*Stochastic Modeling and Applications*
*Vol. 24 No. 2 (July-December, 2020)*

*Received: 09th August 2020* *Revised: 07th September 2020* *Accepted: 15th September 2020*

# TAYLOR WAVELETS OPERATIONAL MATRIX METHOD FOR THE NUMERICAL SOLUTION OF STOCHASTIC VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

S. C. SHIRALASHETTI* AND LATA LAMANI[1]

Abstract. An effective technique is developed in this paper to analyze stochastic Volterra-Fredholm integral equations. We derive the operational matrix of integration of Taylor wavelets and the stochastic operational matrix of integration using Taylor wavelets. These operational matrices are used to achieve the numerical solution of the stochastic Volterra-Fredholm integral equations. In the current investigation, the convergence analysis and error analysis are presented. We give some examples to show the strength and effectiveness of the proposed scheme. Numerical simulations are carried out to ensure the reliability of the potential technique. The numerical and graphical results obtained are disclosed. These obtained results show that the scheme proposed to study and find the solution of stochastic Volterra-Fredholm integral equations is computationally very practical and accurate.

## 1. Introduction

Stochastic integral equations play a major role in physics, mathematics, biology, chemistry, and economics. These problems often depend on a Gaussian white noise, governed by certain laws of probability. An explicit form of the solution of stochastic integral equations is difficult to obtain in many cases. Numerical experiments, therefore, become a practical way to deal with these kinds of problems. The problem of approximating the solution of stochastic integral equations has appeared in many articles. Some are found in [1–15].

Wavelets have found their way into many different science and engineering fields as a powerful tool. Wavelets are mathematical functions that divide data into frequency components and analyze individual components in their respective resolution. As a statistical tool, wavelets can be used to obtain data from various phenomena like earthquakes, seismic waves, signal processing, and fields like acoustics, nuclear engineering, and astronomy. Wavelets allow several functions and operators to be accurately portrayed. These applications of wavelets have

2010 *Mathematics Subject Classification.* 60H05, 60H35, 65T60.

*Key words and phrases.* Brownian motion, Taylor wavelets, Taylor polynomials, Stochastic Volterra-Fredholm integral equations.

*Corresponding author.

*This research is partially supported by UGC-SAP DRS-III for 2016-2021: F.510/3/DRS-III/2016 (SAP-I).

[1]This research is supported Karnatak University Dharwad (KUD) under University Research Studentship (URS) 2016-2019: K. U. 40 (SC/ST)URS/2018-19/32/3/841 Dated: 07/07/2018.

gained considerable interest from many researchers and have been implemented in a wide range of engineering disciplines. In particular, wavelets constructed from orthogonal polynomials are widely used in the pursuit for the numerical solution of different types of differential, integral, and integro-differential equations. In the last few decades, the operational integration matrices for the Haar Wavelets, Chebyshev Wavelets, Legendre Wavelets, and Bernoulli wavelets were used to solve integral and integro-differential equations [16–20]. Likewise, stochastic operational matrices of integration of Haar wavelets, Chebyshev wavelets, Legendre wavelets, and Bernoulli wavelets were used to solve stochastic integral equations [21–24].

Taylor wavelets were introduced by E. Keshavarz et al. in 2018, for solving initial and boundary value problems of Bratu-type equations [25] and fractional integro-differential equations with weakly singular kernels [26] in 2019.

Encouraged by most of the above works, in this article, we introduced a new stochastic operational matrix of integration using Taylor wavelets (SOMITW) for solving stochastic Volterra-Fredholm integral equations (SVFIEs). SVFIEs are previously solved by many authors [27–30].

We consider the following SVFIE:

$$y(x) = f(x) + \int_{\alpha}^{\beta} k_1(x,t)y(t)dt + \int_0^x k_2(x,t)y(t)dt + \int_0^x k_3(x,t)y(t)dW(t), \quad (1.1)$$

where, $y(x)$ is unknown, $f(x)$, $y(x)$, and $k_i(x,t)$, $i = 1,2,3$ are the stochastic processes defined on the same probability space, $\int_0^x k_3(x,t)y(t)dW(t)$ is the Itô-integral, and $W(x)$ is a Brownian motion process. This paper is structured in the following way. Properties of Brownian motion and Taylor wavelets are studied in section 2. In section 3, we study the operational matrix of integration of Taylor wavelets (OMITW), and a new SOMITW is constructed. Convergence and error analysis is studied in section 4. In section 5, a new Taylor wavelets operational matrix method for solving SVFIE is proposed based on a SOMITW. In section 6, numerical examples are presented to justify the efficiency of the proposed method. Ultimately, the conclusion is drawn in section 7.

## 2. Properties of Brownian motion and Taylor wavelets

**2.1. Brownian Motion.** For definitions of Brownian motion see [31].

**2.2. Taylor wavelets.** Taylor wavelets [32] $\psi_{n,m}(x) = \psi(k,n,m,x)$ have four arguments: $n$ ranging from 1 to $2^{k-1}$, where $k$ is assumed to be any positive integer, $m$ is the order for Taylor polynomials, and $x$ is the normalized time. They are defined on the interval $[0,1)$ as follows:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k-1}{2}} \bar{T}_m(2^{k-1}x - n + 1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \quad (2.1)$$

with

$$\bar{T}_m(x) = \sqrt{2m+1}T_m(x).$$

The coefficient $\sqrt{2m+1}$ is for normality and $m = 0,1,...,M-1$. Here, $T_m$ are the well-known Taylor polynomials of order $m$ which can be defined by $T_m(x) = x^m$.

Taylor polynomials form a complete basis over the interval $[0,1)$.
For instance, for $k=2$ and $M=3$, we get:

$$\bar{T}_0(x)=1,$$
$$\bar{T}_1(x)=\sqrt{3}x,$$
$$\bar{T}_2(x)=\sqrt{5}x^2,$$

and therefore, the first six Taylor wavelet bases are given as:

$$\left.\begin{aligned}\psi_{1,0}(x)&=\sqrt{2}\\ \psi_{1,1}(x)&=2\sqrt{6}x\\ \psi_{1,2}(x)&=4\sqrt{10}x^2\end{aligned}\right\} 0\le x<\frac{1}{2},$$

$$\left.\begin{aligned}\psi_{2,0}(x)&=\sqrt{2}\\ \psi_{2,1}(x)&=\sqrt{6}(2x-1)\\ \psi_{2,2}(x)&=\sqrt{10}(4x^2-4x+1)\end{aligned}\right\} \frac{1}{2}\le x<1.$$

**2.3. Function approximation.** Let us consider the set of Taylor wavelets as:

$$\psi(x)=\left[\psi_{1,0}(x),\psi_{1,1}(x),..,\psi_{1,M-1}(x),\psi_{2,0}(x),..,\psi_{2,M-1}(x),..,\psi_{2^{k-1},0}(x),..,\psi_{2^{k-1},M-1}(x)\right]^T \subset L^2[0,1). \tag{2.2}$$

Let us suppose that:

$$B=\operatorname{span}[\psi_{1,0}(x),\psi_{1,1}(x),..,\psi_{1,M-1}(x),\psi_{2,0}(x),..,\psi_{2,M-1}(x),..,\psi_{2^{k-1},0}(x),..,\psi_{2^{k-1},M-1}(x)]. \tag{2.3}$$

Suppose $f$ is an arbitrary function in $L^2[0,1)$. Since $B$ be a finite dimensional vector space, let $f^*\in B$ the best approximation of $f$ out of $B$. This means that for $\forall\ g\in B$, $\|f-f^*\|\le\|f-g\|$. Since, $f^*\in B$, there exist unique coefficients $c_{1,0}$, $c_{1,1},\dots,c_{2^{k-1},M-1}$ such that:

$$f(x)\simeq f^*(x)=\sum_{m=0}^{M-1}\sum_{n=1}^{2^{k-1}}c_{n,m}\psi_{n,m}(x)=C^T\psi(x), \tag{2.4}$$

where $C$ and $\psi(x)$ are $\hat{m}\times 1$ ($\hat{m}=2^{k-1}M$) vectors, and $C$ is given by:

$$C=\left[c_{1,0},c_{1,1},..,c_{1,M-1},c_{2,0},..,c_{2,M-1},..,c_{2^{k-1},0},..,c_{2^{k-1},M-1}\right]^T. \tag{2.5}$$

The coefficients $c_{n,m}$, are computed as:

$$c_{n,m}=\langle f,\psi_{n,m}\rangle=\int_0^1 f(x)\psi_{n,m}(x)dx.$$

Similarly, any arbitrary function $k(x,t)\in[0,1)\times[0,1)$ can be approximated using Taylor wavelets as,

$$k(x,t)\simeq\psi^T(x)K\psi(t)=\psi^T(t)K^T\psi(x),$$

where $K = [k_{n,m}]$ is a $\hat{m} \times \hat{m}$ matrix, where the coefficients $k_{n,m}$ are determined as:

$$k_{n,m} = \langle \psi_{n,m}(x), \langle k(x,t), \psi_{n,m}(t) \rangle \rangle = \int_0^1 \int_0^1 k(x,t)\psi_{n,m}(t)\psi_{n,m}(x)dtdx.$$

Using the collocation point $x_j = \frac{j-0.5}{\hat{m}}$, equation (2.2) reduces to $\hat{m} \times \hat{m}$ Taylor wavelets coefficient matrix. For instance, for $k = 2$ and $M = 3$, we get

$$\psi(x) = \begin{bmatrix} \psi_{1,0}(x) \\ \psi_{1,1}(x) \\ \psi_{1,2}(x) \\ \psi_{2,0}(x) \\ \psi_{2,1}(x) \\ \psi_{2,2}(x) \end{bmatrix} = \begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 \\ 0.4082 & 1.2247 & 2.0412 & 0 & 0 & 0 \\ 0.0878 & 0.7906 & 2.1960 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 0.4082 & 1.2247 & 2.0412 \\ 0 & 0 & 0 & 0.0878 & 0.7906 & 2.1960 \end{bmatrix}.$$

## 3. Stochastic operational matrix of integration of Taylor wavelets

**3.1. Operational matrix of integration of Taylor wavelets (OMITW).** The OMITW $P$ is a $\hat{m} \times \hat{m}$ matrix defined as:

$$\int_0^x \psi(t)dt = P\psi(x). \tag{3.1}$$

In particular, for $M = 3$ and $k = 2$, we have

$$\psi(x) = [\psi_{1,0}(x), \psi_{1,1}(x), \psi_{1,2}(x), \psi_{2,0}(x), \psi_{2,1}(x), \psi_{2,2}(x)]^T. \tag{3.2}$$

The matrix $P$ for these bases is derived as follows:

$$\begin{aligned} \int_0^x \psi_{1,0}(t)dt &= \begin{cases} \sqrt{2}x, & 0 \le x < \frac{1}{2} \\ \frac{\sqrt{2}}{2}, & \frac{1}{2} \le x < 1 \end{cases} \\ &\simeq \frac{1}{2\sqrt{3}}\psi_{1,1}(x) + \frac{1}{2}\psi_{2,0}(x), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \int_0^x \psi_{1,1}(t)dt &= \begin{cases} \sqrt{6}x^2, & 0 \le x < \frac{1}{2} \\ \frac{\sqrt{6}}{4}, & \frac{1}{2} \le x < 1 \end{cases} \\ &\simeq \frac{\sqrt{3}}{4\sqrt{5}}\psi_{1,2}(x) + \frac{\sqrt{3}}{4}\psi_{2,0}(x), \end{aligned} \tag{3.4}$$

$$\begin{aligned} \int_0^x \psi_{1,2}(t)dt &= \begin{cases} \frac{4\sqrt{10}x^3}{3}, & 0 \le x < \frac{1}{2} \\ \frac{\sqrt{10}}{6}, & \frac{1}{2} \le x < 1 \end{cases} \\ &\simeq \frac{\sqrt{5}}{6\sqrt{7}}\psi_{1,3}(x), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \int_0^x \psi_{2,0}(t)dt &= \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ \sqrt{2}x - \frac{\sqrt{2}}{2}, & \frac{1}{2} \le x < 1 \end{cases} \\ &\simeq \frac{1}{2\sqrt{3}}\psi_{2,1}(x), \end{aligned} \tag{3.6}$$

$$\int_0^x \psi_{2,1}(t)dt = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ \frac{\sqrt{6}}{4}(2x-1)^2, & \frac{1}{2} \le x < 1 \end{cases}$$
$$\simeq \frac{\sqrt{3}}{4\sqrt{5}}\psi_{2,2}(x), \tag{3.7}$$

$$\int_0^x \psi_{2,2}(t)dt = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ \frac{\sqrt{10}}{6}(2x-1)^3, & \frac{1}{2} \le x < 1 \end{cases}$$
$$\simeq \frac{\sqrt{5}}{6\sqrt{7}}\psi_{2,3}(x). \tag{3.8}$$

Using equations (3.3) to (3.8), we get

$$\int_0^x \psi(t)dt = \begin{bmatrix} \int_0^x \psi_{1,0}(t)dt \\ \int_0^x \psi_{1,1}(t)dt \\ \int_0^x \psi_{1,2}(t)dt \\ \int_0^x \psi_{2,0}(t)dt \\ \int_0^x \psi_{2,1}(t)dt \\ \int_0^x \psi_{2,2}(t)dt \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{2\sqrt{3}} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{4\sqrt{5}} & \frac{\sqrt{3}}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{4\sqrt{5}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{P} \psi(x).$$

**3.2. Stochastic operational matrix of integration of Taylor wavelets (SOMITW).** The SOMITW $P_S$ is a $\hat{m} \times \hat{m}$ matrix defined as:

$$\int_0^x \psi(t)dW(t) = P_S\psi(x). \tag{3.9}$$

In particular, for $M = 3$ and $k = 2$, the matrix $P_S$ is derived as follows:

$$\int_0^x \psi_{1,0}(t)dW(t) = \begin{cases} \sqrt{2}W(x), & 0 \le x < \frac{1}{2} \\ \sqrt{2}W(\frac{1}{2}), & \frac{1}{2} \le x < 1 \end{cases}$$
$$\simeq W\left(\frac{1}{4}\right)\psi_{1,0}(x) + W\left(\frac{1}{2}\right)\psi_{2,0}(x), \tag{3.10}$$

$$\int_0^x \psi_{1,1}(t)dW(t)dt = \begin{cases} 2\sqrt{6}\left(xW(x) - \int_0^x W(t)dt\right), & 0 \le x < \frac{1}{2} \\ 2\sqrt{6}\left(\frac{1}{2}W\left(\frac{1}{2}\right) - \int_0^{\frac{1}{2}} W(t)dt\right), & \frac{1}{2} \le x < 1 \end{cases}$$
$$\simeq \frac{-1}{\sqrt{2}}\left(\int_0^{\frac{1}{4}} 2\sqrt{6}W(t)dt\right)\psi_{1,0}(x) + W\left(\frac{1}{4}\right)\psi_{1,1}(x)$$
$$+ \frac{1}{\sqrt{2}}\left(\sqrt{6}W\left(\frac{1}{2}\right) - \int_0^{\frac{1}{2}} 2\sqrt{6}W(t)dt\right)\psi_{2,0}(x), \tag{3.11}$$

$$\int_0^x \psi_{1,2}(t)dW(t)dt = \begin{cases} 4\sqrt{10}\left(x^2W(x) - \int_0^x 2tW(t)dt\right), & 0 \le x < \frac{1}{2} \\ 4\sqrt{10}\left(\frac{1}{4}W\left(\frac{1}{2}\right) - \int_0^{\frac{1}{2}} 2tW(t)dt\right), & \frac{1}{2} \le x < 1 \end{cases}$$
$$\simeq -\frac{1}{\sqrt{2}}\left(\int_0^{\frac{1}{4}} 8\sqrt{10}tW(t)dt\right)\psi_{1,0}(x) + W\left(\frac{1}{4}\right)\psi_{1,2}(x)$$
$$+\frac{1}{\sqrt{2}}\left(\sqrt{10}W\left(\frac{1}{2}\right) - \int_0^{\frac{1}{2}} 8\sqrt{10}tW(t)dt\right)\psi_{2,0}(x), \tag{3.12}$$

$$\int_0^x \psi_{2,0}(t)dW(t) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ \sqrt{2}\left(W(x) - W\left(\frac{1}{2}\right)\right), & \frac{1}{2} \le x < 1 \end{cases}$$
$$\simeq \left(W\left(\frac{3}{4}\right) - W\left(\frac{1}{2}\right)\right)\psi_{2,0}(x), \tag{3.13}$$

$$\int_0^x \psi_{2,1}(t)dW(t) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ (2\sqrt{6}x - \sqrt{6})W(x) - \int_{1/2}^x 2\sqrt{6}W(t)dt, & \frac{1}{2} \le x < 1 \end{cases}$$
$$\simeq -\frac{1}{\sqrt{2}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}} 2\sqrt{6}W(t)dt\right)\psi_{2,0}(x) + W\left(\frac{3}{4}\right)\psi_{2,1}(x), \tag{3.14}$$

$$\int_0^x \psi_{2,2}(t)dt = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ \sqrt{10}(4x^2 - 4x + 1)W(x) - \sqrt{10}W\left(\frac{1}{2}\right) - \int_{\frac{1}{2}}^x (8\sqrt{10}t - 4\sqrt{10})W(t)dt, & \frac{1}{2} \le x < 1 \end{cases}$$
$$\simeq -\frac{1}{\sqrt{2}}\left(\sqrt{10}W\left(\frac{1}{2}\right) + \int_{\frac{1}{2}}^{\frac{3}{4}} (8\sqrt{10}t - 4\sqrt{10})W(t)dt\right)\psi_{2,0}(x) + W\left(\frac{3}{4}\right)\psi_{2,2}(x). \tag{3.15}$$

Using equations (3.10) to (3.15), we get

$$\int_0^x \psi(t)dW(t) = \begin{bmatrix} \int_0^x \psi_{1,0}(t)dW(t) \\ \int_0^x \psi_{1,1}(t)dW(t) \\ \int_0^x \psi_{1,2}(t)dW(t) \\ \int_0^x \psi_{2,0}(t)dW(t) \\ \int_0^x \psi_{2,1}(t)dW(t) \\ \int_0^x \psi_{2,2}(t)dW(t) \end{bmatrix}.$$

Therefore,

$$\int_0^x \psi(t)dW(t)$$
$$= \underbrace{\begin{bmatrix} W\left(\frac{1}{4}\right) & 0 & 0 & W\left(\frac{1}{2}\right) & 0 & 0 \\ \frac{-1}{\sqrt{2}}\left(\int_0^{\frac{1}{4}} 2\sqrt{6}W(t)dt\right) & W\left(\frac{1}{4}\right) & 0 & \frac{1}{\sqrt{2}}\left(\sqrt{6}W\left(\frac{1}{2}\right) - \int_0^{\frac{1}{2}} 2\sqrt{6}W(t)dt\right) & 0 & 0 \\ -\frac{1}{\sqrt{2}}\left(\int_0^{\frac{1}{4}} 8\sqrt{10}tW(t)dt\right) & 0 & W\left(\frac{1}{4}\right) & \frac{1}{\sqrt{2}}\left(\sqrt{10}W\left(\frac{1}{2}\right) - \int_0^{\frac{1}{2}} 8\sqrt{10}tW(t)dt\right) & 0 & 0 \\ 0 & 0 & 0 & \left(W\left(\frac{3}{4}\right) - W\left(\frac{1}{2}\right)\right) & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}} 2\sqrt{6}W(t)dt\right) & W\left(\frac{3}{4}\right) & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}}\left(\sqrt{10}W\left(\frac{1}{2}\right) + \int_{\frac{1}{2}}^{\frac{3}{4}} (8\sqrt{10}t - 4\sqrt{10})W(t)dt\right) & 0 & W\left(\frac{3}{4}\right) \end{bmatrix}}_{P_S} \psi(x).$$

The OMITW and SOMITW are derived here for $k = 2$ and $M = 3$ i.e. for $\hat{m} = 6$ and the same can be extended for different values of $k$ and $M$ i.e. for different values of $\hat{m}$.

*Remark* 3.1. If $F$ is a $\hat{m}$ vector, then

$$\psi(x)\psi^T(x)F = \tilde{F}\psi(x), \tag{3.16}$$

where, $\psi(x)$ is the Taylor wavelets coefficient matrix for the collocation point $x_j = \frac{j-0.5}{\hat{m}}$ and $\tilde{F}$ is a $\hat{m} \times \hat{m}$ matrix given by

$$\tilde{F} = \psi(x)\bar{F}\psi^{-1}(x), \tag{3.17}$$

where $\bar{F} = \mathrm{diag}(\psi^{-1}(x)F)$. Also, for a $\hat{m} \times \hat{m}$ matrix $C$,

$$\psi^T(x)C\psi(x) = \hat{C}^T\psi(x), \tag{3.18}$$

where $\hat{C}^T = X\psi^{-1}(x)$, in which $X = \mathrm{diag}(\psi^T(x)C\psi(x))$.

## 4. Convergence and error analysis

**Lemma 4.1.** *Let $y(x) \in L^2(\mathbb{R})$ be a continuous function on the interval $[0, 1)$ and $|y(x)| < \delta$, for every $x \in [0, 1)$. Then, the Taylor wavelet bases of $y(x)$ on equation (2.4) are bounded as:*

$$|c_{n,m}| < \frac{\lambda}{2^{\frac{k-1}{2}}}\delta\frac{2}{2m+1}, \tag{4.1}$$

*where, $\delta$ is a constant and $\lambda$ is given by:*

$$\lambda = \sqrt{2m+1}. \tag{4.2}$$

*Proof.* Using Taylor wavelets, any arbitrary function $y(x)$ can be approximated as:

$$y^*(x) \simeq \sum_{m=0}^{M-1}\sum_{n=1}^{2^{k-1}} c_{n,m}\psi_{n,m}(x) = C^T\psi(x), \tag{4.3}$$

where $C$ and $\psi(x)$ are given in equations (2.5) and (2.2) respectively and the coefficients $c_{n,m}$ are determined as:

$$\begin{aligned} c_{n,m} =&< y, \psi_{n,m} >= \int_0^1 y(x)\psi_{n,m}(x)dx \\ &= 2^{\frac{k-1}{2}}\sqrt{2m+1}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} y(x)T_m(2^{k-1}x - 2n + 1). \end{aligned} \tag{4.4}$$

Using the definition of $\psi_{n,m}(x)$ i.e., Taylor wavelets, we have:

$$\psi_{n,m}(x) = 2^{\frac{k-1}{2}}\sqrt{2m+1}T_m(2^{k-1}x - n + 1), \;\; \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}. \tag{4.5}$$

Let $\lambda = \sqrt{2j+1}$. Let $2^{k-1}x - n + 1 = v$, then equation (4.4) becomes:

$$c_{n,m} = \frac{\lambda}{2^{\frac{k-1}{2}}}\int_0^1 y\left(\frac{v+n-1}{2^{k-1}}\right)T_m(v)dv.$$

Therefore,

$$|c_{n,m}| \leq \frac{\lambda}{2^{\frac{k-1}{2}}} \int_0^1 \left| y\left(\frac{v+n-1}{2^{k-1}}\right) \right| |T_m(v)| dv. \tag{4.6}$$

Seeing the properties of Taylor polynomials, we can say that:

$$\int_0^1 |T_m(x)| dx < \frac{2}{2m+1}, \quad j > 0. \tag{4.7}$$

Using the assumption $|f(x)| < \delta$, equations (4.6) and (4.7), equation (4.6) becomes:

$$|c_{n,m}| < \frac{\lambda}{2^{\frac{k-1}{2}}} \delta \frac{2}{2m+1}. \tag{4.8}$$

□

**Theorem 4.2.** *Let $y(x) \in L^2(\mathbb{R})$ be a continuous function on the interval $[0,1)$ and $|y(x)| < \delta$ for every $x \in [0,1)$. By using the Taylor wavelet expansion we approximate this function. Let $y^*(x) = \sum_{m=0}^{M-1} \sum_{n=1}^{2^{k-1}} c_{n,m}\psi_{n,m}(x)$ be the Taylor wavelet series. Then, the bound of the truncated error $E(x)$ is given as:*

$$\|E(x)\|_2 = \|y(x) - y^*(x)\| < \left( \sum_{m=0}^{M-1} \sum_{n=2^{k-1}+1}^{\infty} \alpha_{n,m}^2 \right)^{\frac{1}{2}} + \left( \sum_{m=M}^{\infty} \sum_{n=1}^{\infty} \alpha_{n,m}^2 \right)^{\frac{1}{2}}, \tag{4.9}$$

*where,*

$$\alpha_{n,m} = \frac{\lambda}{2^{\frac{k-1}{2}}} \delta \frac{2}{2m+1}, \tag{4.10}$$

*where $\lambda = \sqrt{2m+1}$.*

*Proof.* Any function $y(x) \in L^2[0,1)$ can be expanded in terms of Taylor wavelets as:

$$y(x) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} c_{n,m}\psi_{n,m}(x).$$

If $y^*(x)$ is the expansion truncated by using Taylor wavelets, then the error obtained by truncating the above function can be computed as:

$$E(x) = y(x) - y^*(x) = \sum_{m=0}^{M-1} \sum_{n=2^{k-1}+1}^{\infty} c_{n,m}\psi_{n,m}(x) + \sum_{m=M}^{\infty} \sum_{n=1}^{\infty} c_{n,m}\psi_{n,m}(x). \tag{4.11}$$

From equation (4.11), we can write

$$\begin{aligned}
\|E(x)\| &\leq \left\| \sum_{m=0}^{M-1} \sum_{n=2^{k-1}+1}^{\infty} c_{n,m}\psi_{n,m}(x) \right\| + \left\| \sum_{m=M}^{\infty} \sum_{n=1}^{\infty} c_{n,m}\psi_{n,m}(x) \right\| \\
&= \left( \int_0^1 \left| \sum_{m=0}^{M-1} \sum_{n=2^{k-1}+1}^{\infty} c_{n,m}\psi_{n,m}(x) \right|^2 dx \right)^{\frac{1}{2}} + \left( \int_0^1 \left| \sum_{m=M}^{\infty} \sum_{n=1}^{\infty} c_{n,m}\psi_{n,m}(x) \right|^2 dx \right)^{\frac{1}{2}} \\
&\leq \left( \sum_{m=0}^{M-1} \sum_{n=2^{k-1}+1}^{\infty} |c_{n,m}|^2 \int_0^1 |\psi_{n,m}(x)|^2 dx \right)^{\frac{1}{2}} + \left( \sum_{m=M}^{\infty} \sum_{n=1}^{\infty} |c_{n,m}|^2 \int_0^1 |\psi_{n,m}(x)|^2 dx \right)^{\frac{1}{2}}.
\end{aligned} \tag{4.12}$$

From lemma 4.1, using the property

$$|c_{n,m}| < \frac{\lambda}{2^{\frac{k-1}{2}}}\delta\frac{2}{2m+1},$$

equation (4.12) reduces to

$$\|E(x)\|_2 < \left(\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty}\left|\frac{\lambda}{2^{\frac{k-1}{2}}}\delta\frac{2}{2m+1}\right|^2\int_0^1|\psi_{n,m}(x)|^2dx\right)^{\frac{1}{2}}$$
$$+\left(\sum_{m=M}^{\infty}\sum_{n=1}^{\infty}\left|\frac{\lambda}{2^{\frac{k-1}{2}}}\delta\frac{2}{2m+1}\right|^2\int_0^1|\psi_{n,m}(x)|^2dx\right)^{\frac{1}{2}}. \tag{4.13}$$

Let us define

$$\alpha_{n,m} = \frac{\lambda}{2^{\frac{k-1}{2}}}\delta\frac{2}{2m+1}. \tag{4.14}$$

Then from equation (4.13) and (4.14), we get

$$\|E(x)\|_2 < \left(\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty}|\alpha_{n,m}|^2\int_0^1|\psi_{n,m}(x)|^2dx\right)^{\frac{1}{2}} + \left(\sum_{m=M}^{\infty}\sum_{n=1}^{\infty}|\alpha_{n,m}|^2\int_0^1|\psi_{n,m}(x)|^2dx\right)^{\frac{1}{2}}.$$

Therefore,

$$\|E(x)\|_2 < \left(\sum_{m=0}^{M-1}\sum_{n=2^{k-1}+1}^{\infty}\alpha_{n,m}^2\int_0^1|\psi_{n,m}(x)|^2dx\right)^{\frac{1}{2}} + \left(\sum_{m=M}^{\infty}\sum_{n=1}^{\infty}\alpha_{n,m}^2\int_0^1|\psi_{n,m}(x)|^2dx\right)^{\frac{1}{2}}. \tag{4.15}$$

By the definition of Taylor wavelets:

$$\psi_{n,m}^2(x) = 2^{k-1}(2m+1)T_m^2(2^{k-1}x-n+1), \quad \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}. \tag{4.16}$$

Integrating (4.16) with respect to $x$, we get

$$\int_0^1 \psi_{n,m}^2(x)dx = 2^{k-1}(2m+1)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} T_m^2(2^{k-1}x-n+1)dx. \tag{4.17}$$

Let $2^{k-1}x-n+1=u$, equation (4.17) becomes:

$$\int_0^1 \psi_{n,m}^2(x)dx = (2m+1)\int_0^1 T_m^2(u)du. \tag{4.18}$$

From the definition of Taylor polynomials, we have

$$\int_0^1 T_m^2(u)du = \int_0^1 u^{2m}du = \frac{1}{2m+1}. \tag{4.19}$$

Substituting equation (4.19) in (4.18), we get

$$\int_0^1 \psi_{n,m}^2(x)dx = 1. \tag{4.20}$$

From equations (4.15) and (4.20), we get

$$\|E(x)\|_2 = \|y(x) - y^*(x)\| < \left( \sum_{m=0}^{M-1} \sum_{n=2^{k-1}+1}^{\infty} \alpha_{n,m}^2 \right)^{\frac{1}{2}} + \left( \sum_{m=M}^{\infty} \sum_{n=1}^{\infty} \alpha_{n,m}^2 \right)^{\frac{1}{2}}, \tag{4.21}$$

where $\alpha_{n,m}$ is given in (4.14), in which $\lambda = \sqrt{2j+1}$. □

**Lemma 4.3.** *Let $k(x,t) \in L^2(\mathbb{R} \times \mathbb{R})$ be a continuous function on $[0,1) \times [0,1)$ and $|k(x,t)| < \mu$, for each $[x,t) \in [0,1) \times [0,1)$. Then, the Taylor wavelet bases of $k(x,t)$ are bounded as:*

$$|k_{n,m}| < \sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^{k-1}} \sum_{m_2=0}^{M-1} \sum_{n_2=1}^{2^{k-1}} \frac{\eta_1 \eta_2}{2^{\frac{k-1}{2}}} \mu \frac{4}{(2m_1+1)(2m_2+1)}, \tag{4.22}$$

*where, $\mu$ is any constant and*

$$\eta_p = \sqrt{2m_p + 1}, \; p = 1, 2. \tag{4.23}$$

*Proof.* Let us approximate $k(x,t)$ as $k^*(x,t) = \psi^T(t) K \psi(x)$. Here $K = [k_{n,m}]$ is a matrix of order $\hat{m} \times \hat{m}$ and

$$|k_{n,m}| = \sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^{k-1}} \sum_{m_2=0}^{M-1} \sum_{n_2=1}^{2^{k-1}} \langle \langle k(x,t), \psi_{n_1,m_1}(x) \rangle, \psi_{n_2,m_2}(t) \rangle.$$

Hence,

$$|k_{n,m}| \leq \sum_{m_1=0}^{M-1} \sum_{n_1=1}^{2^{k-1}} \sum_{m_2=0}^{M-1} \sum_{n_2=1}^{2^{k-1}} |\langle \langle k(x,t), \psi_{n_1,m_1}(x) \rangle, \psi_{n_2,m_2}(t) \rangle|. \tag{4.24}$$

By the definition of inner product,

$$\langle \langle k(x,t), \psi_{n_1,m_1}(x) \rangle, \psi_{n_2,m_2}(t) \rangle = \int_0^1 \left[ \int_0^1 k(t,x) \psi_{n_1,m_1}(x)\, dx \right] \psi_{n_2,m_2}(t)\, dt. \tag{4.25}$$

By the definition of Taylor wavelets, equation (4.25) reduces to:

$$\begin{aligned} &\langle \langle k(x,t), \psi_{n_1,m_1}(x) \rangle, \psi_{n_2,m_2}(t) \rangle \\ &= 2^{k-1} \eta_1 \eta_2 \int_{\frac{n_2-1}{2^{k-1}}}^{\frac{n_2}{2^{k-1}}} \left[ \int_{\frac{n_1-1}{2^{k-1}}}^{\frac{n_1}{2^{k-1}}} k(t,x) T_{m_1}\left(2^{k-1}x - n_1 + 1\right) dx \right] T_{m_2}\left(2^{k-1}t - n_2 + 1\right) dt. \end{aligned} \tag{4.26}$$

Let $2^{k-1}x - n_1 + 1 = v$ and $2^{k-1}t - n_2 + 1 = u$. Then equation (4.26) becomes:

$$\begin{aligned} &\langle \langle k(x,t), \psi_{n_1,m_1}(x) \rangle, \psi_{n_2,m_2}(t) \rangle \\ &= \frac{\eta_1 \eta_2}{2^{k-1}} \int_0^1 \left[ \int_0^1 \left[ k\left( \frac{v+n_1-1}{2^{k-1}}, \frac{u+n_2-1}{2^{k-1}} \right) \right] T_{m_1}(v)\, dv \right] T_{m_2}(u)\, du. \end{aligned}$$

Therefore,

$$\begin{aligned} &\langle \langle k(x,t), \psi_{n_1,m_1}(x) \rangle, \psi_{n_2,m_2}(t) \rangle \\ &\leq \frac{\eta_1 \eta_2}{2^{k-1}} \int_0^1 \int_0^1 \left| k\left( \frac{v+n_1-1}{2^{k-1}}, \frac{u+n_2-1}{2^{k-1}} \right) \right| |T_{m_1}(v)| \, |T_{m_2}(u)|\, dv du. \end{aligned} \tag{4.27}$$

In the hypothesis it is assumed that $|k(x,t)| < \mu$ and hence, equation (4.27) becomes:

$$\langle\langle k(x,t), \psi_{n_1,m_1}(x)\rangle, \psi_{n_2,m_2}(t)\rangle < \frac{\eta_1\eta_2}{2^{k-1}}\mu\int_0^1\int_0^1 |T_{m_1}(v)|\,|T_{m_2}(u)|\,dvdu. \tag{4.28}$$

Using the properties of Taylor polynomials, we have,

$$\int_0^1 T_{m_1}(v)\,dv < \frac{2}{2m_1+1}, \tag{4.29}$$

and

$$\int_0^1 T_{m_2}(u)\,du < \frac{2}{2m_2+1}. \tag{4.30}$$

And therefore equation (4.28) reduces to,

$$\langle\langle k(x,t), \psi_{n_1,m_1}(x)\rangle, \psi_{n_2,m_2}(t)\rangle < \frac{\eta_1\eta_2}{2^{k-1}}\mu\frac{2}{2m_1+1}\frac{2}{2m_2+1}. \tag{4.31}$$

And from equation (4.24), we get

$$|k_{n,m}| < \sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{2^{k-1}}\frac{\eta_1\eta_2}{2^{\frac{k-1}{2}}}\mu\frac{4}{(2m_1+1)(2m_2+1)}. \tag{4.32}$$

□

**Theorem 4.4.** *Let $k(x,t) \in L^2(\mathbb{R}\times\mathbb{R})$ be a continuous function on $[0,1)\times[0,1)$ and $|k(x,t)| < \mu$ for all $[x,t) \in [0,1)\times[0,1)$. By using the Taylor wavelet expansion we approximate this function. Let*

$$k^*(x,t) = \sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{2^{k-1}} k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)$$

*be the Taylor wavelet series. Then, the bound of the truncated error $E(x,t)$ can be given as:*

$$\begin{aligned}\|E(x,t)\|_2 &= \|k(x,t) - k^*(x,t)\|_2\\ &< \left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}} + \left(\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}}\\ &+ \left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}} + \left(\sum_{m_1=M}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}},\end{aligned} \tag{4.33}$$

*where,*

$$\rho_{n,m} = \sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{2^{k-1}}\frac{\eta_1\eta_2}{2^{\frac{k-1}{2}}}\mu\frac{4}{(2m_1+1)(2m_2+1)}. \tag{4.34}$$

*where $\eta_1$ and $\eta_2$ are given in equation (4.23).*

*Proof.* Any function $k(x,t) \in L^2(\mathbb{R}\times\mathbb{R})$ can be expanded in terms of Taylor wavelets as:

$$k(x,t)=\sum_{m_1=0}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t).$$

If this expansion is truncated by using Taylor wavelets, then then the error obtained by truncating the above function can be computed as:

$$\begin{aligned}
E(x,t) &= k(x,t)-k^*(x,t)\\
&=\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=2^{k-1}+1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\\
&+\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\\
&+\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\\
&+\sum_{m_1=M}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\\
&\le\left\|\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=2^{k-1}+1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\right\|_2\\
&+\left\|\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\right\|_2\\
&+\left\|\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\right\|_2\\
&+\left\|\sum_{m_1=M}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\right\|_2\\
&\le\left(\int_0^1\int_0^1\left|\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=2^{k-1}+1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\right|^2dxdt\right)^{\frac{1}{2}}\\
&+\left(\int_0^1\int_0^1\left|\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\right|^2dxdt\right)^{\frac{1}{2}}\\
&+\left(\int_0^1\int_0^1\left|\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\right|^2dxdt\right)^{\frac{1}{2}}\\
&+\left(\int_0^1\int_0^1\left|\sum_{m_1=M}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}k_{n,m}\psi_{n_1,m_1}(x)\,\psi_{n_2,m_2}(t)\right|^2dxdt\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\|E(x,t)\|_2 \leq & \left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\int_0^1\psi_{n_1,m_1}^2(x)\,dx\int_0^1\psi_{n_2,m_2}^2(t)\,dt\right)^{\frac{1}{2}}\\ &+\left(\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\int_0^1\psi_{n_1,m_1}^2(x)\,dx\int_0^1\psi_{n_2,m_2}^2(t)\,dt\right)^{\frac{1}{2}}\\ &+\left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\int_0^1\psi_{n_1,m_1}^2(x)\,dx\int_0^1\psi_{n_2,m_2}^2(t)\,dt\right)^{\frac{1}{2}}\\ &+\left(\sum_{m_1=M}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\int_0^1\psi_{n_1,m_1}^2(x)\,dx\int_0^1\psi_{n_2,m_2}^2(t)\,dt\right)^{\frac{1}{2}}.\end{aligned} \tag{4.35}$$

From equation (4.20), we have

$$\int_0^1\psi_{n_1,m_1}^2(x)dx = 1,$$

and

$$\int_0^1\psi_{n_2,m_2}^2(x)dx = 1.$$

And hence equation (4.35) reduces to,

$$\begin{aligned}\|E(x,t)\|_2 \leq & \left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}}+\left(\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}}\\ &+\left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}}+\left(\sum_{m_1=M}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}|k_{n,m}|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{4.36}$$

Using Lemma 4.3, we have

$$|k_{n,m}| < \sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{2^{k-1}}\frac{\eta_1\eta_2}{2^{\frac{k-1}{2}}}\mu\frac{4}{(2m_1+1)(2m_2+1)}. \tag{4.37}$$

Let

$$\rho_{n,m} = \sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{2^{k-1}}\frac{\eta_1\eta_2}{2^{\frac{k-1}{2}}}\mu\frac{4}{(2m_1+1)(2m_2+1)}. \tag{4.38}$$

From (4.37) and (4.38), we have

$$|k_{n,m}| < \rho_{n,m}. \tag{4.39}$$

From equations (4.36) and (4.39), we get

$$\|E\,(x,t)\|_2 < \left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{2^{k-1}}\sum_{m_2=0}^{M-1}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}} + \left(\sum_{m_1=0}^{M-1}\sum_{n_1=1}^{2^{k-1}}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}}$$
$$+\left(\sum_{m_1=0}^{M-1}\sum_{n_1=2^{k-1}+1}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}} + \left(\sum_{m_1=M}^{\infty}\sum_{n_1=1}^{\infty}\sum_{m_2=M}^{\infty}\sum_{n_2=1}^{\infty}\rho_{n,m}^2\right)^{\frac{1}{2}}. \tag{4.40}$$

□

**Theorem 4.5.** *Let $y(x)$ and $y^*(x)$ are the exact and approximate solution of equation (1.1), respectively. Also, let us assume that*

(1) $\|y(x)\|_2 \le \xi$,
(2) $\|k_i\,(x,t)\|_2 \le M_i$, $i = 1, 2$, *and* $3$,
(3) $(\beta-\alpha)\,(M_1+\gamma_1) + (M_2+\gamma_2) + \|W\,(x)\|_\infty\,(M_3+\gamma_3) < 1$,

*then,*

$$\|y(x)-y^*(x)\|_2 \le \frac{\Lambda + \gamma_1(\beta-\alpha)\xi + \gamma_2\xi + \|W\,(x)\|_\infty\,\gamma_3\xi}{1-(\beta-\alpha)\,(M_1+\gamma_1)-(M_2+\gamma_2)-\|W\,(x)\|_\infty\,(M_3+\gamma_3)},$$

*where*

$$\Lambda = \max\|f(x)-f^*(x)\|_2\,,$$
$$\gamma_i = \max\|k_i(x,t)-k_i^*(x,t)\|_2\,,\ \ i = 1, 2\,and\ 3,$$

*and $\Lambda$ and $\gamma_i$ are given in Theorem 4.2 and Theorem 4.4, respectively.*

*Proof.* Let us approximate all the known and unknown functions of equation (1.1) using Taylor wavelets. Let us suppose that $f^*(x)$, $k_1^*(x,t), k_2^*(x,t)$, and $k_3^*(x,t)$ are approximations of $f(x)$, $k_1(x,t)$ and $k_2(x,t)$, respectively. Then,

$$\begin{aligned} y(x)-y^*(x) = f(x)-f^*(x) &+ \int_\alpha^\beta \left(k_1\,(x,t)\,y(x)-k_1^*\,(x,t)\,y^*(x)\right)dx \\ &+ \int_0^x \left(k_2\,(x,t)\,y(t)-k_2^*\,(x,t)\,y^*(t)\right)dt \\ &+ \int_0^x \left(k_3\,(x,t)\,y(t)-k_3^*\,(x,t)\,y^*(t)\right)dW(t). \end{aligned}$$

Thus,

$$\begin{aligned} \|y(x)-y^*(x)\|_2 \le \|f(x)-f^*(x)\|_2 &+ (\beta-\alpha)\,\|(k_1\,(x,t)\,y(x)-k_1^*\,(x,t)\,y^*(x))\|_2 \\ &+ \|(k_2\,(x,t)\,y(x)-k_2^*\,(x,t)\,y^*(x))\|_2 \\ &+ \|W(x)\|_\infty\,\|(k_3\,(x,t)\,y(x)-k_3^*\,(x,t)\,y^*(x))\|_2\,. \end{aligned} \tag{4.41}$$

For $i = 1, 2, 3$, we have

$$\begin{aligned} \|k_i\,(x,t)\,y(x)-k_i^*\,(x,t)\,y^*(x)\|_2 \le\ &\|k_i\,(x,t)\|_2\,\|y(x)-y^*(x)\|_2 \\ &+ \|k_i(x,t)-k_i^*(x,t)\|_2\,\|y(x)-y^*(x)\|_2 \\ &+ \|k_i(x,t)-k_i^*(x,t)\|_2\,\|y(x)\|_2\,. \end{aligned}$$

Using Theorem 4.4 and assumptions 1 and 2, we have

$$\|k_i\,(x,t)\,y(x) - k_i^*\,(x,t)\,y^*(x)\|_2 \leq (M_i + \gamma_i)\,\|y(x) - y^*(x)\|_2 + \gamma_i \xi. \tag{4.42}$$

Using Theorem 4.2 and equations (4.41) and (4.42), we get

$$\begin{aligned}\|y(x) - y^*(x)\|_2 \leq \Lambda &+ (\beta - \alpha)\,(M_1 + \gamma_1)\,\|y(x) - y^*(x)\|_2 + \gamma_1 \xi \\ &+ (M_2 + \gamma_2)\,\|y(x) - y^*(x)\|_2 + \gamma_2 \xi \\ &+ \|W(x)\|_\infty\,(M_3 + \gamma_3)\,\|y(x) - y^*(x)\|_2 + \gamma_3 \xi.\end{aligned}$$

By using the assumption 3, we finally conclude that

$$\|y(x) - y^*(x)\|_2 \leq \frac{\Lambda + \gamma_1(\beta - \alpha)\xi + \gamma_2 \xi + \|W\,(x)\|_\infty\,\gamma_3 \xi}{1 - (\beta - \alpha)\,(M_1 + \gamma_1) - (M_2 + \gamma_2) - \|W\,(x)\|_\infty\,(M_3 + \gamma_3)}.$$

□

**Theorem 4.6.** *If $y(x)$ is a sufficiently smooth function defined on the interval $[0,1)$ and $y^*(x)$ is the Taylor wavelet approximation of $y(x)$. Then*

$$|y(x) - y^*(x)| \simeq O\left(\frac{1}{n}\right).$$

*Proof.* If the points $x$ is selected as the root of the Taylor polynomial of order $n$, then

$$|y(x) - y^*(x)| \leq \frac{1}{n} \max_{x \in [0,1)} |y^*(x)|\,. \tag{4.43}$$

Let us assume that $\sigma$ is any constant such that

$$\max_{x \in [0,1)} |y^*(x)| \leq \sigma. \tag{4.44}$$

Then, from (4.43) and (4.44), we conclude that

$$|y(x) - y^*(x)| \leq \frac{\sigma}{n},$$

or

$$|y(x) - y^*(x)| \simeq O\left(\frac{1}{n}\right).$$

□

## 5. Taylor wavelets operational matrix method

In this section, we use the newly derived SOMITW for the numerical solution of SVFIE. Here we consider the SVFIE given in equation (1.1) as,

$$y(x) = f(x) + \int_\alpha^\beta y(t)k_1(x,t)dt + \int_0^x y(t)k_2(x,t)dt + \int_0^x y(t)k_3(x,t)dW(t). \tag{5.1}$$

For the sake of simplicity, we assume that $(\alpha, \beta) = (0,1)$. Approximating $f(x)$, $y(x)$ and $k_i(x,t)$, $i = 1,2,3$ with respect to Taylor wavelets as follows:

$$y(x) \simeq C^T \psi(x) = C \psi^T(x), \tag{5.2}$$

where $C$ is given in equation (2.5) and is the unknown vector to be determined.

$$f(x) \simeq F^T \psi(x) = F \psi^T(x), \tag{5.3}$$

$$k_1(x,t) \simeq \psi^T(x) K_1 \psi(t) = \psi^T(t) K_1^T \psi(x), \tag{5.4}$$

$$k_2(x,t) \simeq \psi^T(x)K_2\psi(t) = \psi^T(t)K_2^T\psi(x), \tag{5.5}$$
$$k_3(x,t) \simeq \psi^T(x)K_3\psi(t) = \psi^T(t)K_3^T\psi(x), \tag{5.6}$$

where $C$ and $F$ are Taylor wavelet coefficient vectors, and $K_1$, $K_2$, and $K_3$ are Taylor wavelet matrices. Substituting (5.2), (5.3), (5.4), (5.5), and (5.6) in (5.1), we get

$$C^T\psi(x) = F^T\psi(x) + C^T\left(\int_0^1 \psi(t)\psi^T(t)dt\right)K_1\psi(x) + \psi^T(x)K_2^T\left(\int_0^x \psi(t)\psi^T(t)Cdt\right) + \psi^T(x)K_3^T\left(\int_0^x \psi(t)\psi^T(t)CdW(t)\right).$$

Using the relation $\int_0^1 \psi(t)\psi^T(t)dt = 1$ and the remark 3.1, we get

$$C^T\psi(x) = F^T\psi(x) + C^TK_1\psi(x) + \psi^T(x)K_2^T\left(\int_0^x \tilde{C}\psi(t)dt\right) + \psi^T(x)K_3^T\left(\int_0^x \tilde{C}\psi(t)dW(t)\right),$$

where $\tilde{C}$ is a $\hat{m}\times\hat{m}$ matrix. Using the OMITW and SOMITW, we get

$$C^T\psi(x) = F^T\psi(x) + C^TK_1\psi(x) + \psi^T(x)K_2^T\tilde{C}P\psi(x) + \psi^T(x)K_3^T\tilde{C}P_s\psi(x).$$

Let $X_2 = K_2^T\tilde{C}P$ and $X_3 = K_3^T\tilde{C}P_s$. Again using remark 3.1, we get

$$C^T\psi(x) - C^TK_1\psi(x) - \hat{X}_2^T\psi(x) - \hat{X}_3^T\psi(x) = F^T\psi(x),$$

where $\hat{X}_2$ and $\hat{X}_2$ are $\hat{m}\times\hat{m}$ matrices and are linear functions of vector $C$ and this equation is applicable for all $x \in [0,1)$, hence

$$C^T - C^TK_1 - \hat{X}_2^T - \hat{X}_3^T = F^T. \tag{5.7}$$

Solving this linear system of equations we get the unknown vector C. Substituting this unknown vector in equation (5.2), we get the solution the SVFIE given in equation (5.1).

## 6. Numerical experiments

*Test problem* 6.1. Consider the SVFIE [28],

$$\begin{aligned} y(x) = f(x) &+ \int_0^1 \cos(x+t)y(t)dt + \int_0^x (x+t)y(t)dt \\ &+ \int_0^x \exp(-(x+t))y(t)dW(t),\ x,t\in[0,1), \end{aligned} \tag{6.1}$$

where

$$f(x) = x^2 + \sin(1+x) - 2\cos(1+x) - 2\sin(x) - \frac{7x^4}{12} + \frac{1}{40}W(x). \tag{6.2}$$

Here, $y(x)$ is the unknown stochastic process defined on the probability space $(\Omega, F, P)$, and $W(x)$ is the Brownian motion process. The exact solution of this integral equation is unknown. Table 1 shows the numerical results obtained by Taylor wavelet method (TWM) described in section 5, Haar wavelet method (HWM) [28], and Block pulse functions (BPFs) [33], and figure 1 shows the comparison of obtained numerical results with HWM [28] and BPFs [33] for $\hat{m} = 8$ for test problem 6.1.

## Method of Implementation

For $\hat{m} = 6$:

Comparing equation (6.1) with equation (5.1), we get

$$k_1(x,t) = \cos(x+t), \tag{6.3}$$

$$k_2(x,t) = x+t, \tag{6.4}$$

$$k_3(x,t) = \exp(-(x+t)). \tag{6.5}$$

Approximating equations (6.2), (6.3), (6.4), and (6.5) using Taylor wavelets, we obtain the vector $F$, and matrices $K_1$, $K_2$, and $K_3$. Substituting the obtained vector $F$, matrices $K_1$, $K_2$, $K_3$, and the approximated unknown solution $C$ in (6.1), and by the use of OMITW and the SOMITW, we obtain the unknown vector $C$ as

$$C = \begin{bmatrix} -0.0090951 & 0.047419 & 0.049192 & 0.143 & 0.1214 & 0.071002 \end{bmatrix}.$$

Substituting this in $y(x) \simeq C^T\psi(x) = C\psi^T(x)$, we obtain the solution as

$$y(x) = \begin{bmatrix} 0.0108132 & 0.0841029 & 0.191955 & 0.258026 & 0.407048 & 0.605956 \end{bmatrix}.$$

TABLE 1. Approximate solution for test problem 6.1.

| $x$ | TWM for $\hat{m} = 6$ | HWM for $\hat{m} = 2^5$ [28] | BPFs for $\hat{m} = 2^5$ [33] | TWM for $\hat{m} = 8$ | HWM for $\hat{m} = 2^6$ [28] | BPFs for $\hat{m} = 2^6$ [33] |
|---|---|---|---|---|---|---|
| 0.1 | 0.01814217 | 0.01894037 | 0.01991100 | 0.1579785 | 0.018461086 | 0.01551376 |
| 0.3 | 0.11645853 | 0.10263681 | 0.11746767 | 0.1539505 | 0.103326990 | 0.05832510 |
| 0.5 | 0.22499050 | 0.24699817 | 0.27412074 | 0.2523345 | 0.246273470 | 0.27753509 |
| 0.7 | 0.36234140 | 0.46248377 | 0.51447080 | 0.4152632 | 0.464473170 | 0.48867600 |
| 0.9 | 0.58606520 | 0.76428458 | 0.76857228 | 0.8694996 | 0.746050990 | 0.82223316 |

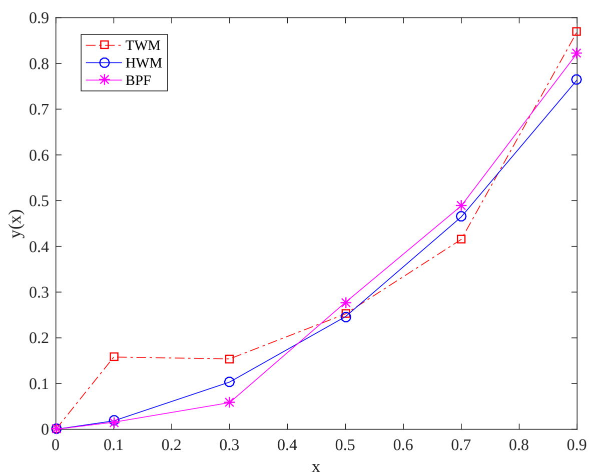


FIGURE 1. Approximate solution of test problem 6.1 for $\hat{m} = 8$.

*Test problem* 6.2. Consider the SVFIE [28],

$$y(x) = f(x) + \int_0^1 (x+t)y(t)dt + \int_0^x (x-t)y(t)dt + \frac{1}{125}\int_0^x \sin(x+t)y(t)dW(t), \ \ x, t \in [0,1), \quad (6.6)$$

where,

$$f(x) = 2 - \cos(1) - (1+x)\sin(1) + \frac{1}{250}\sin(W(x)).$$

Here, $y(x)$ is the unknown stochastic process defined on the probability space $(\Omega, F, P)$, and $W(x)$ is the Brownian motion process. The exact solution of this integral equation is unknown. Table 2 shows the numerical results obtained by TWM described in section 5, HWM [28], and BPFs [33], and figure 2 shows the comparison of obtained numerical results with HWM [28] and BPFs [33] for $\hat{m} = 8$ for test problem 6.2. Method of implementation is shown in test problem 6.1.

TABLE 2. Approximate solution for test problem 6.2.

| $x$ | TWM for $\hat{m}=6$ | HWM for $\hat{m}=2^5$ [28] | BPFs for $\hat{m}=2^5$ [33] | TWM for $\hat{m}=8$ | HWM for $\hat{m}=2^6$ [28] | BPFs for $\hat{m}=2^6$ [33] |
|---|---|---|---|---|---|---|
| 0.1 | 0.9080739 | 0.95261751 | 0.99832325 | 0.8064844 | 0.95351151 | 0.99586771 |
| 0.3 | 0.9364542 | 0.90442995 | 0.94271558 | 0.7857089 | 0.90583300 | 0.96183409 |
| 0.5 | 0.7428050 | 0.81494616 | 0.89309254 | 0.4800800 | 0.81603608 | 0.85038394 |
| 0.7 | 0.5160346 | 0.69226490 | 0.76959231 | 0.3425870 | 0.69438254 | 0.75669689 |
| 0.9 | 0.3634411 | 0.54802651 | 0.69244110 | 0.4047484 | 0.54967133 | 0.61203566 |

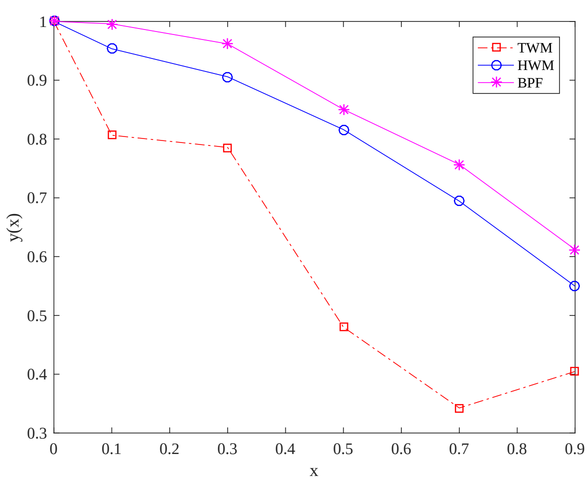


FIGURE 2. Approximate solution of test problem 6.2 for $\hat{m} = 8$.

## 7. Conclusion

In this paper, we used the Taylor wavelets and their stochastic operational integration matrix in a profitable way to solve stochastic Volterra-Fredholm integral equations. This technique used to solve stochastic Volterra-Fredholm integral equations using Taylor wavelets is free of difficulties. In the present study, the convergence and error analysis is offered to show reliability and usefulness. Numerical experiments are performed in order to show the efficiency of the proposed method. We can conclude that the projected algorithm is very efficient and well structured to solve the stochastic Volterra-Fredholm integral equations and are presented numerically to systematically and better describe the real world problems.

## Acknowledgment

[*]We thank University Grants Commission (UGC), New Delhi, for supporting this work partially through UGC-SAP DRS-III for 2016-2021: F.510/3/DRS-III/2016 (SAP-I).

[1]Also, we thank Karnatak University Dharwad (KUD) for supporting this work under University Research Studentship (URS) 2016-2019: K. U. 40 (SC/ST) URS/2018-19/32/3/841 Dated: 07/07/2018.

## References

1. Khodabin, M., Maleknejad, K., and Damercheli, T.: Approximate solution of the stochastic Volterra integral equations via expansion method, *Int. J. Ind. Math.* **6(1)** (2014) 41–48.
2. Momenzade, N., Vahidi, A. R., and Babolian, E.: A computational method for solving stochastic Itô-Volterra integral equation with multi-stochastic terms, *M*ath. Sci. **12(4)** (2018) 295–303.
3. Hashemi, B. H., Khodabin, M., and Maleknejad, K.: Numerical method for solving linear stochastic Itô-Volterra integral equations driven by fractional Brownian motion using hat functions, *T*urk. J. Math. **41(3)** (2017) 611–624.
4. Shekarabi, F. H., Khodabin, M., and Maleknejad, K.: The Petrov-Galerkin method for numerical solution of stochastic Volterra integral equations, *D*iffer. Equations. **14** (2014) 15.
5. Maleknejad, K., Khodabin, M., and Hosseini Shekarabi, F.: Modified block pulse functions for numerical solution of stochastic Volterra integral equations, *J*. Appl. Math. (2014).
6. Heydari, M. H., Hooshmandasl, M. R., Ghaini, F. M., and Cattani, C.: A computational method for solving stochastic Itô-Volterra integral equations based on stochastic operational matrix for generalized hat basis functions, *J*. Comput. Phys. **270** (2014) 402–415.
7. Mohammadi, F.: Second kind Chebyshev wavelet Galerkin method for stochastic Itô-Volterra integral equations, *M*editerr. J. Math. **13(5)** (2016) 2613–2631.
8. Mirzaee, F., and Samadyar, N.: Application of operational matrices for solving system of linear Stratonovich Volterra integral equation, *J*. Comput. Appl. Math. **320** (2017) 164–175.
9. Maleknejad, K., Khodabin, M., and Rostami, M.: A numerical method for solving m-dimensional stochastic Itô-Volterra integral equations by stochastic operational matrix, *C*omput. Math. Appl. **63(1)** (2012) 133–143.
10. Khodabin, M., Maleknejad, K., and Hossoini, F.: Shckarabi, Application of triangular functions to numerical solution of stochastic volterra integral equations, *I*AENG Int. J. Appl. Math. **43(1)** (2013) 1–9.
11. Maleknejad, K., Khodabin, M., and Rostami, M.: Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block pulse functions, *M*ath. Comput. Modell. **55(3-4)** (2012) 791–800.
12. Mohammadi, F.: A wavelet-based computational method for solving stochastic Itô-Volterra integral equations, *J*. Comput. Phys. **298** (2015) 254–265.
13. Mohammadi, F.: Numerical Solution of Stochastic Itô-Volterra Integral Equation using Haar wavelets, *N*umer. Math. Theor. Meth. Appl. **9(3)** (2016) 416–431.
14. Mohammadi, F.: Haar Wavelets Approach For Solving Multidimensional Stochastic Itô-Volterra Integral Equations, *A*ppl. Math. E-Notes. **15** (2015) 80–96.
15. Singh, S., and Saha Ray, S.: Stochastic operational matrix of Chebyshev wavelets for solving multi-dimensional stochastic Itô-Volterra integral equations, *I*nt. J. Wavelets Multiresolution Inf. Process. **17(03)** (2019) 1950007.

16. Lepik, Ü.: Application of the Haar wavelet transform to solving integral and differential equations, *I*n Proc. Est. Acad. Sci. Phys. Math. **56(1)** (2007).
17. Babolian, E., and Fattahzadeh, F.: Numerical computation method in solving integral equations by using Chebyshev wavelet operational matrix of integration, *A*ppl. Math. Comput. **188(1)** (2007) 1016–1022.
18. Yousefi, S., and Razzaghi, M.: Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations,*M*ath. Comput. Simul. **70(1)** (2005) 1–8.
19. Araghi, F., Daliri, S., and Bahmanpour, M.: Numerical solution of integro-differential equation by using Chebyshev wavelet operational matrix of integration, *I*nt. J. Math. Modell. Comput. **2(2 (SPRING))** (2016) 127–136.
20. Keshavarz, E., Ordokhani, Y., and Razzaghi, M.: The Bernoulli wavelets operational matrix of integration and its applications for the solution of linear and nonlinear problems in calculus of variations, *A*ppl. Math. Comput. **351** (2019) 83–98.
21. Mohammadi, F.: Numerical solution of stochastic Itô-Volterra integral equations using Haar wavelets, *N*umer. Math. Theory. Method. Appl. **9(3)** (2016) 416–431.
22. Kajani, M. T., and Vencheh, A. H.: Solving linear integro-differential equation with Legendre wavelets, *I*nt. J. Comput. Math. **81(6)** (2004) 719–726.
23. Mohammadi, F.: A efficient computational method for solving stochastic Itô-Volterra integral equations, *T*WMS. J. Appl. Eng. Math. **5(2)** (2015) 286.
24. Mirzaee, F., and Samadyar, N.: Application of Bernoulli wavelet method for estimating a solution of linear stochastic Itô-Volterra integral equations, *M*ultidiscip. Model. Mater. Struct. **15(3)** (2018) 575–598.
25. Keshavarz, E., and Ordokhani, Y., and Razzaghi, M.: The Taylor wavelets method for solving the initial and boundary value problems of Bratu-type equations, *A*ppl. Numer. Math. **128** (2018) 205–216.
26. Keshavarz, E., and Ordokhani, Y.: A fast numerical algorithm based on the Taylor wavelets for solving the fractional integrodifferential equations with weakly singular kernels, *M*ath. Methods Appl. Sci. **42** (2019) 4427–4443.
27. Mohammadi, F.: A Chebyshev wavelet operational method for solving stochastic Volterra-Fredholm integral equations, *I*nt. J. Appl. Math. Res. **4(2)** (2015) 217–227.
28. Mohammadi, F.: Numerical solution of stochastic Volterra-Fredholm integral equations using Haar wavelets, *U*. P. B. Sci. Bull., Series A. **78(2)** (2016) 111–126.
29. Mohammadi, F.: A computational wavelet method for numerical solution of stochastic Volterra-Fredholm integral equations, *W*avelet. Linear Algebra. **3(1)**, (2016) 13–25.
30. Mohammadi, F., and Adhami, P.: Numerical study of stochastic Volterra-Fredholm integral equations by using second kind Chebyshev wavelets, *R*andom. Operators. Stochastic. Equations. **24(2)** (2016) 129–141.
31. Shiralashetti, S. C., and Lamani, L.: Numerical solution of stochastic integral equations using CAS wavelets, *M*alaya Journal of Matematik (MJM), **1** (2020), 183–186.
32. Toan, P. T., Vo, T. N., and Razzaghi, M.: Taylor wavelet method for fractional delay differential equations, *E*ng. Comput. (2019) 1–10.
33. Khodabin, M., Maleknejad, K., Rostami, M., and Nouri, M.: Numerical approach for solving stochastic Volterra-Fredholm integral equations by stochastic operational matrix, *C*omput. Math. Appl. **64(6)** (2012) 1903–1913.

S. C. Shiralashetti: Department of Mathematics, Karnatak University, Dharwad-580 003, Karnataka, India
*E-mail address*: `shiralashettisc@gmail.com`

Lata Lamani: Department of Mathematics, Karnatak University, Dharwad-580 003, Karnataka, India
*E-mail address*: `latalamani@gmail.com`