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NOTE ON LEGER COHOMOLOGY OF LIE ALGEBRA BUNDLES

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Abstract. In analogy with the theory of cohomology of Lie algebras, we develop the concept of module bundle and Lie algebra bundle extensions.

1. Introduction

Let $\xi = (\xi, p, X)$ be a Lie algebra bundle and $\eta = (\eta, q, X)$ a ξ-module bundle with the corresponding representation $\rho : \xi \to \text{End}\eta$. Let $M^n(\xi, \eta)$ denote the $C(X)$-module of all symmetric morphisms from $\xi^n = \xi \oplus \xi \oplus \cdots \oplus \xi$ to η, for all $n \geq 1$ and $M^0(\xi, \eta) = \text{HOM}(X \times R,\ \eta)$, where $X \times R$ is the trivial vector bundle over X. The space $M^0(\xi, \eta)$ can be identified with $\Gamma(\eta)$. Now we define differentials $\partial^n : M^n(\xi, \eta) \to M^{n+1}(\xi, \eta)$ for all $n \geq 0$, Consider S in $M^0(\xi, \eta)$. Let us define $f : \xi \to \eta$ by $f(u) = \rho(u)\ S(x)$ if $u \in \xi_x$. Then the function f is in $M^1(\xi, \eta)$ since it is the composition of the morphisms,

$$\xi \xrightarrow{(Id,p)} \xi \times X \xrightarrow{(Id,S)} \xi \oplus \eta \xrightarrow{\rho} \eta$$
$$u \to (u, p(u) = x) \to (u, S(x)) \to \rho(u)(S(x)).$$

Hence we can define $\partial^\circ : M^0(\xi, \eta) \to M^1(\xi, \eta)$ by $\partial^\circ(S) = f$ and it can be easily verified that ∂^0 is a $C(X)-$module homomorphism. Given f in $M^n(\xi, \eta)$, we define $g : \xi^{n+1} \to \eta$ by

$$\begin{aligned} g(u_0, \cdots, u_n) = \ & \textstyle\sum_{i=0}^{n} \ (-1)^i \rho(u_i) f(u_0, \cdots, \hat{u_i} \cdots, u_n) \\ + & \sum_{i<j} (-1)^{i+j} f([u_i, u_j], u_0, \cdots, \hat{u_i}, \cdots, \hat{u_j}, \cdots, u_n), \end{aligned}$$

where $\hat{u_i}$ denotes that the i^{th} term is omitted. The map

$$(u_0, \cdots, u_n) \to \rho(u_i) f(u_1, \cdots \hat{u_i}, \cdots, u_n)$$

is a vector bundle morphism being the composition of the vector bundle morphisms

$$\xi^{n+1} \to \text{End}\,\eta \oplus \eta \to \eta$$
$$(u_0, \cdots, u_n) \to (\rho(u_i), f(u_1, \cdots \hat{u_i}, \cdots, u_n)) \to \rho(u_i) f(u_1, \cdots, \hat{u_i}, \cdots, u_n).$$

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The mapping $(u_0, \cdots, u_n) \to f([u_i, u_j], u_0, \cdots, u_n)$ from $\xi^{n+1} \to \eta$ is again a vector bundle morphism being the composition,

$$\xi^{n+1} \to \xi^n \to \eta$$

$$(u_0, \cdots, u_n) \to ([u_i, u_j], u_2, \cdots, u_n) \to f([u_i, u_j], u_2, \cdots, u_n).$$

Thus the mapping $g \in M^{n+1}(\xi, \eta)$ and for all $n \geq 1$, we define $\partial^n : M^n(\xi, \eta) \to M^{n+1}(\xi, \eta)$ by $\partial^n(f) = g$, where

$$\begin{aligned} g(u_0, \cdots, u_n) = & \textstyle\sum_{i=0}^{n} \quad (-1)^i \rho(u_i) f(u_0, \cdots, \hat{u_i} \cdots, u_n) \\ & + \quad \sum_{i<j} (-1)^{i+j} f([u_i, u_j], u_0, \cdots, \hat{u_i}, \cdots, \hat{u_j}, \cdots, u_n), \end{aligned}$$

Then it can be easily verified that $\partial^n \partial^{n-1} = 0$ for all $n \geq 1$. So $\{M^n(\xi, \eta), \partial^n\}$ is a cochain complex.

Given a Lie algebra bundle ξ and a ξ-module bundle η, we define the n^{th} cohomology group of ξ with coefficients in η by

$$H^n(\xi, \eta) = ker\partial^n / Im\partial^{n-1} = Z^n(\xi, \eta)/B^n(\xi, \eta) \text{ for } n \geq 1$$

and

$$H^0(\xi, \eta) = ker\partial^0.$$

The elements of $ker\partial^n$ are called $n-$cocycles and the elements of $Im\partial^{n-1}$ are called $n-$coboundaries.

Some recent developments in the theory of algebra bundles can be found in [1, 2, 8, 9, 10]. Theory of cohomology of algebra bundles over compact Hausdarff space can found in [4, 6]. In this paper, we develop the theory of Leger cohomology of real Lie algebra bundles over a topological space, though much of the elementary theory is similar for Lie algebras [5, 7].

All underlying vector spaces are finite dimensional. The total space of a bundle $\xi = (\xi, p, X)$ is denoted by ξ itself and fibres by ξ_x.

2. Module Bundle Extensions

Let η and $\eta^{'}$ be $\xi-$module bundles. If $\eta \subset E$ and if the sequence

$$0 \to \eta \xrightarrow{i} E \xrightarrow{\phi} \eta^{'} \to 0$$

of $\xi-$module bundles is exact we say that the pair (E, ϕ) ia an extension of η by $\eta^{'}$. Two extensions (E_1, ϕ_1) and (E_2, ϕ_2) of η by $\eta^{'}$, they are called equivalent if there is an isomorphism σ of E_1 onto E_2 such that $\phi_2\sigma = \phi_1$. We denote by $\{(E, \phi)\}$ the class of extensions equivalent to (E, ϕ). Let $M = \cup_{x \in X} M_x$, where M_x is the vector space of all module homomorphism of $\eta^{'}_x$ into η_x. Then M is a vector bundle [4]. We make M into $\xi-$module bundle by defining

$$u.f(q) = u.f(q) - f(u.q) \text{ for all } u \in \xi_x, \ f \in M_x \text{ and } q \in \eta^{'}_x.$$

Proposition 2.1. *Let ξ be a Lie algebra bundle over compact Hausdorff space X. Then there is a bijection correspondence between first cohomology group $H^1(\xi, M)$ and* Ext $[\eta, \eta^{'}]$*, the classes of extensions of the $\xi-$module bundle η by $\eta^{'}$.*

Proof. Consider an extension

$$(E, \phi) \; : \; 0 \to \eta \xrightarrow{i} E \xrightarrow{\phi} \eta^{'} \to 0.$$

Since X is compact Hausdorff, there is a vector bundle splitting morphism $\gamma : \eta^{'} \to E$ such that $\beta\gamma = Id_{\xi}$ [3, page 25]. Let $\Delta : \xi \to M$, $(\Delta_u : q \to u.\gamma(q) - \gamma(u.q))$ for $u \in \xi_x$ and $q \in \eta^{'}$. Then Δ will give the structure of a $\xi-$module bundle on E. For,

$$\rho : \xi \oplus E \to E$$

defined by

$$\rho(u, (p \oplus q)) = \{u.p + \Delta_u(q)\} \oplus u.q,$$

for all $u \in \xi_x$, $p \in \eta$, $q \in \eta^{'}$ is a representation of ξ if and only if $\Delta \in H^1(\xi, M)$. □

Now we give $\xi-$module structure on Ext $[\eta, \eta^{'}]$. Let $u \in \xi_x$, and (E, ϕ) an module extension of η by $\eta^{'}$. We set

$$E^* = \cup_{x \in X} E^*_x, \text{ where } E^*_x = \{(e_1, e_2) \mid e_1, e_2 \in E_x, \text{ and } u.\phi(e_1) = \phi(e_2)\}.$$

The map $\psi : E \oplus E \to \eta^{'}$ defined by $\psi(e_1, e_2) = u.\phi(e_1) - \phi(e_2)$ is a surjective vector bundle morphism whose kernel is E^*. Hence E^* is a sub bundle of $E \oplus E$ [3]. Make E^* into $\xi-$module bundle such that $v.(e_1, e_2) = (v.e_1, v.e_2 + [u, v].e_1)$ for all $v \in \xi_x$ and $(e_1, e_2) \in E^*$. Define an onto map $\psi^* : E^* \to \eta^{'}$ by $\psi^*(e_1, e_2) = \phi(e_1)$. The kernel of ψ^* is (η, η) and the morphism $\omega : (\eta, \eta) \to \eta$ such that $\omega(p_1, p_2) = u.p_1 - p_2$ is (η, η) onto η. The kernel N of ω consists of all pairs (p_1, p_2) for which $p_2 = u.p_1$. Since N is submodule of E^* so that we identify $(\eta, \eta)/N$ with η. Then $\phi^* : E^*/N \to \eta^{'}$ defined by $\phi^*[(e_1, e_2) + N] = \phi(e_1)$ is a module bundle morphism of E^*/N onto $\eta^{'}$ whose kernel is $(\eta, \eta)/N$ which we have identified with η.

Thus we define

$$u.(E, \phi) = (E^*/N, \phi^*).$$

Suppose that (E_1, ϕ_1) is an extension of η by $\eta^{'}$ which is equivalent to (E, ϕ). Let $\sigma : E \to E_1$ be an isomorphism. Then the map $\sigma^* : E^* \to E_1^*$, define by

$$\sigma^*(e_1, e_2) = (\sigma(e_1), \sigma(e_2))$$

is an isomorphism which takes N onto N_1 and induce an isomorphism of E^*/N onto E_1^*/N_1. Thus we define

$$u.\{(E, \phi)\} = \{u.(E, \phi)\}.$$

3. Lie Algebra Bundle Extensions

Let η be $\xi-$module bundle which we consider as abelian Lie algebra bundle. An extension of η by ξ is a pair (ζ, Φ) such that η is an ideal bundle of ζ and such that the sequence

$$0 \to \eta \xrightarrow{i} \zeta \xrightarrow{\Phi} \xi \to 0$$

is exact. Two such extensions (ζ_1, Φ_1) and (ζ_2, Φ_2) are called equivalent if there is an isomorphism σ of ζ_1 onto ζ_2 such that $\Phi_2\sigma = \Phi_1$. We denote by $\{(\zeta, \phi)\}$ the class of extensions equivalent to (ζ, Φ).

Proposition 3.1. *Let ξ be a Lie algebra bundle over compact Hausdorff space X. Then there is a bijection between second cohomology group $H^2(\xi, \eta)$ of ξ with coefficients in $\xi-$module bundle η and* $\mathrm{Ext}(\xi, \eta)$*, the collections of all equivalence classes of all extensions of η by ξ.*

Proof. Consider the short exact sequence of Lie algebra bundles $0 \to \eta \xrightarrow{\alpha} \zeta \xrightarrow{\beta} \xi \to 0$. Here η is consider as abelian Lie algebra bundle. Since X compact Hausdorff above short exact sequence splits $\zeta = \xi \oplus \eta$ as a vector bundles [3, page 25]. Now define, for any $u, v \in \xi_x$ and $a, b \in \eta_x$

$$\theta_\zeta(u + a, v + b) = \theta(u, v) + \{\rho(u)b - \rho(v)a + \kappa(u, v)\}.$$

The Jacobi identity holds if and only if $\kappa : \xi \oplus \xi \to \eta$ is a 2-cocycle.

□

We give $\xi-$module bundle structure on $\mathrm{Ext}(\xi, \eta)$ as follows; Let consider $u \in \xi_x$ and (ζ, Φ) an extension of η by ξ. Define

$$\zeta^* = \cup_{x \in X} \zeta^*_x, \text{ where } \zeta^* = \{(u_1, u_2) \mid u_1, u_2 \in \zeta, \theta(u, \Phi(u_1)) = \Phi(u_2)\}.$$

Define a morphism $\Psi : \zeta \oplus \zeta \to \eta$ defined by $\Psi(u_1, u_2) = \theta(u, \Phi(u_1)) - \Phi(u_2)$ is a surjective vector bundle morphism whose kernel is ζ^*. Hence ζ^* is a sub bundle of $\zeta \oplus \zeta$ [3]. We make ζ^* into a Lie algebra bundle by defining

$$\Theta((u_1, u_2), (u_1^{'}, u_2^{'})) = (\theta(u_1, u_2^{'}), \theta(u_1, u_2^{'}) + \theta(u_2, u_1^{'})).$$

Define an onto map $\Psi^* : \zeta^* \to \xi$ by $\Psi^*(u_1, u_2) = \Phi(u_1)$. The kernel of Ψ^* is (η, η) and the morphism $\Omega : (\eta, \eta) \to \eta$ such that $\Omega(m_1, m_2) = u.m_1 - m_2$ is (η, η) onto η. The kernel N of Ω consists of all pairs (m_1, m_2) for which $m_2 = u.m_1$. Since N is submodule of ζ^* so that we identify $(\eta, \eta)/N$ with η. Then $\Phi^* : \zeta^*/N \to \xi$ defined by $\Phi^*[(u_1, u_2) + N] = \Phi(u_1)$ is a vector bundle morphism of ζ^*/N onto ξ whose kernel is $(\eta, \eta)/N$ which we have identified with η.

Thus we define

$$u.(\zeta, \phi) = (\zeta^*/N, \Phi^*).$$

Suppose that (ζ_1, Φ_1) is an extension of η by ξ which is equivalent to (ζ, Φ). Let $\sigma : \zeta \to \zeta_1$ be an isomorphism. Then the map $\sigma^* : \zeta^* \to \zeta_1^*$, define by

$$\sigma^*(u_1, u_2) = (\sigma(u_1), \sigma(u_2))$$

is an isomorphism which takes N onto N_1 and induce an isomorphism of ζ^*/N onto ζ_1^*/N_1. Thus we define

$$u.\{(\zeta, \Phi)\} = \{u.(\zeta, \Phi)\}.$$

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