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CONTROLLABILITY OF FRACTIONAL INTEGRO-DIFFERENTIAL DAMPED DYNAMICAL SYSTEMS WITH CONTROL DELAY

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Abstract. The primary objective of this research paper is to investigate the characteristics of the controllability problem in both linear and nonlinear fractional integro-differential damped dynamical systems with control delay. We have successfully derived necessary and sufficient conditions for establishing controllability in linear fractional integro-differential damped dynamical systems with control delay. Furthermore, by employing Schauder's fixed point theorem, we have demonstrated sufficient conditions for achieving controllability in nonlinear fractional integro-differential damped dynamical systems with control delay. To illustrate the theoretical aspects, several examples have been provided.

1. Introduction

Differential equations of fractional order arise more commonly in various science and applied engineering research fields, such as fluid flow [23], signal processing [16], and so many different applied fields [11, 12, 15, 20, 24]. We see to [22], for a new pamphlet on fractional calculus. On the other hand, controllability is one of the essential vital tools in mathematical control theory and the most important structural property of the dynamical system. In the brief overview of the dynamical systems' controllability development, the reader can see the literature review [14]. Extensive research has been conducted on the controllability of nonlinear systems in finite-dimensional spaces, primarily employing fixed-point principles, refer to [2, 13], for further details. Utilizing fractional-order derivatives and integrals in control theory has shown superior outcomes compared to traditional integer-order approaches. Numerous authors have examined the controllability of fractional dynamical systems in finite-dimensional spaces, including references to [1, 7, 17, 21] and additional sources mentioned in [6] and [18]. Recently, Balachandran et al. [3, 5] and other researchers have established adequate conditions for the controllability of nonlinear fractional dynamical systems by employing Schauder's fixed-point theorem.

The kernel of the classical controllability operator e^{At} has several useful properties, including uniform convergence and the semigroup property. However, when considering the corresponding fractional system $(t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha}), 0 < \alpha < 1)$, the

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controllability operator's kernel becomes singular at $t = 0$ and does not adhere to the property of a semigroup. This makes it challenging to generalize the theory to all fractional systems. Many constraints and restrictions must be imposed to ensure the solvability of these problems. One of these problems arises from the presence of the term $t^{\alpha-1}$, which can be avoided by excluding it from the kernel of the controllability operator. For example, in [9], the author utilized a fractional integrator $I_0^{\alpha-1}$ in the control and nonlinear terms of the given fractional system to mitigate these issues. The issue of approximate controllability has been successfully resolved by utilizing the analytic resolvent method and capitalizing on the continuity of a resolvent. On the other hand, the author Matar [19] investigated the controllability of linear and nonlinear fractional integro-differential systems of order $0 < \alpha < 1$.

The controllability of linear and nonlinear fractional damped dynamical systems has been investigated in multiple studies. Balachandran et al. explored this topic in their research [4]. They provide insights into the controllability of these systems. Moreover, He et al. [10] mainly focused on the linear fractional damped dynamical system with a time delay in control. They established a necessary and sufficient condition for the controllability of this system. Zhongyang and Feng investigated the controllability of the following nonlinear fractional damped dynamical system with a time delay in control [25]

$$^{C}D_0^{\alpha}x(t) - A\ ^{C}D_0^{\beta}x(t) = Bu(t) + Cu(t-\iota) + f(t, x(t), u(t)), \text{ for } t \geq 0,$$

where $A\ ^{C}D_0^{\beta}x(t)$ represents the fractional damped term, u denotes the control input, and ι is the time control delay.

In light of the above discussion and mainly motivated by [9, 10, 19, 25], we study the controllability of the following fractional integro-differential damped dynamical system with a delay in control:

$$\begin{cases} ^{C}D_0^{p}\vartheta(\xi) - \mathcal{F}\ ^{C}D_0^{q}\vartheta(\xi) = \mathcal{G}I^{1-q}\nu(\xi) + \mathcal{M}\nu(\xi-\iota) \\ \qquad\qquad\qquad\qquad\qquad + f(\xi, \vartheta(\xi), \nu(\xi)), \quad \xi \geq 0, \\ \vartheta(0) = \psi_0,\ \vartheta'(0) = \psi_1, \\ \nu(\xi) = 0, \quad \xi \in [-\iota, 0], \end{cases} \tag{1.1}$$

where $^{C}D_0^{p}\vartheta,\ ^{C}D_0^{q}\vartheta$ are Caputo fractional derivatives of ϑ of order $1 < p < 2$ and $0 < q < 1$, respectively. $\vartheta(\cdot) \in \mathbb{R}^n$ is a state vector, $\nu(\cdot) \in \mathbb{R}^m$ is a control vector, $\mathcal{F} \in \mathbb{R}^{n\times n}$, and $\mathcal{G}, \mathcal{M} \in \mathbb{R}^{n\times m}$ are any matrices; $\iota > 0$ is the time delay; and $f : [0, \ell] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous nonlinear function. We establish the controllability results of the system (1.1) using Schauder's fixed point theorem.

2. Preliminaries

Definition 2.1. The system represented by equation (1.1) is controllable over the interval $[0, \ell]$, if for any given vectors $\psi_0, \psi_1, \psi_2 \in \mathbb{R}^n$, there exists a control input $\nu \in L^1(-\iota, \ell)$ that can steer the system's corresponding solution with initial conditions $\vartheta(0) = \psi_0$ and $\vartheta'(0) = \psi_1$ to satisfy the condition $\vartheta(\ell) = \psi_2$.

Lemma 2.2. [10] *Let $\alpha > -1$ be a given parameter. Assuming that $\varphi(\ell_2, s) \geq 0$ is a continuous function with respect to s on the interval $[\ell_1, \ell_2]$, and satisfying the* *following condition:*

$$\int_{\ell_1}^{\ell_2} (\ell_2 - s)^{\alpha} \varphi(\ell_2, s) ds = 0, \tag{2.1}$$

then it follows that $\varphi(\ell_2, s)$ is identically equal to zero for all $s \in [\ell_1, \ell_2]$.

3. Controllability of linear systems

Consider the following linear system corresponding to (1.1)

$$\begin{cases} {}^C D_0^p \vartheta(\xi) - \mathcal{F}\, {}^C D_0^q \vartheta(\xi) = \mathcal{G} I^{1-q} \nu(\xi) + \mathcal{M}\nu(\xi - \iota), \quad \xi \geq 0, \\ \vartheta(0) = \psi_0, \ \vartheta'(0) = \psi_1, \\ \nu(\xi) = 0, \quad \xi \in [-\iota, 0]. \end{cases} \tag{3.1}$$

In this section, we talk about the solution and controllability of the system (3.1) that needed in the next section.

Lemma 3.1. *The solution of system (3.1) for $\xi \in (0, \iota]$ is expressed as follows:*

$$\begin{aligned} \vartheta(\xi) &= \psi_0 + \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1 \\ &\quad + \int_0^{\xi} (\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G}\nu(s) ds, \end{aligned} \tag{3.2}$$

and for $\xi > \iota$

$$\begin{aligned} \vartheta(\xi) &= \psi_0 + \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1 \\ &\quad + \int_0^{\xi-\iota} \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G} \\ &\quad + (\xi - s - \iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi - s - \iota)^{p-q}\big)\mathcal{M}\Big)\nu(s) ds \\ &\quad + \int_{\xi-\iota}^{\xi} \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\Big)\mathcal{G}\nu(s) ds. \end{aligned} \tag{3.3}$$

Proof. Taking the Laplace transform of system (3.1), we have

$$\begin{aligned} & s^p \mathcal{L}\{\vartheta(\xi)\} - s^{p-1}\vartheta(0) - s^{p-2}\vartheta'(0) - \mathcal{F}s^q \mathcal{L}\{\vartheta(\xi)\} + \mathcal{F}s^{q-1}\vartheta(0) \\ &\quad = \mathcal{G}s^{q-1}\mathcal{L}\nu(\xi) + \mathcal{M}\mathcal{L}\{\nu(\xi - \iota)\}, \end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{\vartheta(\xi)\} &= (s^pI-\mathcal{F}s^q)^{-1}s^p\mathcal{L}\{\psi_0\}-(s^pI-\mathcal{F}s^q)^{-1}\mathcal{F}s^q\mathcal{L}\{\psi_0\}\\
&\quad +(s^pI-\mathcal{F}s^q)^{-1}s^{p-2}\psi_1+(s^pI-\mathcal{F}s^q)^{-1}s^{q-1}\mathcal{L}\{\mathcal{G}\nu(\xi)\}\\
&\quad +(s^pI-\mathcal{F}s^q)^{-1}\mathcal{L}\{\mathcal{M}\nu(\xi-\iota)\}\\
&= \mathcal{L}\{\psi_0\}+(s^{p-q}I-\mathcal{F})^{-1}s^{p-q-2}\psi_1+(s^{p-q}I-\mathcal{F})^{-1}s^{-1}\mathcal{L}\{\mathcal{G}\nu(\xi)\}\\
&\quad +(s^{p-q}I-\mathcal{F})^{-1}s^{-q}\mathcal{L}\{\mathcal{M}\nu(\xi-\iota)\}\\
&= \mathcal{L}\{\psi_0\}+\mathcal{L}(\xi E_{p-q,2}(\mathcal{F}\xi^{p-q}))\psi_1\\
&\quad +\mathcal{L}(\xi^{p-q}E_{p-q,p-q+1}(\mathcal{F}\xi^{p-q}))\mathcal{L}\{\mathcal{G}\nu(\xi)\}\\
&\quad +\mathcal{L}(\xi^{p-1}E_{p-q,p}(\mathcal{F}\xi^{p-q}))\mathcal{L}\{\mathcal{M}\nu(\xi-\iota)\}\\
&= \mathcal{L}\{\psi_0\}+\mathcal{L}(\xi E_{p-q,2}(\mathcal{F}\xi^{p-q}))\psi_1\\
&\quad +\mathcal{L}(\xi^{p-q}E_{p-q,p-q+1}(\mathcal{F}\xi^{p-q})*\mathcal{G}\nu(\xi))\\
&\quad +\mathcal{L}(\xi^{p-1}E_{p-q,p}(\mathcal{F}\xi^{p-q})*\mathcal{M}\nu(\xi-\iota)). \qquad (3.4)
\end{aligned}$$

By applying the convolution theorem to (3.4), we obtain

$$\begin{aligned}
\mathcal{L}\{\vartheta(\xi)\} &= \mathcal{L}\{\psi_0\}+\mathcal{L}\big(\xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\big)\psi_1\\
&\quad +\mathcal{L}\int_0^\xi\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds\\
&\quad +\mathcal{L}\int_0^\xi\Big((\xi-s)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{M}\nu(s-\iota)ds. \qquad (3.5)
\end{aligned}$$

Applying the inverse Laplace transform to (3.5), we get

$$\begin{aligned}
\vartheta(\xi) &= \psi_0+\xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1+\\
&\quad \int_0^\xi\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds\\
&\quad +\int_0^\xi\Big((\xi-s)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{M}\nu(s-\iota)ds.
\end{aligned}$$

For $\xi\in(0,\iota]$

$$\begin{aligned}
\vartheta(\xi) &= \psi_0+\xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1+\\
&\quad \int_0^\xi\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds,
\end{aligned}$$

and for $\xi>\iota$

$$\begin{aligned}
\vartheta(\xi) &= \psi_0+\xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1\\
&\quad +\int_0^{\xi-\iota}\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds\\
&\quad +\int_{\xi-\iota}^{\xi}\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds\\
&\quad +\int_0^{\xi-\iota}\Big((\xi-s-\iota)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\Big)\mathcal{M}\nu(s)ds,
\end{aligned}$$

$$
\begin{aligned}
\vartheta(\xi) &= \psi_0 + \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1 \\
&+ \int_0^{\xi-\iota} \Big[(\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G} \\
&+ (\xi-s-\iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\mathcal{M}\Big]\nu(s)ds \\
&+ \int_{\xi-\iota}^{\xi} \Big((\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds.
\end{aligned}
$$

□

Define

$$
\begin{aligned}
\langle \mathcal{F}|\mathcal{G},\mathcal{M}\rangle = \delta + \mathcal{F}\delta + \mathcal{F}^2\delta + \mathcal{F}^3\delta + \cdots + \mathcal{F}^{n-1}\delta \\
+ \eta + \mathcal{F}\eta + \mathcal{F}^2\eta + \mathcal{F}^3\eta + \cdots + \mathcal{F}^{n-1}\eta, \qquad (3.6)
\end{aligned}
$$

where n is order of $\mathcal{F}$ and $\delta = \mathrm{Im}(\mathcal{G})$, $\eta = \mathrm{Im}(\mathcal{M})$. Then the space $\langle \mathcal{F}|\mathcal{G},\mathcal{M}\rangle$ is spanned by the columns of the matrix

$$
[\mathcal{G}, \mathcal{F}\mathcal{G}, \mathcal{F}^2\mathcal{G}, \ldots, \mathcal{F}^{n-1}\mathcal{G}, \mathcal{M}, \mathcal{F}\mathcal{M}, \mathcal{F}^2\mathcal{M}, \ldots, \mathcal{F}^{n-1}\mathcal{M}].
$$

Lemma 3.2. *For any $z \in \mathbb{R}^n$ let us define the function $Q(\xi) : \mathbb{R}^n \to \mathbb{R}^n$ as follows:*
For $\xi \in (0, \iota]$,

$$
\begin{aligned}
Q(\xi)z &= \int_0^{\xi} \Big[(\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G}\mathcal{G}^T \\
&\times (\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)^T\Big] z ds, \qquad (3.7)
\end{aligned}
$$

and for $\xi > \iota$,

$$
\begin{aligned}
Q(\xi)z &= \int_0^{\xi-\iota} \Big[(\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G} \\
&+ (\xi-s-\iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\mathcal{M}\Big] \\
&\times \Big[(\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G} \\
&+ (\xi-s-\iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\mathcal{M}\Big]^T z ds \\
&+ \int_{\xi-\iota}^{\xi} \Big[(\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G}\mathcal{G}^T \\
&\times (\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)^T\Big] z ds. \qquad (3.8)
\end{aligned}
$$

Then $Im\big(Q(\xi)\big) = \langle \mathcal{F}|\mathcal{G},\mathcal{M}\rangle$ for $\xi > 0$.

Proof. We know that, $\mathrm{Im}\big(Q(\xi)\big) = \langle \mathcal{F}|\mathcal{G}, \mathcal{M}\rangle$ is equivalent to

$$\mathrm{Ker}Q(\xi) = \bigcap_{i=0}^{n-1} \mathrm{Ker}\mathcal{G}^T(\mathcal{F}^T)^i \bigcap_{j=0}^{n-1} \mathrm{Ker}\mathcal{M}^T(\mathcal{F}^T)^j. \tag{3.9}$$

We will only present the proof for the scenario when $\xi > \iota$. The proof for the case when ξ belongs to the interval $(0, \iota]$ follows a similar approach and will be omitted here.
To proof (3.9), first we show that

$$\mathrm{Ker}Q(\xi) \subset \bigcap_{i=0}^{n-1} \mathrm{Ker}\mathcal{G}^T(\mathcal{F}^T)^i \bigcap_{j=0}^{n-1} \mathrm{Ker}\mathcal{M}^T(\mathcal{F}^T)^j. \tag{3.10}$$

If $z \in \mathrm{Ker}Q(\xi)$ and $z \neq 0$, then

$$\begin{aligned} 0 = z^T Q(\xi) z \;\;=\;\; & z^T \int_0^{\xi-\iota} \Big[(\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G} \\ & + (\xi - s - \iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi - s - \iota)^{p-q}\big)\mathcal{M}\Big] \\ & \times \Big[(\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G} \\ & + (\xi - s - \iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi - s - \iota)^{p-q}\big)\mathcal{M}\Big]^T z ds \\ & + z^T \int_{\xi-\iota}^{\xi} \Big[(\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G}\mathcal{G}^T \\ & \times (\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)^T\Big] z ds. \end{aligned} \tag{3.11}$$

$$\begin{aligned} 0 \;\;=\;\; & \int_0^{\xi-\iota} \bigg\| \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G} \\ & + (\xi - s - \iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi - s - \iota)^{p-q}\big)\mathcal{M}\Big)^T z \bigg\|^2 ds \\ & + \int_{\xi-\iota}^{\xi} \Big\| \mathcal{G}^T (\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)^T z \Big\|^2 ds, \end{aligned} \tag{3.12}$$

which implies that

$$\begin{aligned} 0 \;\;=\;\; & \int_0^{\xi-\iota} \bigg\| \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G} \\ & + (\xi - s - \iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi - s - \iota)^{p-q}\big)\mathcal{M}\Big)^T z \bigg\|^2 ds, \end{aligned} \tag{3.13}$$

and

$$0 \;\;=\;\; \int_{\xi-\iota}^{\xi} \Big\| \mathcal{G}^T (\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)^T z \Big\|^2 ds. \tag{3.14}$$

By using Lemma 2.2 and equation (3.15), we have

$$0=\mathcal{G}^T E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)^T z=\mathcal{G}^T\sum_{k=0}^{\infty}\frac{(\mathcal{F}^T)^k(\xi-s)^{k(p-q)}}{\Gamma\big(k(p-q)+p-q+1\big)}z,$$
$$\text{on } s\in[\xi-\iota,\xi]. \quad (3.15)$$

By taking $s=\xi$ in (3.15), we get $\mathcal{G}^T z=0$. Further, it follows from (3.14) that

$$0 = \int_{\xi-\iota}^{\xi}(\xi-s)^{p-q}\left\|\mathcal{G}^T\sum_{k=0}^{\infty}\frac{(\mathcal{F}^T)^k(\xi-s)^{k(p-q)}}{\Gamma\big(k(p-q)+p-q+1\big)}z\right\|^2 ds,$$
$$0 = \int_{\xi-\iota}^{\xi}(\xi-s)^{p-q}\left\|\frac{\mathcal{G}^T}{\Gamma(p-q+1)}z+\mathcal{G}^T\sum_{k=1}^{\infty}\frac{(\mathcal{F}^T)^k(\xi-s)^{k(p-q)}}{\Gamma\big(k(p-q)+p-q+1\big)}z\right\|^2 ds,$$

which means that

$$0 = \int_{\xi-\iota}^{\xi}(\xi-s)^{p-q}\left\|\frac{\mathcal{G}^T}{\Gamma(p-q+1)}z\right\|^2 ds,$$

and

$$0 = \int_{\xi-\iota}^{\xi}(\xi-s)^{p-q}\left\|\mathcal{G}^T\sum_{k=1}^{\infty}\frac{(\mathcal{F}^T)^k(\xi-s)^{k(p-q)}}{\Gamma\big(k(p-q)+p-q+1\big)}z\right\|^2 ds,$$
$$0 = \int_{\xi-\iota}^{\xi}(\xi-s)^{2(p-q)}\left\|\mathcal{G}^T\sum_{k=1}^{\infty}\frac{(\mathcal{F}^T)^k(\xi-s)^{(k-1)(p-q)}}{\Gamma\big(k(p-q)+p-q+1\big)}z\right\|^2 ds. \quad (3.16)$$

By Lemma 2.2 and equation (3.16), we get

$$0 = \mathcal{G}^T\sum_{k=1}^{\infty}\frac{(\mathcal{F}^T)^k(\xi-s)^{(k-1)(p-q)}}{\Gamma\big(k(p-q)+p-q+1\big)}z, \text{ on } s\in[\xi-\iota,\xi]. \quad (3.17)$$

Taking $s=\xi$ in (3.17), yields

$$\mathcal{G}^T\mathcal{F}^T z=0.$$

By mathematical induction, we obtain

$$\mathcal{G}^T(\mathcal{F}^T)^k z=0 \text{ for } k=2,3,\dots,n-1. \quad (3.18)$$

According to the Cayley-Hamilton Theorem, there exist functions $\Omega_0(\xi),\Omega_1(\xi),\Omega_2(\xi),\dots,\Omega_{n-1}(\xi)$ defined over the interval $[0,\infty)$, such that

$$E_{p-q,p-q+1}\big(\mathcal{F}\xi^{p-q}\big)=\sum_{i=0}^{n-1}\Omega_i(\xi)\mathcal{F}^i. \quad (3.19)$$

From (3.18) and (3.19), we get

$$\Big(E_{p-q,p-q+1}\big(\mathcal{F}\xi^{p-q}\big)\mathcal{G}\Big)^T z\equiv 0, \text{ for all } \xi\geq 0. \quad (3.20)$$

By using (3.13) and (3.20), we get

$$0 = \int_0^{\xi-\iota}\left\|\Big((\xi-s-\iota)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\mathcal{M}\Big)^T z\right\|^2 ds. \quad (3.21)$$

Similar to (3.15), (3.17) and (3.18), we have

$$\mathcal{M}^T(\mathcal{F}^T)^k z = 0,\ \ k = 0, 1, 2, \ldots, n-1. \tag{3.22}$$

From the Cayley-Hamilton Theorem, there exist functions $\pi_0(\xi), \pi_1(\xi), \pi_2(\xi), \ldots, \pi_{n-1}(\xi)$ defined on $[0, \infty)$, such that

$$E_{p-q,p}\big(\mathcal{F}(\xi)^{p-q}\big) = \sum_{j=0}^{n-1} \pi_j(\xi)\mathcal{F}^j. \tag{3.23}$$

From (3.18) and (3.22), we have

$$z \in \bigcap_{i=0}^{n-1} \mathrm{Ker}\mathcal{G}^T(\mathcal{F}^T)^i \bigcap_{j=0}^{n-1} \mathrm{Ker}\mathcal{M}^T(\mathcal{F}^T)^j,$$

that is,

$$\mathrm{Ker}Q(\xi) \subset \bigcap_{i=0}^{n-1} \mathrm{Ker}\mathcal{G}^T(\mathcal{F}^T)^i \bigcap_{j=0}^{n-1} \mathrm{Ker}\mathcal{M}^T(\mathcal{F}^T)^j. \tag{3.24}$$

Conversely, we show that

$$\bigcap_{i=0}^{n-1} \mathrm{Ker}\mathcal{G}^T(\mathcal{F}^T)^i \bigcap_{j=0}^{n-1} \mathrm{Ker}\mathcal{M}^T(\mathcal{F}^T)^j \subset \mathrm{Ker}Q(\xi).$$

Let $z \in \bigcap_{i=0}^{n-1} \mathrm{Ker}\mathcal{G}^T(\mathcal{F}^T)^i \bigcap_{j=0}^{n-1} \mathrm{Ker}\mathcal{M}^T(\mathcal{F}^T)^j$, (3.18) and (3.22) are true. For $\xi - \iota < s \leq \xi$,

$$\begin{aligned} & (\xi - s)^{p-q}\Big(E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G}\Big)^T z \\ = \ & \sum_{i=0}^{n-1} \Omega_i(\xi - s)(\xi - s)^{p-q}\mathcal{G}^T(\mathcal{F}^T)^i z \\ = \ & 0. \end{aligned} \tag{3.25}$$

And for $0 \leq s \leq \xi - \iota$,

$$\begin{aligned} & \Bigg((\xi - s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G} + \\ & (\xi - s - \iota)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi - s - \iota)^{p-q}\big)\mathcal{M}\Bigg)^T z \\ = \ & \sum_{i=0}^{n-1} \Omega_i(\xi - s)(t - s)^{p-q}\mathcal{G}^T(\mathcal{F}^T)^i z + \\ & \sum_{j=0}^{n-1} \pi_j(\xi - s - \iota)(\xi - s - \iota)^{p-1}\mathcal{M}^T(\mathcal{F}^T)^j z \\ = \ & 0. \end{aligned} \tag{3.26}$$

By using (3.25) and (3.26), we obtain (3.12), after that we can say that $0 = z^T Q(\xi)z$. Therefore, $z \in \mathrm{Ker}Q(\xi)$, that is

$$\bigcap_{i=0}^{n-1} \mathrm{Ker}\mathcal{G}^T(\mathcal{F}^T)^i \bigcap_{j=0}^{n-1} \mathrm{Ker}\mathcal{M}^T(\mathcal{F}^T)^j \subset \mathrm{Ker}Q(\xi). \tag{3.27}$$

From (3.24) and (3.27), we have that (3.9) is true and the proof of Lemma 3.2 is completed. □

Lemma 3.3. $R(0,0) = \langle \mathcal{F}|\mathcal{G}, \mathcal{M}\rangle$.

Lemma 3.4. $R(0,0,0) = \langle \mathcal{F}|\mathcal{G}, \mathcal{M}\rangle$.

See, [10] for the proof of Lemma 3.3 and 3.4.

4. Main results

The main results of this section can be stated as follows:

Theorem 4.1. *The systems (3.1) is controllable if and only if*

$$rank[\mathcal{G}, \mathcal{F}\mathcal{G}, \mathcal{F}^2\mathcal{G}, \dots, \mathcal{F}^{n-1}\mathcal{G}, \mathcal{M}, \mathcal{F}\mathcal{M}, \mathcal{F}^2\mathcal{M}, \dots, \mathcal{F}^{n-1}\mathcal{M}] = n. \tag{4.1}$$

Proof. From (3.6), the condition (4.1) is equivalent to $\langle \mathcal{F}|\mathcal{G}, \mathcal{M}\rangle = \mathbb{R}^n$.
First, we show that $\mathbb{R}^n \subset \langle \mathcal{F}|\mathcal{G}, \mathcal{M}\rangle$.

Case I: If the control arrived time $\ell > \iota$.

Then assuming that system (3.1) is controllable, for any $\vartheta \in \mathbb{R}^n$ and initial state $\psi_0 = 0$, $\psi_1 = 0$, and initial control $\phi = 0$, according to the definition (2.1), there exists a control $\nu(s)$ such that

$$\begin{aligned}\vartheta(\xi) \;=\; & \int_0^{\xi-\iota} \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G} \\ & +(\xi - s - \iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi - s - \iota)^{p-q}\big)\mathcal{M}\Big)\nu(s)ds \\ & + \int_{\xi-\iota}^{\xi} \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds. \end{aligned} \tag{4.2}$$

From (3.19), (3.23) and (4.2), we have

$$\begin{aligned}
\vartheta(\xi) &= \int_0^{\xi-\iota}\Big((\xi-s)^{p-q}\sum_{i=0}^{n-1}\Omega_i(\xi-s)\mathcal{F}^i\mathcal{G} \\
&\quad +(\xi-s-\iota)^{p-1}\sum_{j=0}^{n-1}\pi_j(\xi-s-\iota)\mathcal{F}^j\mathcal{M}\Big)\nu(s)ds \\
&\quad +\int_{\xi-\iota}^{\xi}(\xi-s)^{p-q}\sum_{i=0}^{n-1}\Omega_i(\xi-s)\mathcal{F}^i\mathcal{G}\nu(s)ds, \\
\vartheta(\xi) &= \sum_{i=0}^{n-1}\mathcal{F}^i\mathcal{G}\Big(\int_0^{\xi-\iota}(\xi-s)^{p-q}\Omega_i(\xi-s) \\
&\quad +\int_{\xi-\iota}^{\xi}(\xi-s)^{p-q}\Omega_i(\xi-s)\Big)\nu(s)ds \\
&\quad +\sum_{j=0}^{n-1}\mathcal{F}^j\mathcal{M}\Big(\int_0^{\xi-\iota}(\xi-s-\iota)^{p-1}\pi_j(\xi-s-\iota)\Big)\nu(s)ds, \\
\vartheta(\xi) &= \sum_{i=0}^{n-1}\mathcal{F}^i\mathcal{G}\Big(\int_0^{\xi}(\xi-s)^{p-q}\Omega_i(\xi-s)\Big)\nu(s)ds \\
&\quad +\sum_{j=0}^{n-1}\mathcal{F}^j\mathcal{M}\Big(\int_0^{\xi-\iota}(\xi-s-\iota)^{p-1}\pi_j(\xi-s-\iota)\Big)\nu(s)ds. \qquad (4.3)
\end{aligned}$$

Assume that

$$\Big(\int_0^{\xi}(\xi-s)^{p-q}\Omega_i(\xi-s)\Big)\nu(s)ds = F(\xi), \qquad (4.4)$$

and

$$\Big(\int_0^{\xi-\iota}(\xi-s-\iota)^{p-1}\pi_j(\xi-s-\iota)\Big)\nu(s)ds = G(\xi-\iota). \qquad (4.5)$$

From (4.3), (4.4) and (4.5), we get

$$\vartheta(\xi) = \sum_{i=0}^{n-1}\mathcal{F}^i\mathcal{G}F(\xi) + \sum_{j=0}^{n-1}\mathcal{F}^j\mathcal{M}G(\xi-\iota). \qquad (4.6)$$

From (4.6), we conclude that $\vartheta \in \langle\mathcal{F}|\mathcal{G},\mathcal{M}\rangle$. Thus, $\mathbb{R}^n \subset \langle\mathcal{F}|\mathcal{G},\mathcal{M}\rangle$, and (4.1) holds.

Case II: If the control arrived time $\ell \in (0,\iota]$, the proof follows a similar approach and will be omitted here.

For the converse part, assume that condition (4.1) holds, then we have to show that the system (3.1) is controllable. There are two cases.

Case I: The control arrived time $\ell \in (0,\iota]$. The condition (4.1) holds, then $\mathbb{R}^n \subset \langle\mathcal{F}|\mathcal{G},\mathcal{M}\rangle$. For any $\bar{\vartheta} \in \mathbb{R}^n$ and any initial state ψ_0, ψ_1, let

$$k = \bar{\vartheta} - \psi_0 - \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1. \qquad (4.7)$$

For $k \in \mathbb{R}^n = \langle \mathcal{F}|\mathcal{G}, \mathcal{M}\rangle$, by Lemma 3.3, we have $k \in R(0,0)$. From [10, Definition 2.1], there exists a control $\nu^* \in L^1(0,\ell)$ with initial control zero such that

$$k = \int_0^{\xi} (\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G}\nu^*(s)ds. \tag{4.8}$$

From (4.7) and (4.8), we have

$$\begin{aligned}\bar{\vartheta} &= \psi_0 + \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1 + \\ &\quad \int_0^{\xi} (\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G}\nu^*(s)ds,\end{aligned} \tag{4.9}$$

which means that the system with control delay is controllable for control arrived time $\ell \in (0, \iota]$.

Case II: The control arrived time $\ell > \iota$. Assume that condition (4.1) holds, then $\mathbb{R}^n \subset \langle \mathcal{F}|\mathcal{G}, \mathcal{M}\rangle$. For any $\bar{\vartheta} \in \mathbb{R}^n$ and any initial state ψ_0, ψ_1 with zero initial control, let

$$k = \bar{\vartheta} - \psi_0 - \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1 \tag{4.10}$$

For $k \in \mathbb{R}^n = \langle \mathcal{F}|\mathcal{G}, \mathcal{M}\rangle$, by Lemma 3.4, we have $k \in R(0,0,0)$. By [10, Definition 2.1], there exists a control $\nu(\xi) \in L^1(0,\iota)$ such that

$$\begin{aligned}k &= \int_0^{\xi-\iota} \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G} \\ &\quad + (\xi - s - \iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi - s - \iota)^{p-q}\big)\mathcal{M}\Big)\nu(s)ds \\ &\quad + \int_{\xi-\iota}^{\xi} \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds.\end{aligned} \tag{4.11}$$

Then from (4.10) and (4.11), we have

$$\begin{aligned}\bar{\vartheta} &= \psi_0 + \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1 + \int_0^{\xi-\iota} \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G} \\ &\quad + (\xi - s - \iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi - s - \iota)^{p-q}\big)\mathcal{M}\Big)\nu(s)ds \\ &\quad + \int_{\xi-\iota}^{\xi} \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds,\end{aligned} \tag{4.12}$$

which means that the system with control delay is controllable for control arrived time $\ell > \iota$. The sufficiency is shown and thus, the proof is completed. □

Remark 4.2. Let Q be invertible, then the control $\bar{\nu}(\xi) \in \mathbb{R}^m$ is defined by for $\ell \in (0, \iota]$

$$\bar{\nu}(\xi) = \begin{cases} 0, & \xi \in [-\iota, \ell - \iota], \\ 0, & \xi \in (\ell - \iota, 0], \\ \big[(\ell - \xi)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\ell - \xi)^{p-q}\big)\mathcal{G}\big]^T Q^{-1} y_1, & \xi \in (0, \ell], \end{cases} \tag{4.13}$$

where

$$y_1 = \psi_2 - \psi_0 - \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1,$$

and the control $\bar{\nu}(\xi) \in \mathbb{R}^m$ is defined by for $\ell > \iota$

$$\bar{\nu}(\xi) = \begin{cases} 0, \quad \xi \in [-\iota, 0], \\ \Big[\Big((\ell-\xi)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\ell-\xi)^{p-q}\big)\mathcal{G} \\ +(\ell-\xi-\iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\ell-\xi-\iota)^{p-q}\big)\mathcal{M}\Big)\Big]^T Q^{-1} y_2, \\ \hfill \xi \in (0, \ell-\iota], \\ \big[(\ell-\xi)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\ell-\xi)^{p-q}\big)\mathcal{G}\big]^T Q^{-1} y_2, \quad \xi \in (\ell-\iota, \ell], \end{cases} \tag{4.14}$$

where

$$y_2 = \psi_2 - \psi_0 - \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1,$$

is optimal and steers the system (3.1) from initial state ψ_0, ψ_1 to final state ψ_2 at time ℓ. That means under the control (4.13) or control (4.14), respectively, the solution of systems (3.1) satisfies $\vartheta(0) = \psi_0, \vartheta'(0) = \psi_1$ and $\vartheta(\ell) = \psi_2$.

Corollary 4.3. *The system (3.1) is controllable if and only if $Q(\xi)$ is invertible.*

Proof. For the proof of the above corollary, see [25]. □

5. Controllability of nonlinear fractional systems with control delay

Using Lemma 3.1, the general solution of nonlinear fractional integro-differential damped dynamical system (1.1) is given by

$$\begin{aligned} \vartheta(\xi) \quad = \quad & \psi_0 + \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1 + \int_0^{\xi-\iota} \Big((\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G} \\ & +(\xi-s-\iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\mathcal{M}\Big) u(s)ds \\ & + \int_{\xi-\iota}^{\xi} \Big((\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds \\ & + \int_0^{\xi} \Big((\xi-s)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big) f(s, \vartheta(s), \nu(s))ds. \end{aligned} \tag{5.1}$$

Let W be the Banach space of all continuous $\mathbb{R}^n \times \mathbb{R}^m$-valued functions defined on interval $J = [0, \ell]$, denoted as

$$W = \{(z, v) : z \in \mathcal{M}(J, \mathbb{R}^n), v \in \mathcal{M}(J, \mathbb{R}^m)\},$$

equipped with uniform norm $\|(z, v)\| = \|z\| + \|v\|$, where $\|z\| = \sup\{|z(\xi)|,\ \xi \in J\}$ and $\|v\| = \sup\{|v(\xi)|,\ \xi \in J\}$, that is, $W = \mathcal{M}(J, \mathbb{R}^n) \times \mathcal{M}(J, \mathbb{R}^m)$, where $\mathcal{M}(J, \mathbb{R}^n) = \{z : J \to \mathbb{R}^n | z \text{ is continuous on } J\}$, $\mathcal{M}(J, \mathbb{R}^m) = \{v : J \to \mathbb{R}^n | v \text{ is continuous on } J\}$ are Banach spaces. For $\mathbb{R}^n$ and $\mathbb{R}^m$ we denote the max norm by $|\cdot|_n$ and $|\cdot|_m$ and use the notation $|\cdot|$, if there is no confusion.
For every $(z, v) \in W$, let the following nonlinear fractional integro-differential damped dynamical system:

$$\begin{cases} {}^C D_0^p \vartheta(\xi) - \mathcal{F}\ {}^C D_0^q \vartheta(\xi) = \mathcal{G} I^{1-q}\nu(\xi) + \mathcal{M}\nu(\xi-\iota) + f(\xi, z, v), \quad \xi \geq 0, \\ \vartheta(0) = \psi_0, \ \vartheta'(0) = \psi_1, \\ \nu(\xi) = 0, \quad \xi \in [-\iota, 0]. \end{cases} \tag{5.2}$$

By using (5.1), the solution of system (5.2) can be written as

$$\begin{aligned}\vartheta(\xi) &= \psi_0+\xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1+\int_0^{\xi-\iota}\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G}\\ &\quad +(\xi-s-\iota)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\mathcal{M}\Big)\nu(s)ds\\ &\quad +\int_{\xi-\iota}^{\xi}\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds\\ &\quad +\int_0^{\xi}\Big((\xi-s)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)f(s,z,v)ds. \qquad (5.3)\end{aligned}$$

For crispness, let us acquaint the following notations and constants:

$$\begin{aligned}H_1(\xi,s) &= (\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G}+\\ &\quad (\xi-s-\iota)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\mathcal{M},\\ H_2(\xi,s) &= (\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G},\\ b_1 &= \sup_{\xi\in[0,\ell]}\|\psi_0+\xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1\|,\\ b_2 &= \sup_{\xi\in[0,\ell]}\|E_{p-q,p}\big(\mathcal{F}(\xi-s)^{p-q}\|,\\ b_3 &= \sup_{\xi\in[0,\ell]}\|E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\|,\\ b_4 &= \sup_{\xi\in[0,\ell]}\Big\|\int_0^{\xi-\iota}\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G}\\ &\quad +(\xi-s-\iota)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\mathcal{M}\Big)ds\mathcal{M}\\ &\quad +\int_{\xi-\iota}^{\xi}\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}ds\Big\|,\\ b_5 &= \max_{i=1,2}\sup_{\xi\in[0,\ell]}\|H_i^T(\ell,\xi)\|,\\ \sup|f| &= \sup\{|f(s,z(s),v(s))|; s\in J\},\\ d_1 &= 4b_5|Q^{-1}|\big[|\psi_2|+b_1\big],\\ d_2 &= 4b_1,\\ c_1 &= 4b_5|Q^{-1}|\big[b_2\ell^p p^{-1}\sup|f|\big],\\ c_2 &= 4b_2\ell^p p^{-1},\\ d &= \max\{d_1,d_2,b_4d_1\},\\ c &= \max\{c_1,c_2,b_4c_1\}.\end{aligned}$$

We also define the control function

$$\nu(\xi)=\begin{cases} H_1^T(\ell,\xi)Q^{-1}y, & \xi\in[0,\ell-\iota),\\ H_2^T(\ell,\xi)Q^{-1}y, & \xi\in[\ell-\iota,\ell],\end{cases} \qquad (5.4)$$

where

$$\begin{aligned} y &= \psi_2 - \psi_0 - \ell E_{p-q,2}(\mathcal{F}\ell^{p-q})\psi_1 \\ &\quad - \int_0^{\ell} \Big((\ell - s)^{p-1} E_{p-q,p}\big(\mathcal{F}(\ell - s)^{p-q}\big)\Big) f(s, z, v) ds, \end{aligned}$$

where $\psi_2 \in \mathbb{R}^n$ are chosen arbitrary.

Lemma 5.1. [8] *Assume that the function f is locally bounded in v and satisfies the condition such that*

$$\lim_{|v|\to\infty} \frac{|f(w, v)|}{|v|} = 0 \tag{5.5}$$

uniformly in $w \in J$, we can conclude that for any given pair of constants c and d, there exists a constant r such that if $|v| \leq r$, then $c|f(w, v)|+d \leq r$ for all $w \in J$.

The main results of this section are presented in the following theorems.

Theorem 5.2. *Assume that f is a continuous function that satisfies the condition*

$$\lim_{|(\vartheta,\nu)|\to\infty} \frac{|f(\xi, \vartheta, \nu)|}{|\vartheta, \nu|} = 0 \tag{5.6}$$

uniformly in $\xi \in J$, and the linear fractional integro-differential damped dynamical system (3.1) with delay in control is controllable. Then the nonlinear fractional integro-differential damped dynamical system (5.2) with delay in control is controllable on J.

Proof. By the assumption, system (3.1) is controllable. By Corollary 4.3, Q is given in (3.2) is non-singular. We define the operator $\Theta : W \to W$, such that

$$\Theta(z, v) = (\vartheta, \nu), \quad (z, v) \in W,$$

where $\vartheta(\xi)$ is given by (5.3), such as

$$\begin{aligned} \vartheta(\xi) &= \psi_0 + \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1 + \int_0^{\xi-\iota} \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\mathcal{G} \\ &\quad +(\xi - s - \iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi - s - \iota)^{p-q}\big)\mathcal{M}\Big)\nu(s) ds \\ &\quad + \int_{\xi-\iota}^{\xi} \Big((\xi - s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi - s)^{p-q}\big)\Big)\mathcal{G}\nu(s) ds \\ &\quad + \int_0^{\xi} \Big((\xi - s)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi - s)^{p-q}\big)\Big) f(s, z, v) ds, \end{aligned} \tag{5.7}$$

where $\nu(\xi)$ is given by (5.4). We indicate $\nu_i(\xi)$, where $i = 1, 2$ such as

$$\begin{aligned} \nu_i(\xi) &= H_i^T(\ell, \xi) Q^{-1} y, \\ \nu_i(\xi) &= H_i^T(\ell, \xi) Q^{-1}\Big[(\psi_2 - \psi_0 - \ell E_{p-q,2}(\mathcal{F}\ell^{p-q})\psi_1 \\ &\quad - \int_0^{\ell} \Big((\ell - s)^{p-1} E_{p-q,p}\big(\mathcal{F}(\ell - s)^{p-q}\big)\Big) f(s, z, v) ds\Big]. \end{aligned} \tag{5.8}$$

The function f satisfies the condition (5.6), hence by Lemma 5.1, for every pair of constants c, d there exists a constant $r > 0$ such that if $|(z, v)| \leq r$, then $d + c|f(\xi, z, v)| \leq r$, for all $\xi \in J$.
Let $L(r) = \{(z, v) \in W : \|(z, v)\| \leq r\}$. If $(z, v) \in L(r)$, then from (5.8) and (5.7), we have

$$\begin{aligned}
|\nu_i(\xi)| &= \Big\| H_i^T(\ell,\xi) Q^{-1}\Big[(\psi_2 - \psi_0 - \ell E_{p-q,2}(\mathcal{F}\ell^{p-q})\psi_1 \\
&\quad - \int_0^{\ell} \Big((\ell-s)^{p-1} E_{p-q,p}\big(\mathcal{F}(\ell-s)^{p-q}\big)\Big) f(s,z,v)ds\Big]\Big\| \\
|\nu_i(\xi)| &\leq \|H_i^T(\ell,\xi)\| |Q^{-1}| \big[|\psi_2| + b_1 + b_2 \ell^p p^{-1} \sup|f|\big] \\
|\nu_i(\xi)| &\leq b_5|Q^{-1}|\big[|\psi_2| + b_1\big] + b_5|Q^{-1}|\big[b_2\ell^p p^{-1}\sup|f|\big] \\
|\nu_i(\xi)| &\leq \frac{d_1}{4} + \frac{c_1}{4}\sup|f| \\
|\nu_i(\xi)| &\leq \frac{1}{4}\big(d + c\sup|f|\big) \\
|\nu_i(\xi)| &\leq \frac{r}{4}, \qquad (5.9)
\end{aligned}$$

and

$$\begin{aligned}
|\vartheta(\xi)| &= \Big\|\psi_0 + \xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1 + \\
&\quad \int_0^{\xi-\iota} \Big((\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G} \\
&\quad + (\xi-s-\iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\mathcal{M}\Big)\nu(s)ds \\
&\quad + \int_{\xi-\iota}^{\xi} \Big((\xi-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds \\
&\quad + \int_0^{\xi} \Big((\xi-s)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big) f(s,z,v)ds\Big\| \\
|\vartheta(\xi)| &\leq b_1 + b_4|\nu_i(s)| + b_2\ell^p p^{-1}\sup|f| \\
|\vartheta(\xi)| &\leq \frac{d_2}{4} + b_4\Big[\frac{d_1}{4} + \frac{c_1}{4}\sup|f|\Big] + \frac{c_2}{4}\sup|f| \\
|\vartheta(\xi)| &\leq \frac{d}{2} + \frac{c}{2}\sup|f| \\
|\vartheta(\xi)| &\leq \frac{r}{2}. \qquad (5.10)
\end{aligned}$$

Therefore, if $\|z\| \leq \frac{r}{2}$ and $\|v\| \leq \frac{r}{2}$, then $|(z, v)| = \|z\| + \|v\| \leq r$, for all $s \in J$. Thus, (5.9) and (5.4) give us $\|\nu\| \leq \frac{r}{4}$ and (5.10) gives us $\|\vartheta\| \leq \frac{r}{2}$.
Therefore, the mapping Θ maps $L(r)$ into itself, implying $\Theta(L(r)) \subset L(r)$. Since the continuity of f ensures the continuity of the operator Θ, we can conclude that Θ is completely continuous, as guaranteed by the Arzela-Ascoli theorem. Consequently, $L(r)$ is closed, bounded, and convex. By applying the Schauder fixed-point theorem, we can assert that the operator Θ has a fixed point $(z, v) \in$

$L(r)$ such that $\Theta(z,v)=(z,v)=(\vartheta,\nu)$. Hence, we obtain the following such that

$$\begin{aligned}\vartheta(\xi) &= \psi_0+\xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1+\int_0^{\xi-\iota}\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G}\\&+(\xi-s-\iota)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\mathcal{M}\Big)\nu(s)ds\\&+\int_{\xi-\iota}^{\xi}\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds\\&+\int_0^{\xi}\Big((\xi-s)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)f(s,z,v)ds.\end{aligned}$$

Thus, $\vartheta(\xi)$ is the solution the system (5.2) and it is verify that $\vartheta(\ell)=\psi_2$. Hence the system (5.2) is controllable on J. □

If $\wp_j\in\mathcal{L}^1(J),\ j=1,2,\ldots,\alpha$, then $\|\wp_j\|$ is $\mathcal{L}^1$ norm of $\wp_j(\cdot)$, defined as

$$\|\wp_j\|=\int_J|\wp_j(s)|ds.$$

For crispness, let us acquaint the following notations and constants:

$$\begin{aligned}g_1 &= \sup_{\xi_1,\xi_2\in[0,\ell]}\Big\|\big(\xi_1E_{p-q,2}(\mathcal{F}\xi_1^{p-q})-\xi_2E_{p-q,2}(\mathcal{F}\xi_2^{p-q})\big)\psi_1\Big\|,\\ g_2 &= \sup_{\xi_1,\xi_2\in[0,\ell]}\Big\|\Big((\xi_1-s)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi_1-s)^{p-q}\big)\\&\quad-(\xi_2-s)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi_2-s)^{p-q}\big)\Big)\Big\|,\\ g_3 &= \sup_{\xi_2\in[0,\ell]}\Big\|(\xi_2-s)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi_2-s)^{p-q}\big)\Big\|,\\ g_4 &= \sup_{\xi_1,\xi_2\in[0,\ell]}\Big\|\int_0^{\xi_1-\iota}\Big((\xi_1-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi_1-s)^{p-q}\big)\mathcal{G}\\&\quad+(\xi_1-s-\iota)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi_1-s-\iota)^{p-q}\big)\mathcal{M}\\&\quad-(\xi_2-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi_2-s)^{p-q}\big)\mathcal{G}\\&\quad-(\xi_2-s-\iota)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi_2-s-\iota)^{p-q}\big)\mathcal{M}\Big)ds\Big\|,\\ g_5 &= \sup_{\xi_1,\xi_2\in[0,\ell]}\Big\|\int_{\xi_1-\iota}^{\xi_2-\iota}\Big((\xi_2-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi_2-s)^{p-q}\big)\mathcal{G}\\&\quad+(\xi_2-s-\iota)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi_2-s-\iota)^{p-q}\big)\mathcal{M}\Big)ds\Big\|,\end{aligned}$$

$$\begin{aligned}
g_6 &= \sup_{\xi_1\in[0,\ell]}\left\|\int_{\xi_1-\iota}^{\xi_1}\Big((\xi_1-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi_1-s)^{p-q}\big)\Big)\mathcal{G}ds\right\|,\\
g_7 &= \sup_{\xi_2\in[0,\ell]}\left\|\int_{\xi_2-\iota}^{\xi_2}\Big((\xi_2-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi_2-s)^{p-q}\big)\Big)\mathcal{G}ds\right\|,\\
e_j &= \max\big\{4b_2b_5\ell^p p^{-1}|Q^{-1}|\|\wp_j\|, 4b_2\ell^p p^{-1}\|\wp_j\|\big\},\\
\bar{c_j} &= \max\{e_j, b_4e_j\}.
\end{aligned}$$

Theorem 5.3. *Assume that $\varrho_j : \mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^+$ is measurable functions and $\wp_j : J \to \mathbb{R}^+$ is $\mathcal{L}^1(J)$ functions, where $j = 1, 2, \ldots, \alpha$ such that*

$$\|f(\xi,\vartheta,\nu)\| \le \sum_{j=1}^{\alpha}\wp_j(\xi)\varrho_j(\vartheta,\nu). \tag{5.11}$$

Then the controllability of (3.1) implies the controllability of (5.2), if

$$\overline{\lim_{r\to\infty}}\left(r-\sum_{j=1}^{\alpha}\bar{c}_j\sup\varrho_j(\vartheta,\nu) : \|(\vartheta,\nu)\|\le r\right) = +\infty. \tag{5.12}$$

Proof. Define the operator $\Theta : W \to W$ such as $\Theta(z, v) = (\vartheta, \nu)$, where ϑ and ν are given by (5.3) and (5.4), respectively.
Assume that $\Psi(r) = \sup\{\varrho_j(\vartheta,\nu) : \|(z, v)\| \le r\}$.
From (5.12), $\exists\ r_0$ such that

$$r_0-\sum_{j=1}^{\alpha}\bar{c_j}\psi(r_0)\ge d,$$

This implies that

$$\sum_{j=1}^{\alpha}\bar{c_j}\psi(r_0)+d\le r_0.$$

Also,

$$L_{r_0} = \{(z, v) \in W : \|(z, v)\| \le r_0\}.$$

If $(z, v) \in L_{r_0}$, then by (5.4) and (5.3), we have

$$\begin{aligned}
|\nu_i(\xi)| &= \bigg\|H_i^T(\ell,\xi)Q^{-1}\Big[(\psi_2-\psi_0-\ell E_{p-q,2}(\mathcal{F}\ell^{p-q})\psi_1\\
&\quad -\int_0^{\ell}\Big((\ell-s)^{p-1}E_{p-q,p}\big(\mathcal{F}(\ell-s)^{p-q}\big)\Big)f(s,z,v)ds\Big]\bigg\|\\
|\nu_i(\xi)| &\le \|H_i^T(\ell,\xi)\||Q^{-1}|\big[|\psi_2|+b_1+b_2\ell^p p^{-1}\sum_{j=1}^{\alpha}\|\wp_j\|\Psi_j(r_0)\big]\\
|\nu_i(\xi)| &\le b_5|Q^{-1}|\big[|\psi_2|+b_1\big]+b_5|Q^{-1}|\big[b_2\ell^p p^{-1}\sum_{j=1}^{\alpha}\|\wp_j\|\Psi_j(r_0)\big]\\
|\nu_i(\xi)| &\le \frac{d_1}{4}+\frac{e_j}{4}\sum_{j=1}^{\alpha}\Psi_j(r_0)
\end{aligned}$$

$$
\begin{aligned}
|\nu_i(\xi)| &\leq \frac{1}{4}\Big(d+\bar{c}_j\sum_{j=1}^{\alpha}\Psi_j(r_0)\Big)\\
|\nu(\xi)| &\leq \frac{r_0}{4},\ \forall\ \xi\in J.
\end{aligned}
$$

This implies that

$$
\|\nu\|\leq\frac{r_0}{4},
$$

and

$$
\begin{aligned}
|\vartheta(\xi)| &= \Big\|\psi_0+\xi E_{p-q,2}(\mathcal{F}\xi^{p-q})\psi_1\\
&\quad+\int_0^{\xi-\iota}\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\mathcal{G}\\
&\quad+(\xi-s-\iota)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s-\iota)^{p-q}\big)\mathcal{M}\Big)\nu(s)ds\\
&\quad+\int_{\xi-\iota}^{\xi}\Big((\xi-s)^{p-q}E_{p-q,p-q+1}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds\\
&\quad+\int_0^{\xi}\Big((\xi-s)^{p-1}E_{p-q,p}\big(\mathcal{F}(\xi-s)^{p-q}\big)\Big)f(s,z,v)ds\Big\|\\
|\vartheta(\xi)| &\leq b_1+b_4|\nu_i(s)|+b_2\ell^pp^{-1}\sum_{j=1}^{\alpha}\|\wp_j\|\Psi_j(r_0)\\
|\vartheta(\xi)| &\leq \frac{d_2}{4}+b_4\Big[\frac{d_1}{4}+\frac{e_j}{4}\sum_{j=1}^{\alpha}\Psi_j(r_0)\Big]+b_2\ell^pp^{-1}\sum_{j=1}^{\alpha}\|\wp_j\|\Psi_j(r_0)\\
|\vartheta(\xi)| &\leq \frac{d}{2}+\frac{\bar{c}_j}{2}\sum_{j=1}^{\alpha}\Psi_j(r_0)\\
\|\vartheta\| &\leq \frac{r_0}{2}.
\end{aligned}
$$

Hence, Θ maps L_{r_0} into itself, that means $\Theta(L_{r_0})\subset L_{r_0}$. Further, we need to show that $\Theta(L(r))$ is equicontinous $\forall\ r>0$. Let for every $(z,v)\in L(r)$ and $\xi_1,\xi_2\in J,\ \xi_1<\xi_2$, we have

$$
\begin{aligned}
|\nu_i(\xi_1)-\nu_i(\xi_2)| &= \Big\|\big(H_i^T(\ell,\xi_1)-H_i^T(\ell,\xi_2)\big)Q^{-1}\Big[(\psi_2-\psi_0-\ell E_{p-q,2}(\mathcal{F}\ell^{p-q})\psi_1\\
&\quad-\int_0^{\ell}\Big((\ell-s)^{p-1}E_{p-q,p}\big(\mathcal{F}(\ell-s)^{p-q}\big)\Big)f(s,z,v)ds\Big]\Big\|\\
|\nu_i(\xi_1)-\nu_i(\xi_2)| &\leq \Big\|\big(H_i^T(\ell,\xi_1)-H_i^T(\ell,\xi_2)\big)\Big\|\\
&\quad\times\Big\|Q^{-1}\Big\|\Big[|\psi_2|+b_1+b_2\ell^pp^{-1}\sum_{j=1}^{\alpha}\|\wp_i\|\Psi_j(r)\Big], \qquad (5.13)
\end{aligned}
$$

and

$$
\begin{aligned}
|\vartheta(\xi_1)-\vartheta(\xi_2)| \leq\ & \Big\|\big(\xi_1 E_{p-q,2}(\mathcal{F}\xi_1^{p-q}) - \xi_2 E_{p-q,2}(\mathcal{F}\xi_2^{p-q})\big)\psi_1\Big\| \\
& + \Big\| \int_0^{\xi_1-\iota} \Big((\xi_1-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi_1-s)^{p-q}\big)\mathcal{G} \\
& + (\xi_1-s-\iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi_1-s-\iota)^{p-q}\big)\mathcal{M} \\
& - (\xi_2-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi_2-s)^{p-q}\big)\mathcal{G} \\
& - (\xi_2-s-\iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi_2-s-\iota)^{p-q}\big)\mathcal{M}\Big)\nu(s)ds\Big\| \\
& + \Big\| \int_{\xi_1-\iota}^{\xi_2-\iota} \Big((\xi_2-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi_2-s)^{p-q}\big)\mathcal{G} \\
& + (\xi_2-s-\iota)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi_2-s-\iota)^{p-q}\big)\mathcal{M}\Big)\nu(s)ds\Big\| \\
& + \Big\| \int_{\xi_1-\iota}^{\xi_1} \Big((\xi_1-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi_1-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds\Big\| \\
& + \Big\| \int_{\xi_2-\iota}^{\xi_2} \Big((\xi_2-s)^{p-q} E_{p-q,p-q+1}\big(\mathcal{F}(\xi_2-s)^{p-q}\big)\Big)\mathcal{G}\nu(s)ds\Big\| \\
& + \Big\| \int_0^{\xi_1} \Big((\xi_1-s)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi_1-s)^{p-q}\big) \\
& - (\xi_2-s)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi_2-s)^{p-q}\big)\Big) f(s,z,v)ds\Big\| \\
& + \Big\| \int_{\xi_1}^{\xi_2} (\xi_2-s)^{p-1} E_{p-q,p}\big(\mathcal{F}(\xi_2-s)^{p-q}\big) f(s,z,v)ds\Big\|, \\
\leq\ & g_1 + g_4|\nu(s)| + g_5|\nu(s)| + g_6|\nu(s)| + g_7|\nu(s)| \\
& + g_2 \sum_{j=1}^{\alpha} \|\wp_j\| \Psi_j(r) + g_3 \sum_{j=1}^{\alpha} \|\wp_j\| \Psi_j(r) \\
\leq\ & g_1 + \big(g_4+g_5+g_6+g_7\big)\frac{r}{2} \\
& + g_2 \sum_{j=1}^{\alpha} \|\wp_j\| \Psi_j(r) + g_3 \sum_{j=1}^{\alpha} \|\wp_j\| \Psi_j(r). \qquad (5.14)
\end{aligned}
$$

Hence, it can be observed that the right-hand sides of equations (5.13) and (5.14) do not rely on specific choices of (z, v). Consequently, it is evident that $\Theta(L(r))$ exhibits equicontinuity for all $r > 0$. By applying the Arzela-Ascoli theorem, we can establish that Θ is a compact operator. Since, $L(r)$ is a nonempty, closed, bounded, and convex set. Therefore, the Schauder fixed-point theorem guarantees the existence of solution. Moreover, since $\vartheta(\ell) = \psi_2$, it follows that the system described by equation (5.2) is controllable on J. □

6. Applications

Example.1. Consider the following system

$$\begin{cases} {}^C D_0^p \vartheta(\xi) - \mathcal{F}\, {}^C D_0^q \vartheta(\xi) = \mathcal{G} I^{1-q} \nu(\xi) + \mathcal{M}\nu(\xi - \iota) \\ \qquad\qquad\qquad\qquad\qquad +f(\xi, \vartheta(\xi), \nu(\xi)), \quad \xi \geq 0, \\ \vartheta(0) = \psi_0, \ \vartheta'(0) = 0, \\ \nu(\xi) = 0, \quad \xi \in [-\iota, 0], \end{cases} \tag{6.1}$$

with $1 < p < 2$, $0 < q < 1$,

$$\mathcal{F} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{G} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \vartheta(\xi) = \begin{pmatrix} \vartheta_1(\xi) \\ \vartheta_2(\xi) \end{pmatrix},$$

$$\nu(\xi) = \begin{pmatrix} \nu_1(\xi) \\ \nu_2(\xi) \end{pmatrix}^T \quad \text{and } f(\xi, \vartheta, \nu) = \begin{pmatrix} \frac{\vartheta_1}{1+\vartheta_2^2+\nu_2^2} \\ 0 \end{pmatrix}.$$

A simple calculation shows that

$$\begin{pmatrix} \mathcal{G} & \mathcal{F}\mathcal{G} & \mathcal{M} & \mathcal{F}\mathcal{M} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{6.2}$$

and

$$\operatorname{rank} \begin{pmatrix} \mathcal{G} & \mathcal{F}\mathcal{G} & \mathcal{M} & \mathcal{F}\mathcal{M} \end{pmatrix} = 2. \tag{6.3}$$

From Theorem 4.1, we say that corresponding linear system of (6.1) is controllable. We see that $f(\xi, \vartheta, \nu)$ is continuous and satisfies the condition (5.6), and thus the system (6.1) is controllable by Theorem 5.2.

Example.2. Consider the following system

$$\begin{cases} {}^C D_0^p \vartheta(\xi) - \mathcal{F}\, {}^C D_0^q \vartheta(\xi) = \mathcal{G} I^{1-q} \nu(\xi) + \mathcal{M}\nu(\xi - \iota) \\ \qquad\qquad\qquad\qquad\qquad +f(\xi, \vartheta(\xi), \nu(\xi)), \quad \xi \geq 0, \\ \vartheta(0) = \psi_0, \ \vartheta'(0) = 0, \\ \nu(\xi) = 0, \quad \xi \in [-\iota, 0], \end{cases} \tag{6.4}$$

with

$$\mathcal{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{G} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \vartheta(\xi) = \begin{pmatrix} \vartheta_1(\xi) \\ \vartheta_2(\xi) \\ \vartheta_3(\xi) \end{pmatrix},$$

$$\nu(\xi) = \begin{pmatrix} \nu_1(\xi) \\ \nu_2(\xi) \\ \nu_3(\xi) \end{pmatrix}^T \quad \text{and} \quad f(\xi, \vartheta, \nu) = \begin{pmatrix} \frac{\vartheta_1+\vartheta_3}{1+\vartheta_2^2+\nu_2^2+\nu_3^2} \\ 0 \\ 0 \end{pmatrix}.$$

A simple calculation shows that

$$\begin{pmatrix} \mathcal{G} & \mathcal{F}\mathcal{G} & \mathcal{F}^2\mathcal{G} & \mathcal{M} & \mathcal{F}\mathcal{M} & \mathcal{F}^2\mathcal{M} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.5}$$

and

$$\operatorname{rank} \begin{pmatrix} \mathcal{G} & \mathcal{F}\mathcal{G} & \mathcal{F}^2\mathcal{G} & \mathcal{M} & \mathcal{F}\mathcal{M} & \mathcal{F}^2\mathcal{M} \end{pmatrix} = 3. \tag{6.6}$$

From Theorem 4.1, we say that corresponding linear system of (6.4) is controllable. We see that $f(\xi,\vartheta,\nu)$ is continuous and satisfies the condition (5.6), and thus the system (6.4) is controllable by Theorem 5.2.

Example.3. Consider the following system

$$\begin{cases} {}^{C}D_0^p\vartheta(\xi) - \mathcal{F}\,{}^{C}D_0^q\vartheta(\xi) = \mathcal{G}I^{1-q}\nu(\xi) + \mathcal{M}\nu(\xi-\iota) \\ \qquad\qquad\qquad\qquad\qquad\qquad +f(\xi,\vartheta(\xi),\nu(\xi)), \quad \xi \geq 0, \\ \vartheta(0)=\psi_0, \ \vartheta'(0)=0, \\ \nu(\xi)=0, \quad \xi\in[-\iota,0], \end{cases} \tag{6.7}$$

with

$$\mathcal{F}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \quad \mathcal{G}=\begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \mathcal{M}=\begin{pmatrix}2\\0\\0\end{pmatrix}, \quad \vartheta(\xi)=\begin{pmatrix}\vartheta_1(\xi)\\\vartheta_2(\xi)\\\vartheta_3(\xi)\end{pmatrix},$$

$$\nu(\xi)=\begin{pmatrix}\nu_1(\xi)\\\nu_2(\xi)\\\nu_3(\xi)\end{pmatrix}^T, \quad f(\xi,\vartheta,\nu)=\begin{pmatrix}0\\\frac{2\vartheta_3}{1+\vartheta_2^2+\nu_1^2+\nu_2^2}\\0\end{pmatrix}.$$

A simple calculation shows that

$$\begin{pmatrix}\mathcal{G} & \mathcal{F}\mathcal{G} & \mathcal{F}^2\mathcal{G} & \mathcal{M} & \mathcal{F}\mathcal{M} & \mathcal{F}^2\mathcal{M}\end{pmatrix}=\begin{pmatrix}0&0&0&2&2&2\\0&0&0&0&0&0\\1&0&0&0&0&0\end{pmatrix}, \tag{6.8}$$

and

$$\operatorname{rank}\begin{pmatrix}\mathcal{G} & \mathcal{F}\mathcal{G} & \mathcal{F}^2\mathcal{G} & \mathcal{M} & \mathcal{F}\mathcal{M} & \mathcal{F}^2\mathcal{M}\end{pmatrix}=2. \tag{6.9}$$

In this case, by Theorem 4.1, we see that the corresponding linear system of (6.7) is not controllable.

Moreover, $f(\xi,\vartheta,\nu)$ is continuous and satisfies the condition (5.6), but the system (6.7) is not controllable.

Example.4. Consider the following system

$$\begin{cases} {}^{C}D_0^p\vartheta(\xi) - \mathcal{F}\,{}^{C}D_0^q\vartheta(\xi) = \mathcal{G}I^{1-q}\nu(\xi) + \mathcal{M}\nu(\xi-\iota) \\ \qquad\qquad\qquad\qquad\qquad\qquad +f(\xi,\vartheta(\xi),\nu(\xi)), \quad \xi \geq 0, \\ \vartheta(0)=\psi_0, \ \vartheta'(0)=0, \\ \nu(\xi)=0, \quad \xi\in[-\iota,0], \end{cases} \tag{6.10}$$

with

$$\mathcal{F}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \quad \mathcal{G}=\begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \mathcal{M}=\begin{pmatrix}2\\1\\0\end{pmatrix}, \quad \vartheta(\xi)=\begin{pmatrix}\vartheta_1(\xi)\\\vartheta_2(\xi)\\\vartheta_3(\xi)\end{pmatrix},$$

$$\nu(\xi)=\begin{pmatrix}\nu_1(\xi)\\\nu_2(\xi)\\\nu_3(\xi)\end{pmatrix}^T, \quad f(\xi,\vartheta,\nu)=\begin{pmatrix}\frac{\vartheta_1}{1+\vartheta_1+\nu_1}\\\frac{\vartheta_2}{1+\vartheta_2+\nu_2}\\\frac{\vartheta_3}{1+\vartheta_3+\nu_3}\end{pmatrix}.$$

A simple calculation shows that

$$\begin{pmatrix}\mathcal{G} & \mathcal{F}\mathcal{G} & \mathcal{F}^2\mathcal{G} & \mathcal{M} & \mathcal{F}\mathcal{M} & \mathcal{F}^2\mathcal{M}\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.11}$$

and

$$\operatorname{rank}\begin{pmatrix}\mathcal{G} & \mathcal{F}\mathcal{G} & \mathcal{F}^2\mathcal{G} & \mathcal{M} & \mathcal{F}\mathcal{M} & \mathcal{F}^2\mathcal{M}\end{pmatrix} = 3. \tag{6.12}$$

From Theorem 4.1, we say that the corresponding linear system of (6.10) is controllable. Furthermore,

$$|f(\xi, \vartheta, \nu)| \leq \frac{1}{\|\vartheta(\xi)\|}.$$

In order to obtain desire results, it is enough to show that the condition (5.12) holds under the following settings:

$$\alpha = 1, \ \varrho(\vartheta, \nu) = \frac{1}{\vartheta}.$$

Hence,

$$\overline{\lim_{r\to\infty}} \left(r - \sup c \frac{1}{\vartheta} \right) = +\infty,$$

and thus by Theorem 5.3, the system (6.10) is controllable on J.

7. Conclusions

This study revolves around investigating controllability, a crucial qualitative characteristic of fractional dynamical systems. Controllability pertains to the system's ability to navigate the entirety of the configuration space through admissible actions. Our research paper provides proof of a set of necessary and sufficient conditions that establish the controllability of linear and nonlinear fractional integrodifferential damped dynamical systems with control delay in finite-dimensional spaces. We have used the MittagLeffler matrix function, Arzela-Ascoli Theorem, and Schauder fixed-point theorem as essential analytical tools in our analysis. By leveraging these tools, we have obtained significant insights into the controllability properties of fractional dynamical systems. Moreover, in the future, this study can be extended for multiple delays in control under various settings.

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