbb{R}^m$-valued, $\sigma(t,x)$, $\varphi(t,x,y)$, and $f(t,x,y,z)$ take values in $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$, and $\mathbb{R}^m$, respectively, and the arguments of these functions are of appropriate dimensions.

It is well known that the FBSDE associated to (2.31) takes the form (see e.g. [15])

$$\begin{cases} X_t^{\tau,x} = x + \int_\tau^t \varphi(s, X_s^{\tau,x}, Y_s^{\tau,x})ds + \int_\tau^t \sigma(s, X_s^{\tau,x})dW_s, \\ Y_t^{\tau,x} = h(X_T^{\tau,x}) + \int_t^T f(s, X_s^{\tau,x}, Y_s^{\tau,x}, Z_s^{\tau,x})ds - \int_t^T Z_s^{\tau,x} dW_s,\end{cases} \tag{2.32}$$

where $\tau\in[r,T]$, W_t is an n-dimensional Brownian motion.

Consider a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and for each fixed $\tau\in[0,T]$, define the filtration $\mathcal{F}^W_{\tau,t} = \sigma\{W_s - W_\tau, s\in[\tau,t]\}\vee\mathcal{N}$, where $\mathcal{N}$ is the collection of $\mathbb{P}$-null sets. The solution $(X_t^{\tau,x}, Y_t^{\tau,x}, Z_t^{\tau,x})$ to (2.32) is understood in the same way as in [15].

In the remainder of this section, we make use of the following assumptions.

(B1) The functions f, φ, σ, and h, are differentiable with respect to their spatial variables; the derivatives $\partial_x\sigma$ and ∇h are bounded by a constant K, and the other derivatives satisfy the linear growth condition on $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n}$:

$$|\partial_{(x,y)}\varphi| + |\partial_{(x,y,z)} f| \leqslant K(1+|y|).$$

(B2) Assume there exists a constant $L>0$ such that for all $(t,x,y,z)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times n}$,

$$|h(x)| + |\sigma(t,x)| \leqslant L; \quad |\varphi(t,x,y)| \leqslant L(1+|x|+|y|);$$
$$|f(t,x,y,z)| \leqslant L(1+|y|+|z|).$$

(B3) Finally, assume there exists a constant $\lambda > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and $\zeta \in \mathbb{R}^n$,

$$(\sigma(t,x)\sigma(t,x)^\top \zeta, \zeta) > \lambda |\zeta|^2.$$

Lemma 2.9. *Assume $y(t,x)$ is a $\mathrm{C}_b^{1,2}([0,T] \times \mathbb{R}^n)$-solution to final value problem* (2.31) *on $[r,T] \times \mathbb{R}^n$. Then, for any $\tau \in [r,T]$,*

$$\big(X_t^{\tau,x}, y(t, X_t^{\tau,x}), \partial_x y(t, X_t^{\tau,x})\sigma(t, X_t^{\tau,x})\big) \tag{2.33}$$

is a solution to FBSDE (2.32) *on $[\tau, T]$.*

Proof. The existence and uniqueness of solution to the SDE

$$X_t^{\tau,x} = x + \int_\tau^t \varphi(s, X_s^{\tau,x}, y(s, X_s^{\tau,x}))ds + \int_\tau^t \sigma(s, X_s^{\tau,x})\, dW_s \tag{2.34}$$

is a classical result under (B1) and (B2).

Now assume that $(X_t^{\tau,x}, Y_t^{\tau,x}, Z_t^{\tau,x})$ is given by (2.33). Then, the forward SDE in (2.32) is satisfied. Applying Itô's formula to $y(t, X_t^{\tau,x})$ at times t and T, we can easily check that the above triple verifies the backward SDE in (2.32). $\square$

Our main result in this subsection is the following.

Theorem 2.10. *Assume (B1)–(B3). Further assume that $y(t,x)$ is a $\mathrm{C}_b^{1,2}$-solution to final value problem* (2.31) *on $[r,T] \times \mathbb{R}^n$. Then, there exists a constant $\gamma_{T,K,L,\lambda}$, that depends only on T, K, L, and λ, such that for all $(x,t) \in \mathbb{R}^n \times [r,T]$,*

$$|\partial_x y(t,x)| \leqslant \gamma_{T,K,L,\lambda}. \tag{2.35}$$

In particular, the constant $\gamma_{T,K,L,\lambda}$ does not depend on r.

Proof. Everywhere throughout the proof, $\gamma_{\mathcal{A}}^{(i)}$, $i = 1, 2, \ldots$, will denote constants depending only on the set of parameters $\mathcal{A}$.

Step 1. Boundedness of $y(t,x)$. Let $(X_t^{\tau,x}, Y_t^{\tau,x}, Z_t^{\tau,x})$ be the solution to (2.32) on $[\tau, T]$ given by (2.33). For simplicity of notations, in what follows, we skip the upper index τ, x using it just where it is necessary.

Itô's formula and the backward SDE in (2.32) imply

$$\mathbb{E}|Y_t|^2 + \int_t^T \mathbb{E}|Z_s|^2 ds = \mathbb{E}|h(X_T)|^2 + \mathbb{E}\int_t^T 2(f(s, X_s, Y_s, Z_s), Y_s) ds. \tag{2.36}$$

By Assumption (B2), there exists a constant $\gamma_L^{(1)}$ such that

$$\mathbb{E}|Y_t|^2 + \int_t^T \mathbb{E}|Z_s|^2 ds \leqslant L^2 + \gamma_L^{(1)} \int_t^T \mathbb{E}|Y_s|^2 ds + \frac{1}{2}\int_t^T \mathbb{E}|Z_s|^2 ds.$$

By Gronwall's inequality, for all $t \in [\tau, T]$,

$$\mathbb{E}|Y_t|^2 \leqslant \gamma_{L,T}^{(2)}.$$

Since $Y_t^{\tau,x} = y(t, X_t^{\tau,x})$, where $X_t^{\tau,x}$ is the unique solution to (2.34), then

$$|y(\tau, x)| \leqslant M_{L,T}, \tag{2.37}$$

where $M_{L,T}$ is a constant that depends only on L and T.

Step 2. Transformation of the PDE. Rewrite PDE (2.31) with respect to

$$\tilde{y}(t,x) = \frac{1}{\alpha}\, y(t,x), \tag{2.38}$$

where $\alpha = 3M_{L,T}$. We obtain

$$\begin{cases} \partial_t \tilde{y}(t,x) + \frac{1}{2}\mathrm{tr}(\partial^2_{xx}\tilde{y}(t,x)(\sigma\sigma^\top)(t,x)) + \big(\varphi(t,x,\alpha\,\tilde{y}(t,x)),\partial_x\big)\tilde{y}(t,x) \\ +\frac{1}{\alpha} f\big(t,x,\alpha y(t,x),\alpha\,\partial_x\tilde{y}(t,x)\sigma(t,x)\big) = 0, \\ \tilde{y}(T,x) = \frac{1}{\alpha}h(x). \end{cases} \tag{2.39}$$

Let X_t be the solution to SDE (2.40) below

$$X_t = x + \int_\tau^t \varphi(s,X_s,\alpha\tilde{y}(s,X_s))ds + \int_\tau^t \sigma(s,X_s)dW_s. \tag{2.40}$$

By Lemma 2.9, the triple

$$X_t, \quad Y_t = \tilde{y}(t,X_t), \quad Z_t = \partial_x\tilde{y}(t,X_t)\sigma(t,X_t) \tag{2.41}$$

is the solution to the associated FBSDE

$$\begin{cases} X_t = x + \int_\tau^t \varphi(s,X_s,\alpha Y_s)ds + \int_\tau^t \sigma(s,X_s)dW_s, \\ Y_t = \frac{1}{\alpha}h(X_T) + \int_t^T \frac{1}{\alpha} f(s,X_s,\alpha Y_s,\alpha Z_s)ds - \int_t^T Z_s dW_s. \end{cases} \tag{2.42}$$

Although the solution triple, defined by (2.41), is different than the triple defined by (2.33)–(2.34), we denote it again by (X_t, Y_t, Z_t) for simplicity of notation.

Step 3. Boundedness of $\mathbb{E}\exp\Big\{\frac{\lambda}{4}\int_\tau^T |\nabla\tilde{y}(s,X_s)|^2 ds\Big\}$. Note that (2.37) and (2.38) imply that $|\tilde{y}(\tau,x)| \leqslant \frac{1}{3}$ for all $\tau \in [r,T]$ by the choice of α, and, therefore, by (2.41),

$$|Y_t| \leqslant \frac{1}{3} \quad \text{for all } t \in [\tau,T] \quad \text{a.s.} \tag{2.43}$$

By Itô's product formula and (2.42), we obtain

$$\begin{aligned} |Y_t|^2 + \int_t^T |Z_s|^2 ds &= \frac{1}{\alpha^2}|h(X_T)|^2 + 2\int_t^T (\frac{1}{\alpha} f(s,X_s,\alpha Y_s,\alpha Z_s),Y_s)ds \\ &+ 2\int_t^T (Y_s, Z_s dW_s) \leqslant \gamma^{(3)}_{L,\alpha}\Big(1 + \int_t^T |Y|_s\, ds + \int_t^T |Y|^2_s\, ds\Big) + \frac{1}{2}\int_t^T |Z_s|^2 ds \\ &+ 2\int_t^T (Y_s, Z_s dW_s). \end{aligned}$$

By (2.43), there exists a constant $\gamma^{(4)}_{L,T}$ such that

$$\frac{1}{2}\int_t^T |Z_s|^2 ds \leqslant \gamma^{(4)}_{L,T} + 2\int_t^T (Y_s, Z_s dW_s).$$

This implies

$$\begin{aligned} \exp\Big\{\frac{1}{2}\int_t^T |Z_s|^2 ds\Big\} &\leqslant \gamma^{(5)}_{L,T}\, \exp\Big\{2\int_t^T (Y_s, Z_s dW_s) - 2\sum_{i=1}^n \int_t^T (Y_s, Z^i_s)^2 ds\Big\} \\ &\times \exp\Big\{\frac{2}{9}\int_t^T |Z_s|^2 ds\Big\}. \end{aligned}$$

Therefore,

$$\exp\Big\{\frac{1}{4}\int_t^T |Z_s|^2 ds\Big\} \leqslant \gamma^{(5)}_{L,T}\, \exp\Big\{2\int_t^T (Y_s, Z_s dW_s) - 2\sum_{i=1}^n \int_t^T (Y_s, Z_s^i)^2 ds\Big\}. \tag{2.44}$$

Note that on the right-hand side we have a Doléans-Dade exponential of a martingale considered as a process with respect to T while t is fixed. Indeed, by (B2) and (2.41), the Novikov condition $\mathbb{E}\big[\exp\{\sum_{i=1}^n \int_t^T (Y_s, Z_s^i)^2 ds\}\big] < \infty$ is fulfilled. Therefore, the expectation of the exponential on the right-hand side of (2.44) equals to one. Finally, representation (2.41) for Z_s and (B3) imply

$$\mathbb{E}\exp\Big\{\frac{\lambda}{4}\int_\tau^T |\nabla\tilde{y}(s, X_s)|^2 ds\Big\} \leqslant \gamma^{(5)}_{L,T}. \tag{2.45}$$

Step 4. Obtaining an a priori bound for $\partial_x y(t,x)$. Since any solution to the final value problem (2.31) is bounded by $M_{T,L}$, introduce $\hat{\varphi}$ and $\hat{f}$ as follows

$$\hat{\varphi}(t,x,y) = \varphi(t,x,y\xi_{M_{T,L}}(y)) \quad \text{and} \quad \hat{f}(t,x,y,z) = f(t,x,y\xi_{M_{T,L}}(y),z),$$

where $\xi_{M_{T,L}}(y)$ is a C^∞-cutting function for the ball $B_{M_{T,L}}$ introduced in Remark 2.2. Note that by (B1), $\hat{\varphi}$ and $\hat{f}$ possess bounded derivatives w.r.t. the spacial variables. Let $\gamma^{(6)}_{K,L,T}$ be the common bound for these spatial derivatives. This bound depends on K, and on T, L via the constant $M_{T,L}$. Observe that the solution (X_t, Y_t, Z_t) to FBSDE (2.42), given by (2.41), is also a solution to

$$\begin{cases} X_t = x + \int_\tau^t \hat{\varphi}(s, X_s, \alpha Y_s) ds + \int_\tau^t \sigma(s, X_s) dW_s, \\ Y_t = \frac{1}{\alpha} h(X_T) + \int_t^T \frac{1}{\alpha}\hat{f}(s, X_s, \alpha Y_s, \alpha Z_s) ds - \int_t^T Z_s dW_s. \end{cases} \tag{2.46}$$

Let $(\partial_x X_s, \partial_x Y_s, \partial_x Z_s)$ denote the derivative of the solution to FBSDE (2.46) w.r.t. the initial data x. Further, for the function $\hat{f}(t,x,y,z)$, $\nabla_2\hat{f} = \partial_x\hat{f}$, $\nabla_3\hat{f} = \partial_y\hat{f}$, and $\nabla_4\hat{f} = \partial_z\hat{f}$. For the function $\hat{\varphi}$, the derivatives ∇_2 and ∇_3 are defined similarly. In case of just one spatial variable, as in the function σ, we skip the index 2. Remark that under (B1)-(B2), the differentiability of the solution $X_t^{\tau,x}$ to SDE (2.34) is well known and the derivative process satisfies

$$\partial_x X_t = I + \int_\tau^t \nabla\tilde{\varphi}_s \partial_x X_s ds + \int_\tau^t \nabla\sigma_s \partial_x X_s dW_s,$$

where

$$\tilde{\varphi}(t,x) = \hat{\varphi}(t,x,\alpha\tilde{y}(t,x)),$$

$\tilde{\varphi}_s$ and σ_s are abbreviations for $\tilde{\varphi}(s, X_s)$ and $\sigma(s, X_s)$, respectively. An application of Itô's formula gives

$$|\partial_x X_t|^2 = 1 + 2\int_\tau^t (\nabla\tilde{\varphi}_s \partial_x X_s, \partial_x X_s) ds + 2\sum_{k=1}^n \int_\tau^t (\nabla\sigma_s^k \partial_x X_s, \partial_x X_s) dW_s^k + \sum_{k=1}^n \int_\tau^t |\nabla\sigma_s^k \partial_x X_s|^2 ds, \tag{2.47}$$

where $\sigma_s^k = (\sigma_s, e_k)$. Define

$$\vartheta_s = \begin{cases} \frac{\partial_x X_s}{|\partial_x X_s|} & \text{if } \partial_x X_s \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Equation (2.47) becomes

$$|\partial_x X_t|^2 = 1 + 2\int_\tau^t (\nabla\tilde\varphi_s \vartheta_s, \vartheta_s)|\partial_x X_s|^2 ds + \sum_{k=1}^n \int_\tau^t |\nabla \sigma_s^k \vartheta_s|^2 |\partial_x X_s|^2 ds$$
$$+ 2\sum_{k=1}^n \int_\tau^t (\nabla\sigma_s^k \vartheta_s, \vartheta_s)|\partial_x X_s|^2 dW_s^k.$$

This implies the following representation for $|\partial_x X_t|^2$ via the Doléans-Dade exponential:

$$|\partial_x X_t|^2 = e^{-1} \exp\Big\{ \int_\tau^t \big[2(\nabla\tilde\varphi_s\vartheta_s, \vartheta_s) + \sum_{k=1}^n \big(|\nabla\sigma_s^k\vartheta_s|^2 + 2(\nabla\sigma_s^k\vartheta_s, \vartheta_s)^2\big)\big] ds \Big\}$$
$$\times \exp\Big\{ 2\sum_{k=1}^n \int_\tau^t (\nabla\sigma_s^k\vartheta_s, \vartheta_s) dW_s^k - 4\sum_{k=1}^n \int_\tau^t (\nabla\sigma_s^k\vartheta_s, \vartheta_s)^2 ds \Big\}.$$

In the above expression, the term $2\sum_{k=1}^n \int_\tau^t (\nabla\sigma_s^k\vartheta_s, \vartheta_s)^2 ds$ was added and subtracted so we could get the estimate

$$|\partial_x X_t|^2 \leqslant \exp\Big\{ 2\int_\tau^t \big(2|\nabla\tilde\varphi_s| + 3|\nabla\sigma_s|^2\big) ds \Big\} \tag{2.48}$$
$$+ \exp\Big\{ 4\sum_{k=1}^n \int_\tau^t (\nabla\sigma_s^k\vartheta_s, \vartheta_s) dW_s^k - 8\sum_{k=1}^n \int_\tau^t (\nabla\sigma_s^k\vartheta_s, \vartheta_s)^2 ds \Big\}.$$

Since $\nabla\tilde\varphi(t,x) = \nabla_2\hat\varphi(t,x,\alpha\tilde y(t,x)) + \alpha\nabla_3\hat\varphi(t,x,\alpha\tilde y(t,x))\partial_x\tilde y(t,x)$,

$$|\nabla\tilde\varphi_s| \leqslant \gamma_{K,L,T}^{(6)}(1 + \alpha|\nabla\tilde y(s,X_s)|) \leqslant \gamma_{K,L,T}^{(6)} + \frac{4(\gamma_{K,L,T}^{(6)})^2\alpha^2}{\lambda} + \frac{\lambda}{16}|\nabla\tilde y(s,X_s)|^2.$$

Taking the expectation of the both parts of (2.48), we obtain

$$\mathbb{E}|\partial_x X_t|^2 \leqslant \gamma_{K,L,T,\lambda}^{(7)}\, \mathbb{E}\exp\Big\{\frac{\lambda}{4}\int_\tau^T \big|\nabla\tilde y(s,X_s)\big|^2 ds\Big\} + 1 \leqslant \gamma_{K,L,T,\lambda}^{(8)}, \tag{2.49}$$

where the last inequality holds by (2.45).

Further, let us estimate $\mathbb{E}|\partial_x Y_t|^2$. Applying Itô's product formula and using the backward SDE in (2.46), we obtain that

$$\mathbb{E}|\partial_x Y_t|^2 + \int_t^T \mathbb{E}|\partial_x Z_s|^2 ds = \frac{1}{\alpha^2}\mathbb{E}|\nabla h_T \partial_x X_T|^2$$
$$+ 2\int_t^T \mathbb{E}\big(\frac{1}{\alpha}\nabla_2\hat f_s\partial_x X_s + \nabla_3\hat f_s\partial_x Y_s + \nabla_4\hat f_s\partial_x Z_s, \partial_x Y_s\big) ds \leqslant \gamma_{K,T,L}^{(9)}\Big(\mathbb{E}|\partial_x X_T|^2$$
$$+ \int_t^T \mathbb{E}|\partial_x X_s|^2 ds + \int_t^T \mathbb{E}|\partial_x Y_s|^2 ds\Big) + \frac{1}{2}\int_t^T \mathbb{E}|\partial_x Z_s|^2 ds. \tag{2.50}$$

By (2.49) and Gronwall's inequality,

$$\mathbb{E}|\partial_x Y_t|^2 \leqslant \gamma^{(10)}_{K,L,T,\lambda}.$$

Evaluating at $t = \tau$, and taking into account that $\tilde{y}$ and y are related by the formula $y(t,x) = \alpha\tilde{y}(t,x)$, we obtain the final estimate, i.e., there exists a constant $\gamma_{K,L,T,\lambda}$ such that

$$|\partial_x y(\tau,x)| \leqslant \gamma_{K,L,T,\lambda}.$$

The theorem is proved. □

2.4. Global existence. We start with a lemma on the uniqueness of a $\mathrm{C}^{1,2}_b$-solution to Cauchy problem (2.2).

Lemma 2.11. *Assume (A1)–(A3). Then, problem* (2.2) *can have at most one pathwise* $\mathrm{C}^{1,2}_b([0,T]\times\mathbb{R}^n)$*-solution on* $[0,T]$.

Proof. Assume there are two solutions $y_1, y_2 \in \mathrm{C}^{1,2}_b([0,T]\times\mathbb{R}^n)$ to problem (2.2), and let $y = y_1 - y_2$. Then, $y(t,x)$ solves the problem

$$\begin{cases} \partial_t y(t,x) = \nu\Delta y(t,x) - (\eta(t,x) + y_1, \nabla)y(t,x) \\ \quad +\big(\Phi(t,x) + \partial_x y_2\big)y(t,x) = 0, \qquad y(0,x) = 0, \end{cases} \tag{2.51}$$

where $\Phi(t,x) = \int_0^1 \partial_y F(t,x,\lambda y_1 + (1-\lambda)y_2)d\lambda$. Then, $y(t,x) = 0$ since we can express $y(t,x)$ via the fundamental solution to (2.51). □

Let us proceed with the global existence. Define the sequence of stopping times

$$T_N = T \wedge \inf\big\{t \in (0,T] :\, \|\eta(t,\,\cdot\,)\|_{\mathrm{C}^4_b(\mathbb{R}^n)} > N\big\}, \tag{2.52}$$

where $N > 0$ is an integer. Note that since $\eta \in \mathrm{C}^{0,4}_b([0,T]\times\mathbb{R}^n)$ on Ω_0, then the stopping time T_N is non-zero on Ω_0. Furthermore, we define

$$\eta_N(t,x) = \eta(t\wedge T_N, x) \quad \text{and} \quad h_N(x) = h(x)\,\mathbb{I}_{\{\|h\|_{\mathrm{C}^2_b(\mathbb{R}^n)}\leqslant N\}}. \tag{2.53}$$

Note that for each $\omega \in \Omega_0$, $\|\eta_N\|_{\mathrm{C}^{0,4}_b([0,T]\times\mathbb{R}^n)} \leqslant N$.

The existence and uniqueness of a global solution to (1.1) is case $\eta = \eta_N$ is given by Lemma 2.12 below.

Lemma 2.12. *Let (A1)–(A3) hold. Then, there exists a unique $\mathcal{F}_t$-adapted* $\mathrm{C}^{0,2}_b$*-solution to*

$$y(t,x) = h_N(x) + \int_0^t \big[f(s,x,y) - (y,\nabla)y(s,x) + \nu\Delta y(s,x)\big]ds + \eta_N(t,x). \tag{2.54}$$

Proof. Define $F_N(t,x,y)$ by (2.3) via η_N. Then, $|F_N(t,x,y)| + |\nabla_{(x,y)}F_N(t,x,y)| + |\nabla^2_{(x,y)}F_N(t,x,y)| \leqslant K_N(1+|y|)$, where $K_N > N$ is a deterministic constant depending only on N. Consider the backward equation associated to (2.54) by means of substitution (2.1) and the time change:

$$\bar{y}(t,x) = h_N(x) + \int_t^T \big[\nu\Delta\bar{y}(s,x) - (\bar{\eta}_N(t,x) + \bar{y}, \nabla)\bar{y}(s,x) + \bar{F}_N(s,x,\bar{y})\big]ds. \tag{2.55}$$

Here $\bar{F}_N(t,x,y) = F_N(T-t,x,y)$ and $\bar{\eta}_N(t,x) = \eta_N(T-t,x)$.

By Theorem 2.8, on a deterministic interval $[T-\gamma_{K_N}, T]$, where γ_{K_N} is the small constant defined by Theorem 2.8, there exists an $\mathcal{F}_{T-t}$-adapted $\mathrm{C}_b^{1,2}$-solution $\bar{y}_N(t,x)$ to equation (2.55). Then, $y_N(t,x) = \bar{y}_N(T-t,x) + \eta_N(t,x)$ is an $\mathcal{F}_t$-adapted $\mathrm{C}_b^{0,2}$-solution to (2.54) which exists on some set $\Omega_N \subset \Omega_0$, $\mathbb{P}(\Omega_N) = 1$. Remark that for each $\omega \in \Omega_N$, $\bar{y}_N(t,x,\omega)$ is also a pathwise solution to (2.55). By Theorem 2.10, $\partial_x \bar{y}_N(t,x,\omega)$ is bounded by a constant $\mu_{K_N,T}$ depending only on K_N and T but not depending on the length of the time interval γ_{K_N}. Further remark that $\mu_{K_N,T}$ is the same for all $\omega \in \Omega_N$.

Now take $t_1 = \gamma_{K_N}$ and consider the equation

$$y(t,x) = y_N(t_1,x) + \int_{t_1}^{t} \big[f(s,x,y) - (y,\nabla)y(s,x) + \nu\Delta y(s,x)\big]ds \\ + \eta_N(t,x) - \eta_N(t_1,x). \quad (2.56)$$

Note that $\mathcal{F}_t = \sigma\{B_s, s \in [t_1,t]\} \vee \mathcal{F}_{t_1}$ and $y_N(t_1,x)$ is $\mathcal{F}_{t_1}$-measurable. Further, by what was proved, $\partial_x y_N(t_1,x)$ is bounded by $\mu_{K_N,T}$. Hence, by Theorem 2.8, there exists a constant γ'_{K_N} such that on the time interval $[t_1, t_1+\gamma'_{K_N}]$, there exists a $\mathrm{C}_b^{0,2}$-solution to (2.56). Furthermore, for each $t \in [t_1, t_1+\gamma'_{K_N}]$, this solution is $\mathcal{F}_t$-adapted. In the similar manner, a $\mathrm{C}_b^{0,2}$-solution to (2.54) can be built on the next successive interval $[t_2, t_2+\gamma'_{K_N}]$, where $t_2 = \gamma_{K_N} + \gamma'_{K_N}$. It is important to mention that the initial condition on each short-time interval has a bounded derivative in x (by the constant $\mu_{K_N,T}$) by Theorem 2.10. By glueing the solutions on short-time intervals, we obtain a $\mathrm{C}_b^{0,2}$-solution to (2.54) on $[0,T]$. Remark that this solution is unique by Lemma 2.11 since (2.54) can be reduced to equation of type (2.2) by substitution (2.1). □

The main result of this work is Theorem 2.13 below which gives the existence of an $\mathcal{F}_t$-adapted $\mathrm{C}_b^{0,2}$-solution to equation (1.1).

Theorem 2.13. *Assume (A1)–(A3). Then, there exists a unique $\mathrm{C}_b^{0,2}$-solution to equation* (1.1) *which is $\mathcal{F}_t$-adapted for each $x \in \mathbb{R}^n$.*

Proof. Consider equation (1.1) for a fixed $\omega_0 \in \cap_N \Omega_N$, where Ω_N is the set of ω, where y_N solves (2.54), i.e., we regard (1.1) as a deterministic equation. Then, $\eta(t,x,\omega_0)$ can be regarded as a bounded function in t and x. Applying Lemma 2.12, to deterministic equation (1.1), we obtain the existence and uniqueness of a $\mathrm{C}_b^{0,2}$-solution $y(t,x,\omega_0)$. Pick an integer $N > 0$ such that $\|h(\,\cdot\,,\omega_0)\|_{\mathrm{C}_b^2(\mathbb{R}^n)} \leqslant N$. Then, $h(\,\cdot\,,\omega_0) = h_N(\,\cdot\,,\omega_0)$. Further note that on $[0, T_N(\omega_0)]$, equations (1.1) and (2.54) coincide. By Lemma 2.11, $y_N(t,x,\omega_0) = y(t,x,\omega_0)$ on $[0,T_N(\omega_0)]$. Since $T_N(\omega_0) \to T$ as $N \to \infty$, then $y_N(t,x,\omega_0) \to y(t,x,\omega_0)$. This is valid for any $\omega_0 \in \cap_N \Omega_N$. Therefore, $y(t,x,\omega)$ is $\mathcal{F}_t$-adapted. □

3. Vanishing viscosity limit

Here we investigate the behavior of the solution to (1.1) when the viscosity ν goes to zero. Throughout this section, the C_b^2-norm of the function $h(x)$ is assumed bounded in ω. At first, we assume that $\eta(t,x) = \eta_N(t,x)$, where $\eta_N(t,x)$ is defined by (2.53). This will allow us to prove that the local vanishing viscosity limit for equation (2.2) exists on $[0,\gamma_{K_N}]$, where γ_{K_N} is defined in the proof of Lemma 2.12.

In what follows, β_i, $i=1,2,\dots$, denote positive constants, and $\mathbb{E}_\tau$ denote the conditional expectation with respect to $\mathcal{F}_{T-\tau}$.

Lemma 3.1. *Assume (A1)–(A3). Further assume that $\eta=\eta_N$, and $\|h\|_{\mathrm{C}_b^2}$ is bounded in $\omega\in\Omega_0$. Then, for all $\omega\in\Omega_0$, the system of forward-backward random equations*

$$\begin{cases} X_t^{\tau,x,0} = x - \int_\tau^t \big(\bar\eta(s,X_s^{\tau,x,0}) + Y_s^{\tau,x,0}\big)ds, \\ Y_t^{\tau,x,0} = h(X_T^{\tau,x,0}) + \int_t^T \bar F(s,X_s^{\tau,x,0},Y_s^{\tau,x,0})\,ds \end{cases} \tag{3.1}$$

possesses a unique solution $(X_t^{\tau,x,0},Y_t^{\tau,x,0})$ on $[T-\gamma_{K_N},T]$ which is continuous in (τ,x,t).

Proof. Forward-backward system (3.1) is a particular case of FBSDE (2.5). Therefore, if $T-\tau<\gamma_{K_N}$, then (3.1) has a unique solution $(X_t^{\tau,x,0},Y_t^{\tau,x,0})$ for each fixed $\omega\in\Omega_0$. Further, the uniform boundedness of $Y_t^{\tau,x,0}$ is a direct consequence of the backward equation in (3.1) and Gronwall's inequality. Furthermore, (2.11) can be proved for (3.1) pathwise and without involving expectations. This implies the uniform in $t\in[T-\gamma_{K_N},T]$ continuity of the solution $(X_t^{\tau,x,0},Y_t^{\tau,x,0})$ in (τ,x) (as before, it is assumed that $(X_t^{\tau,x,0},Y_t^{\tau,x,0})$ is extended to $[T-\gamma_{K_N},\tau]$ by $(x,Y_\tau^{\tau,x,0})$). Therefore, the solution $(X_t^{\tau,x,0},Y_t^{\tau,x,0})$ is continuous in $(\tau,x,t)\in[T-\gamma_{K_N},T]\times\mathbb{R}^n\times[T-\gamma_{K_N},T]$. □

For each $(t,x,\omega)\in[T-\gamma_{K_N},T]\times\mathbb{R}^n\times\Omega_0$, we define

$$\bar y_0(t,x) = Y_t^{t,x,0}. \tag{3.2}$$

Let for any viscosity $\nu\in(0,\nu_0]$, where $\nu_0>0$ is a fixed parameter, $\bar y_\nu(t,x)$ denote the unique $\mathcal{F}_{T-t}$-adapted $\mathrm{C}_b^{1,2}$-solution to (2.4). In the lemma below, we will treat ν as a "time" parameter and $\bar y_{\cdot}:[0,\nu_0]\times\Omega\to\mathrm{C}_b([T-\gamma_{K_N},T]\times\mathbb{R}^n)$, $(\nu,\omega)\mapsto\bar y_\nu(\,\cdot\,,\,\cdot\,)$, as a stochastic process with values in $\mathrm{C}_b([T-\gamma_{K_N},T]\times\mathbb{R}^n)$.

Lemma 3.2. *Under assumptions of Lemma 3.1, there exists a constant $\dot\gamma_{K_N}<\gamma_{K_N}$ such that there is a continuous version of*

$$\bar y_{\cdot}:[0,\nu_0]\times\Omega\to\mathrm{C}_b([T-\dot\gamma_{K_N},T]\times\mathbb{R}^n),\quad (\nu,\omega)\mapsto\bar y_\nu(\,\cdot\,,\,\cdot\,). \tag{3.3}$$

Proof. Let $(X_t^{\tau,x,\nu},Y_t^{\tau,x,\nu},Z_t^{\tau,x,\nu})$ be the solution to (2.5) associated to $\nu\in(0,\nu_0]$. As before, sometimes we skip the upper index (τ,x) (but keep ν). We have

$$\begin{cases} X_t^\nu - X_t^{\bar\nu} = \int_\tau^t \big[\bar\eta(s,X_s^\nu)-\bar\eta(s,X_s^{\bar\nu}) + Y_s^\nu - Y_s^{\bar\nu}\big]ds \\ \qquad\qquad\qquad + (\sqrt{2\nu}-\sqrt{2\bar\nu})(W_t-W_\tau), \\ Y_t^\nu - Y_t^{\bar\nu} = h(X_T^\nu)-h(X_T^{\bar\nu}) + \int_t^T(\bar F_\delta(s,X_s^\nu,Y_s^\nu)-\bar F_\delta(s,X_s^{\bar\nu},Y_s^{\bar\nu}))\,ds \\ \qquad\qquad\qquad - \int_t^T(Z_s^\nu - Z_s^{\bar\nu})dW_s, \end{cases} \tag{3.4}$$

where $\bar F_\delta$ is defined by (2.9). Note that $Z_t^{\tau,x,0}=0$. By Gronwall's inequality, the forward SDE implies that a.s.

$$\mathbb{E}_\tau|X_t^\nu - X_t^{\bar\nu}|^2 \leqslant \beta_1\big[(T-\tau)^2\mathbb{E}_\tau|Y_t^\nu - Y_t^{\bar\nu}|^2ds + (T-\tau)|\nu-\bar\nu|\big]. \tag{3.5}$$

Itô's formula applied to the BSDE in (3.4) gives

$$\mathbb{E}_\tau|Y_t^\nu - Y_t^{\bar\nu}|^2 \leqslant \mathbb{E}_\tau|h(X_T^\nu) - h(X_T^{\bar\nu})|^2 + 2\mathbb{E}_\tau\int_t^T (\bar F_\delta(s,X_s^\nu,Y_s^\nu) - \bar F_\delta(s,X_s^{\bar\nu},Y_s^{\bar\nu}), Y_s^\nu - Y_s^{\bar\nu})\big]ds \quad \text{a.s.} \tag{3.6}$$

From (3.5) and (3.6) it follows that there exists a positive constant $\dot\gamma_{K_N} < \gamma_{K_N}$ such that for each fixed ν and $\bar\nu$, a.s.,

$$|y_\nu(\tau,x) - y_{\bar\nu}(\tau,x)| \leqslant \beta_2|\nu - \bar\nu| \quad \text{for all}\, x\in\mathbb{R}^n,\ \tau\in[T-\dot\gamma_{K_N},T]. \tag{3.7}$$

Remark that since for each fixed ν and $\bar\nu$, $Y_\tau^{\tau,x,\nu}$ and $Y^{\tau,x,\bar\nu}$ possess (τ,x)-continuous modifications, (3.7) holds on a set of full $\mathbb{P}$-measure that does not depend on τ and x. Further remark that the constant β_2 on the right-hand side of (3.7) does not depend on τ and x. Therefore, for an integer $p>1$,

$$\mathbb{E}\sup_{x\in\mathbb{R}^n,\tau\in[T-\dot\gamma_{K_N},T]}|y_\nu(\tau,x)-y_{\bar\nu}(\tau,x)|^{2p} \leqslant \beta_3|\nu-\bar\nu|^p. \tag{3.8}$$

By Kolmogorov's continuity theorem ([26], p. 31), there is an a.s. ν-continuous version of the stochastic process $\bar y_\cdot : [0,\nu_0]\times\Omega \to \mathrm{C}_b([T-\dot\gamma_{K_N},T]\times\mathbb{R}^n)$, $(\nu,\omega)\mapsto \bar y_\nu(\,\cdot\,,\,\cdot\,)$. □

Lemma 3.3 below states the existence of a local vanishing viscosity limit of equation (2.4) for $\eta = \eta_N$.

Lemma 3.3. *Let assumptions of Lemma 3.1 be fulfilled. Then, there exists a positive constant $\beta_{K_N} < \dot\gamma_{K_N}$ such that $\bar y_0(t,x)$, defined by* (3.2)*, is a $\mathrm{C}_b^{1,1}$-solution to equation* (2.55) *with $\nu=0$ on $[T-\beta_{K_N},T]$. Moreover, as $\nu\to 0$, a.s., $\bar y_\nu(t,x)\to \bar y_0(t,x)$ uniformly in $(x,t)\in\mathbb{R}^n\times[T-\beta_{K_N},T]$, where $\bar y_\nu$ is the ν-continuous version defined by* (3.3).

Proof. Let us prove that for each fixed $x\in\mathbb{R}^n$ and $\tau\in[T-\beta_{K_N},T]$, we can take a limit in (2.4) as $\nu\to 0$ in the space $L_2(\Omega)$, where β_{K_N} is an appropriate small constant. Note that the proof of differentiability of the FBSDE solution (Step 3 of the proof of Theorem 2.8) holds for the case $\nu=0$ (with $Z_t^{\tau,x,0}=0$). Therefore, $(X_t^{\tau,x,0},Y_t^{\tau,x,0})$ is differentiable in x, and $(\partial_k X_t^{\tau,x,0},\partial_k Y_t^{\tau,x,0},0)$ satisfies (2.22). The FBSDE for the triple $(\partial_k X_t^\nu - \partial_k X_t^0, \partial_k Y_t^\nu - \partial_k Y_t^0, \partial_k Z_t^\nu)$ takes the form

$$\begin{cases} \partial_k X_t^\nu - \partial_k X_t^0 = -\int_\tau^t \big(\nabla\bar\eta(s,X_s^0)(\partial_k X_s^\nu - \partial_k X_s^0) + \partial_k Y_s^\nu - \partial_k Y_s^0 + \xi_\nu^X(s)\big)ds \\ \partial_k Y_t^\nu - \partial_k Y_t^0 = \nabla h(X_T^0)(\partial_k X_T^\nu - \partial_k X_T^0) \\ \quad + \int_t^T \big[\nabla_2\bar F_\delta(s,X_s^0,Y_s^0)\dot(\partial_k X_s^\nu - \partial_k X_s^0) + \nabla_3\bar F_\delta(s,X_s^0,Y_s^0)(\partial_k Y_s^\nu - \partial_k Y_s^0) \\ \qquad\qquad + \xi_\nu^Y(s)\big]ds + \int_t^T \partial_k Z_s^\nu dW_s + \varsigma_{T,\nu}^Y, \end{cases} \tag{3.9}$$

where $\xi_\nu^X(s) = -\big(\nabla\bar\eta(s,X_s^\nu) - \nabla\bar\eta(s,X_s^0)\big)\partial_k X_s^\nu$, $\varsigma_{T,\nu}^Y = \big(\nabla h(X_T^\nu) - \nabla h(X_T^0)\big)\partial_k X_T^\nu$, $\xi_\nu^Y(s) = \big(\nabla_2\bar F_\delta(s,X_s^\nu,Y_s^\nu) - \nabla_2\bar F_\delta(s,X_s^0,Y_s^0)\big)\partial_k X_s^\nu + \big(\nabla_3\bar F_\delta(s,X_s^\nu,Y_s^\nu) - \nabla_3\bar F_\delta(s,X_s^0,Y_s^0)\big)\partial_k Y_s^\nu$. From (3.9), by standard arguments, we obtain that there exists a constant $\beta_{K_N} < \dot\gamma_{K_N}$ such that for all

$\tau \in [T-\beta_{K_N}, T]$, $x \in \mathbb{R}^n$, and $\nu > 0$, a.s.,

$$|\partial_k Y_\tau^{\tau,x,\nu} - \partial_k Y_\tau^{\tau,x,0}|^2 \leqslant \beta_4 \, \mathbb{E}_\tau \Big\{ \int_{T-\beta_{K_N}}^T \big(|\xi_\nu^X(s)|^2 + (|\xi_\nu^Y(s)|^2\big) ds + |\varsigma_{T,\nu}^Y|^2 \Big\}.$$

By what was proved, we can choose continuous versions of the maps $[T-\beta_{K_N}, T] \times \mathbb{R}^n \to \mathrm{C}([T-\beta_{K_N}, T])$, $(\tau, x) \mapsto \partial_k X^{\tau,x}$, $(\tau, x) \mapsto \partial_k Y^{\tau,x}$, $(\tau, x) \mapsto X^{\tau,x}$, $(\tau, x) \mapsto Y^{\tau,x}$, and of the map $(\tau, x) \mapsto Y_\tau^{\tau,x}$. Therefore, the above estimate holds on a set of full $\mathbb{P}$-measure that does not depend on τ and x. Hence,

$$\begin{aligned} &\mathbb{E} \sup_{x\in\mathbb{R}^n, \tau\in[T-\beta_{K_N}, T]} |\partial_x \bar{y}_\nu(\tau, x) - \partial_x \bar{y}_0(\tau, x)|^2 \\ &\quad \leqslant \beta_4 \, \mathbb{E}\Big\{ \sup_{\tau,x} \mathbb{E}_\tau \int_{T-\beta_{K_N}}^T \big(|\xi_\nu^X(s)|^2 + (|\xi_\nu^Y(s)|^2\big) ds + |\varsigma_{T,\nu}^Y|^2 \Big\} \to 0 \quad \text{as } \nu \to 0 \end{aligned}$$

by (2.17), (3.5), and (3.7). Further, by (2.8) and (2.21), the bounds for $\bar{y}_\nu(t,x)$ and $\partial_x \bar{y}_\nu(t,x)$ do not depend on $\nu \in (0, \nu_0]$. Therefore, as $\nu \to 0$,

$$\begin{aligned} &\mathbb{E} \sup_{x\in\mathbb{R}^n, \tau\in[T-\beta_{K_N}, T]} |(\bar{y}_\nu, \partial_x)\bar{y}_\nu(t,x) - (\bar{y}_0, \partial_x)\bar{y}_0(t,x)|^2 \\ &\leqslant \mathbb{E} \sup_{x\in\mathbb{R}^n, \tau\in[T-\beta_{K_N}, T]} \big(|(\bar{y}_\nu - \bar{y}_0), \partial_x)\bar{y}_\nu(t,x)|^2 + |(\bar{y}_0, \partial_x)(\bar{y}_\nu - \bar{y}_0)(t,x)|^2\big) \to 0. \end{aligned} \tag{3.10}$$

Finally, by (2.28), $\Delta \bar{y}_\nu(t,x)$ is bounded uniformly in $\nu \in (0, \nu_0]$ and $(t,x) \in [T-\beta_{K_N}, T] \times \mathbb{R}^n$. This implies that as $\nu \to 0$,

$$\nu \, \mathbb{E} \sup_{x\in\mathbb{R}^n, \tau\in[T-\beta_{K_N}, T]} |\Delta \bar{y}_\nu(t,x)|^2 \to 0. \tag{3.11}$$

Now equation (2.4), together with Lemma 3.2, (3.10), and (3.11) imply that, a.s., for all $(t,x) \in [T-\beta_{K_N}, T] \times \mathbb{R}^n$,

$$\bar{y}_0(t,x) = h(x) + \int_t^T \big[(\bar{y}_0, \nabla)\bar{y}_0(s,x) + \bar{F}_N(s, x, \bar{y}_0(t,x))\big] \, ds. \tag{3.12}$$

Further, by Lemma 3.2, for the ν-continuous version of the process $\bar{y}_\cdot : [0, \nu_0] \times \Omega \to \mathrm{C}([T-\beta_{K_N}, T] \times \mathbb{R}^n)$, $(\nu, \omega) \mapsto \bar{y}_\nu$, it holds that, a.s.,

$$\sup_{x\in\mathbb{R}^n, \tau\in[T-\beta_{K_N}, T]} |\bar{y}_\nu(\tau, x) - \bar{y}_0(\tau, x)| \to 0 \quad \text{as } \nu \to 0.$$

The lemma is proved. □

The following theorem is the main result of this section.

Theorem 3.4. *Assume (A1)–(A3). Further, we assume that $\|h\|_{\mathrm{C}_b^2}$ is bounded in $\omega \in \Omega_0$. Then, there exists a stopping time S, positive a.s., such that on $[0, S]$ there exists a $\mathrm{C}_b^{0,1}$-solution $y_0(t,x)$ to the inviscid stochastic Burgers equation*

$$y(t,x) = h(x) + \int_0^t \big[f(s,x,y) - (y, \nabla)y(s,x)\big] ds + \eta(t,x). \tag{3.13}$$

This solution is $\mathcal{F}_t$-adapted for each $x \in \mathbb{R}^n$. Moreover, if $\tilde{y}_0(t,x)$ is another $\mathrm{C}_b^{0,1}$-solution to (3.13) on $[0, \tilde{S}]$, where $\tilde{S}$ is a positive stopping time, then, a.s.,

$\tilde y_0(t,x) = y_0(t,x)$ on $[0, S\wedge \tilde S]$. Furthermore, if $y_\nu(t,x)$ is the $\mathrm{C}_b^{0,2}$-solution to (1.1) (whose existence has been established by Theorem 2.13), then there exists a ν-continuous version of $y_\cdot : [0,\nu_0]\times\Omega \to \mathrm{C}_b([0,S]\times\mathbb{R}^n)$, $(\nu,\omega)\mapsto y_\nu$. In particular, it holds that $\lim_{\nu\to 0} y_\nu(t,x) = y_0(t,x)$ a.s., where the limit is uniform in $(x,t)\in\mathbb{R}^n\times[0,S]$.

Proof. Let $\bar y_0^N$ be defined by (3.2) and associated to a positive integer N. As it was shown in the proof of Lemma 3.3, $\bar y_0^N$ is a $C_b^{1,1}$-solution to (3.12) on $[T-\beta_{K_N}, T]$. Therefore, $y_0^N(t,x) = \bar y_0^N(T-t,x) + \eta_N(t,x)$ is a $C_b^{0,1}$-solution to

$$y(t,x) = h(x) + \int_0^t \big[f(s,x,y) - (y,\nabla)y(s,x)\big]ds + \eta_N(t,x) \tag{3.14}$$

on $[0,\beta_{K_N}]$. Define $S = \beta_{K_N}\wedge T_N$, where T_N is given by (2.52). By Lemma 2.11, $y_\nu^N(t,x) = y_\nu(t,x)$ on $[0,S]$ for all $\nu\in(0,\nu_0]$, where $y_\nu(t,x)$ is the unique $\mathrm{C}_b^{0,2}$-solution to (1.1). Since, by Lemma 3.3, $\lim_{\nu\to 0} y_\nu^N(t,x) = y_0^N(t,x)$, a.s., in the space $\mathrm{C}_b([0,\beta_{K_N}]\times\mathbb{R}^n)$, then $y_0^N(t,x) = y_0(t,x)$ on $[0,S]$. Thus, we skip the index N when we consider this solution in $[0,S]$. Clearly, on $[0,S]$, $y_0(t,x)$ verifies (3.13) a.s.

Assume, equation (3.13) has another $\mathrm{C}_b^{0,1}$-solution $\tilde y_0(t,x)$ which verifies this equation on a random time interval $[0,\tilde S]$, where the stopping time $\tilde S$ is positive a.s. On $[T-\tilde S, T]$, we define $\check y_0(t,x) = \tilde y_0(T-t,x) - \eta(T-t,x)$, and consider equation (3.15) below pathwise for each $\tau\in[T-\tilde S,T]$:

$$\tilde X_t^{\tau,x,0} = x - \int_\tau^t \big(\bar\eta(s,\tilde X_s^{\tau,x,0}) + \check y_0(s,\tilde X_s^{\tau,x,0})\big)ds. \tag{3.15}$$

Let $\tilde X_t^{\tau,x,0}$ be the solution to (3.15). Then, it is straightforward to verify that $\big(\tilde X_t^{\tau,x,0}, \check y_0(t,\tilde X_t^{\tau,x,0})\big)$ is a solution to (3.1). Indeed, it suffices to note that $\partial_t\check y_0(t,\tilde X_t^{\tau,x,0}) = (\partial_t\tilde X_t^{\tau,x,0},\partial_x)\check y_0(t,\tilde X_t^{\tau,x,0})$ and compute $\partial_t\tilde X_t^{\tau,x,0}$ via (3.15). By the uniqueness of solution to (3.1) on $[T-S\wedge\tilde S, T]$, we conclude that $y_0(t,x) = \tilde y_0(t,x)$ on $[0,S\wedge\tilde S]\times\mathbb{R}^n$ a.s. The theorem is proved. □

References

[1] Assing, S.: A pregenerator for Burgers equation forced by conservative noise, *Commun. Math. Phys.* **225**(2002) 611–632.

[2] Bec, J. and Khanin, K: Burgers turbulence, *Phys. Rep.* **447** (2007) 1–66.

[3] Bec, J.: Universality of velocity gradients in forced Burgers turbulence, *Phys. Rev. Lett.* **87** (2001) 104501

[4] Bec, J., Frisch, U. and Khanin, K:, Kicked Burgers turbulence, *J. Fluid. Mech.* **416** (2000) 239–267.

[5] Blatter, G., Feigelman, M.V., Geshkenbein, V.B., Larkin, A.I. and Vinokur, V. M.: Vortices in high-temperature superconductors, *Rev. Mod. Phys.*, **66**, no. 4 (1994) 1125–1388.

[6] Bertini, L., Cancrini, N. and Jona-Lasinio, G.: The stochastic Burgers equation, *Comm. Math. Phys.* **165** (1994) 211–232.

[7] Boldyrev, S.A.: Turbulence without pressure in *d* dimensions, *Phys. Rev. E* **59** (1999) 29-71.

[8] Boritchev, A.: Multidimensional Potential Burgers Turbulence, *Communications in Mathematical Physics* **342**, issue 2 (2016) 441–489.

[9] Brzezniak, Z., Goldys, B. and Neklyudov, M.: Multidimensional stochastic Burgers equation, *SIAM journal of Mathematical Analysis* **46**, no. 1, (2014), 871–889.

[10] Chen, Y., Fan, E. and Yuen M.:, The Hopf–Cole transformation, topological solitons and multiple fusion solutions for the n-dimensional Burgers system, *Physics Letters A* **380**, issues 1–2 (2016) 9–14.

[11] Catuogno, P., Colombeau, J.F. and Olivera, C.: Generalized solutions of the multidimensional stochastic Burgers equation, *J Math Anal Appl.* **464**, Issue 2 (2018) 1375–1382.

[12] Davoudi, J., Masoudi, A.A., Tabar, M.R.R., Rastegar, A.R. and Shahbazi F.: Three-dimensional forced Burgers turbulence supplemented with a continuity equation, *Phys. Rev. E* **63** (2001) 056308.

[13] Da Prato, G., Debussche, A. and Temam, R.: Stochastic Burgers equation, *Nonlinear Differential Equations Appl. NoDEA* **1**, issue 4, (1994) 389–402.

[14] Da Prato, G. and Zabczyk, J.: *Ergodicity for Infinite Dimensional Systems*, Cambridge Univ. Press, Cambridge, 1996.

[15] Delarue, F.: On the existence and uniqueness of solutions to the FBSDEs in a non-generate case, *Stoch. Proc. and their Appl.* **99** (2002) 209–286.

[16] F. Delarue, Estimates of the Solutions of a System of Quasi-linear PDEs. A Probabilistic Scheme, *Sé minaire de Probabilité* **XXXVII** (2003) 290–332.

[17] Delarue, F.: A forward-backward stochastic algorithm for quasi-linear PDEs, *Ann. Appl. Probab.* **16**, no 1 (2006) 140–184.

[18] Ebin, D. and Marsden, J.:, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.* **92**, no. 1 (1970), 102–163.

[19] Frisch, U. and J. Bec, J.: Burgulence. *New trends in turbulence Turbulence: nouveaux aspects, vol. 74 of the series Les Houches - Ecole d'Ete de Physique Theorique, Springer Berlin Heidelberg* (2002) 341–383.

[20] Gyöngy, I. and Nualart, D.:, On the stochastic Burgers equation in the real line, *Ann. Probab.* **27** (1999) 782–802.

[21] Golovkin, K.K.: Vanishing viscosity in Cauchy's problem for hydromechanics equations, Boundary value problems of mathematical physics, Part 4, *Proc. Steklov Inst. Math.* **92** (1998) 33–53.

[22] Gotoh, T. and Kraichnan, R.H.:. Burgers turbulence with large scale forcing, *Phys. Fluids A* **10** (1998) 2859–2866.

[23] Gurbatov, S., Moshkov, A. and Noullez, A.: Evolution of anisotropic structures and turbulence in the multidimensional Burgers equation, *Phys. Rev. E* **81**, no. 4, (2010) 046312.

[24] Iturriaga, R. and Khanin, K.: *Burgers turbulence and Dynamical Systems*, In: Casacuberta C., Miró-Roig R.M., Verdera J., Xambó-Descamps S. (eds) European Congress of Mathematics. Progress in Mathematics, vol 201, Birkhäuser, Basel (2001) 429–443.

[25] Iturriaga, R. and Khanin, K.:. Burgers turbulence and random Lagrangian systems, *Comm. Math. Phys.* **232**, issue 3 (2003) 377–428.

[26] Kunita H.: *Stochastic flows and stochastic differential equations*, Cambridge university press, 1990

[27] Ladyzhenskaya, O.A.: On the solvability in the small of initial-boundary value problems for non-compressible ideal and viscous liquids and on vanishing viscosity, *Boundary-value problems of mathematical physics and related problems of function theory. Part 5, Zap. Nauchn. Sem. LOMI,* **21** (1971) 65–78.

[28] Ladyženskaya, O.A., Solonnikov, V.A. and Ural'ceva, N.N.: *Linear and quasi-linear equations of parabolic type*, Translations of Mathematical Monographs 23, Providence, RI: American Mathematical Society, 1968.

[29] Landau, L.D. and Lifshitz, E.M.:, *Fluid Mechanics*, Pergamon press, 1987.

[30] Pardoux, E. and Peng, S.: Backward stochastic differential equations and quasilinear parabolic partial differential equations, *In: Rozovskii B.L., Sowers R.B. (eds) Stochastic Partial Differential Equations and Their Applications. Lecture Notes in Control and Information Sciences, vol 176. Springer, Berlin, Heidelberg* (1992) 200–217.

[31] Polyakov, A. M.: Turbulence without pressure, *Phys. Rev. E* **52** (1995) 61–83.

[32] Ton, B.A.: Non-stationary Burgers flows with vanishing viscosity in bounded domains of $\mathbb{R}^3$, *Math. Z.* **145** (1975) 69–79.

[33] Zel'dovich, Ya.B.: Gravitational instability: An approximate theory for large density perturbations, *Astro. Astrophys* **5** (1970) 84–89.

Alberto Ohashi: Assistant professor of Mathematics, Federal University of Paraíba, 13560-970, João Pessoa - Paraíba, Brazil
E-mail address: ohashi@mat.ufpb.br

Evelina Shamarova: Assistant professor of Mathematics, Federal University of Paraíba, 13560-970, João Pessoa - Paraíba, Brazil
E-mail address: evelina@mat.ufpb.br