Journal of Mathematical Control Science and Applications
Vol. 7 No. 1 (January-June 2021)

Received: 21st January 2021 *Revised: 16th February 2021* *Accepted: 25th February 2021*

THE OPEN-LOOP ZERO-SUM LINEAR QUADRATIC INDEX ONE DISCRETE-TIME SOFT-CONSTRAINED DESCRIPTOR DYNAMIC GAMES

MUHAMMAD WAKHID MUSTHOFA

Abstract. This paper deals with discrete-time zero-sum linear quadratic soft-constrained dynamic games in open-loop information structure for index one descriptor systems. The aim is to provide both necessary and sufficient conditions for the existence of an open-loop saddle point (OLSP) equilibrium of the game. To find this equilibrium, the idea is to transform the game from descriptor system into nonsingular system.

1. Introduction

Dynamic game is a scientific formulation to model a clash situation between various parties. In linear quadratic setting, these parties drive a dynamical system and every of them aims to optimize their personal quadratic cost function using actions they give to the system. For that goal, dynamic games in linear quadratic setting have been widely implemented in various areas like resources in environmental economics, marketing, management, industrial organization, and armed conflicts ([1], [2]).

The problem is not every system can be presented by an ordinary dynamic game. In particular, when the systems are presented in a pair of difference and algebraic equations that model the dynamic and static constrains. In this case, such systems can be modelled as a descriptor system [3], or singular systems [4]. These systems have been applied in many fields like large scale interconnected systems ([5], [6]), biological economic systems [7], circuit systems ([8], [9]), chemical processes [10], power systems ([11], [12]), mechanical systems ([13], [14]), and medical robotics ([15], [16]).

This article is the continued work of [17] where the game is set for continuous linear quadratic index one descriptor systems. Here we study its counterpart of the system time settings that is the discrete-time. For this discrete soft-constrained descriptor game, we will find necessary and sufficient conditions under which the game has an open-loop saddle-point (OLSP) solution. To find these conditions our idea is to change the descriptor dynamic game into an ordinary (nonsingular) dynamic game. Having being an ordinary game, hence we use the results from

2000 *Mathematics Subject Classification.* Primary 91A25; Secondary 91A50.
Key words and phrases. soft-constrained linear quadratic dynamic games, descriptor systems, discrete-time, open-loop information structure.

[18] to obtain the necessary and sufficient conditions of OLSP solution for the game. The different method have been worked out by [19] but for closed-loop memoryless information structure and then the result was used to solve the disturbance attenuation control problem. In this paper the players are assumed to act non-cooperatively and the information structure for both players is considered an open-loop, that is the players already set their strategies at the moment the system begins and these strategies cannot be changed once the system is running [20]. A different way for saddle-point problems has been undertaken by [21] and [22], while in feedback information structure the problem studied by [23] and [24]. However, all the contributions presented above only consider the hard-constrained game.

The paper outline is compiled as follows. Section 2 proposes the game to be studied and the main problem to be solved. Section 3 is focused on the solution of OLSP equilibrium of the game and the conclusions are compiled in Section 4.

2. Preliminaries

We consider in this paper a game that presented with the dynamical system of the form

$$Ex_{k+1} = Ax_k + B_1 u_{1k} + B_2 u_{2k},\ x_{k=1} = x_0,\ k \in [1, K], \tag{2.1}$$

where $E, A \in \mathbb{R}^{(n+r)\times(n+r)}$, $\operatorname{rank}(E) = n$, $B_i \in \mathbb{R}^{(n+r)\times m_i}$ and x_0 denotes the consistent initial state [1]. An $m_i \times 1$ vector $u_{ik} \in U_{si} \subset \mathbb{R}^{m_i}$ states the strategy made by the players to control the system, and U_{si} states the set of all possible strategies for the two players. The first player aims to minimize his cost functional J_γ presented in a quadratic form

$$J_\gamma (u_{1k}, u_{2k}) = \sum_{k=1}^{K} \left(x_k^T Q x_k + u_{1k}^T R_1 u_{1k} - \gamma u_{2k}^T R_2 u_{2k} \right) + x_{K+1} \bar{Q}_K x_{K+1}, \tag{2.2}$$

where $Q, \bar{Q}_K \geq 0,\ R_1, R_2 > 0$, and the parameter $\gamma \in \mathbb{R}$ is a weighting for the second player strategy that want to maximize J_γ (or, in other words, to minimize $-J_\gamma$). The game considered above is called a soft-constrained descriptor dynamic game. The soft-constrained term is added to indicate that there is no hard bound for u_2 in this game [18].

We begin this section by reviewing some basic results of descriptor systems. Initially, we call from [25] that if there exists $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) \neq 0$ then the couple (E, A) is called regular, otherwise it is called non-regular. If (E, A) is regular, then the system (1) is said to be regular. Next, when the system (2.1) is regular then there are non-singular matrices $X, Y \in \mathbb{R}^{(n+r)\times(n+r)}$ such that [26]

$$Y^T E X = \begin{bmatrix} I_n & 0 \\ 0 & N \end{bmatrix} \ \textit{and}\ Y^T A X = \begin{bmatrix} A_1 & 0 \\ 0 & I_r \end{bmatrix} \tag{2.3}$$

[1] An initial state is called consistent if with the selection of this initial state the system (2.1) has a solution.

where A_1 and N are appropriate matrices, with N is a nilpotent matrix with index r and A_1 is an $(n \times n)$ Jordan matrix. The I_n and I_r denote identity matrices and 0 is a zero matrix. Applying the coordinates transformation

$$x_k = X \begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix}, \tag{2.4}$$

and using the result of (2.3), the system (2.1) then can be described by the following two subsystems

$$\begin{aligned} x_{1(k+1)} &= A_1 x_{1k} + Y_1 B_1 u_{1k} + Y_1 B_2 u_{2k},\ x_{1(k=1)} = \begin{bmatrix} I_n & 0 \end{bmatrix} X^{-1} x_0 \\ N x_{2(k+1)} &= x_{2k} + Y_2 B_1 u_{1k} + Y_2 B_2 u_{2k} \end{aligned} \tag{2.5}$$

where $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$. The solution of system (2.5) is given by

$$\begin{aligned} x_{1k} &= A_1^q x_{1,0} + \sum_{j=1}^{2} \sum_{i=0}^{q-1} A_1^{q-i-1} Y_1 B_j u_{ji} \\ x_{2k} &= -\sum_{j=1}^{2} \sum_{i=0}^{q-1} N^i Y_2 B_j u_{j(k+i)}. \end{aligned}$$

We define q is the nilpotency degree of N, that is the integer q such that $N^q = 0$ and $N^{q-1} \neq 0$. The index of system (2.1) is the degree q of nilpotency of N. When the matrix E is nonsingular, we specify the index to be zero. If the system (2.1) has index more than one, then the impulses can appear in the system response if the control of the system is not smooth enough. Moreover, since the system (2.1) is usually only an approximation of such nonlinear system then in general the disturbance of the system caused an impulsive solutions when the system is of index more than one. For that reason, here we restrict our system has index no more than one. From [27] the system (2.1) is regular and of index no more than one if and only if $rank\left(\begin{bmatrix} E & AS_\infty(E) \end{bmatrix}\right) = n + r$, where $S_\infty(E)$ satisfies the construction that $S_\infty(E)$ and $T_\infty(E)$ are full rank matrices whose columns span the null spaces $N(E)$ and $N(E^H)$.

Based on the above description, we make the following assumptions.

Assumption 2.1. In this paper, we assume the following applies with respect to system (2.1):

(1) matrix E is singular
(2) system (2.1) is regular
(3) $rank\left(\begin{bmatrix} E & AS_\infty(E) \end{bmatrix}\right) = n + r$.

Next, we describe the open-loop saddle point (OLSP) solution as a main object discussed in this paper.

Definition 2.2. Consider the zero-sum discrete-time soft-constrained descriptor dynamic game (2.1,2.2) with open-loop information structure, where Assumption 2.1 applies and the initial state $x_{1,0}$ is consistent. The set $U_s = U_{s1} \times U_{s2}$ define the set of bounded piecewise continuous functions that represent the set of all admissible actions for both players. The pair (u_{1k}, u_{2k}) are an open-loop saddle-point (OLSP) equilibrium for the game (2.1,2.2) if for every $(u_{1k}, u_{2k}^*), (u_{1k}^*, u_{2k}) \in U_s$, the following inequalities hold

$$J_\gamma (u_{1k}^*, u_{2k}) \leq J_\gamma (u_{1k}^*, u_{2k}^*) \leq J_\gamma (u_{1k}, u_{2k}^*).$$

To find the above OLSP equilibrium the idea is to transform the index one descriptor dynamic game (2.1,2.2) into a reduced nonsingular dynamic game. That there is no hassle in the function of final state, then we consider that [28]

$$X^T \bar{Q}_K X = \begin{bmatrix} Q_K & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } Q_K \in \mathrm{R}^{n\times n}. \tag{2.6}$$

Under the index one assumption, with the coordinate transformation (2.4) the game (2.1,2.2) has a set of OLSP equilibrium actions (u_{1k}, u_{2k}) if and only if (u_{1k}, u_{2k}) are OLSP equilibrium actions for the game [2]

$$x_{1(k+1)} = A_1 x_{1k} + Y_1 B_1 u_{1k} + Y_1 B_2 u_{2k}, \; x_{1(k=1)} = \begin{bmatrix} I_n & 0 \end{bmatrix} X^{-1} x_0 \tag{2.7}$$

$$0 = x_{2k} + Y_2 B_1 u_{1k} + Y_2 B_2 u_{2k} \tag{2.8}$$

where the quadratic cost functional for the first player is

$$\begin{aligned} J_\gamma (u_{1k}, u_{2k}) = \sum_{k=1}^{K} \left(\begin{bmatrix} x_{1k}^T & x_{2k}^T \end{bmatrix} X^T Q X \begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix} + u_{1k}^T R_1 u_{1k} - \gamma u_{2k}^T R_2 u_{2k} \right) \\ + \begin{bmatrix} x_{1(K+1)}^T & x_{2(K+1)}^T \end{bmatrix} X^T \bar{Q}_K X \begin{bmatrix} x_{1(K+1)} \\ x_{2(K+1)} \end{bmatrix}. \end{aligned} \tag{2.9}$$

From (2.8), it follows that

$$x_{2k} = -Y_2 B_1 u_{1k} - Y_2 B_2 u_{2k}. \tag{2.10}$$

Substitution (2.6) and (2.10) into the cost function (2.9) shows that (u_{1k}, u_{2k}) are OLSP equilibrium action for the game (2.1,2.2) if and only if (u_{1k}, u_{2k}) are OLSP equilibrium actions for the game described by the dynamical system (2.7) which the first player has a quadratic cost function of the form

$$\begin{aligned} J_\gamma (u_{1k}, u_{2k}) = \sum_{k=1}^{K} \Bigg(\begin{bmatrix} x_{1k}^T \; u_{1k}^T \; u_{1k}^T \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & -B_1^T Y_2^T \\ 0 & -B_2^T Y_2^T \end{bmatrix} \begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix} Q \begin{bmatrix} X_1 \; X_2 \end{bmatrix} \\ \times \begin{bmatrix} I_n & 0 & 0 \\ 0 & -Y_2 B_1 & -Y_2 B_2 \end{bmatrix} \begin{bmatrix} x_{1k} \\ u_{1k} \\ u_{2k} \end{bmatrix} + u_{1k}^T R_1 u_{1k} - \gamma u_{2k}^T R_2 u_{2k} \Bigg) \\ + x_{1(K+1)}^T Q_K x_{1(K+1)} \\ =: \sum_{k=1}^{N} \left(z_k^T M z_k \right) + x_{1(K+1)}^T Q_K x_{1(K+1)} \end{aligned} \tag{2.11}$$

[2] see [29] for the continuous (hard-constrained) open-loop differential game.

where $z_k = \begin{bmatrix} x_{1k} \\ u_{1k} \\ u_{2k} \end{bmatrix}$ and

$$M = \begin{bmatrix} \tilde{Q} & V & W \\ V^T & R_{11} & N \\ W^T & N^T & R_{22\gamma} \end{bmatrix}. \tag{2.12}$$

The spelling of the matrices defined in (2.12) is presented in the Appendix.

3. The OLSP Solution

In this section we deal with the game (2.1,2.2) that equivalent with (2.7,2.11) under the assumption that K is finite and the consistent initial state of the game to be known, but nor necessary zero. First, we recall a theorem from [18] that states the existence of solution for such game.

Theorem 3.1. *Consider the dynamic game (2.7,2.11). Let U_{si} be convex, $J_\gamma(u_{1k}, u_{2k})$ be convex in $u_{1k} \in U_{s1}$ for every $u_{2k} \in U_{s2}$ and concave in $u_{2k} \in U_{s2}$ for every $u_{1k} \in U_{s1}$. Then there exists a saddle point solution in pure strategies for this game. If, furthermore, $J_\gamma(u_{1k}, u_{2k})$ is strictly convex-concave, the saddle point solution is unique.*

According to Theorem 3.1 above, to find the unique OLSP solution of the game, first we set the open-loop action of the second player, say $\bar{u}_{2k}$, $k \in [1, K]$, and minimize $J_\gamma(u_{1k}, \bar{u}_{2k})$ in (2.11) with respect to u_{1k}, $k \in [1, K]$. This issue mathematically may be described as finding a unique solution for the following linear quadratic optimization problem

$$\min_{u_{1k}} J_\gamma(u_{1k}, \bar{u}_{2k}).$$

Using the linear quadratic optimization control theory (see e.g. [20]), it states that the solution exists and is unique. Furthermore, since $J_\gamma(u_{1k}, \bar{u}_{2k})$, then this solution is strictly convex in u_{1k}. Mathematically, this solution is characterized by the discrete-time Riccati equation

$$\begin{aligned} &\bar{A}_{11} S_k = \tilde{Q}_{11} - A_1^T S_{k+1} \tilde{A}_{11} - A_1^T S_{k+1} \hat{B} \left(\gamma I + \hat{B}^T S_{k+1} \hat{B}\right) \hat{B}^T S_{k+1} \tilde{A}_{11}; \\ &S_{K+1} = Q_{1K}, \end{aligned}$$

where $\bar{A}_{11} = I + Y_1 B_1 R_{11}^{-1} V^T$, $\tilde{A}_{11} = A_1 - Y_1 B_1 R_{11}^{-1} V^T$, $\tilde{Q}_{11} = \bar{Q} - V R_{11}^{-1} V^T$, and $\hat{B}\hat{B}^T = Y_1 B_1 R_{11}^{-1} B_1^T Y_1^T$.

Next, conversely, we set the open-loop action of the first player, say $\bar{u}_{1k}$, $k \in [1, K]$, and minimize $J_\gamma(\bar{u}_{1k}, u_{2k})$ in (2.11) with respect to u_{2k}, $k \in [1, K]$. Commensurate with the explanation above, this issue mathematically may be described as finding a unique solution for the following linear quadratic optimization problem

$$\max_{u_{2k}} J_\gamma(\bar{u}_{1k}, u_{2k}). \tag{3.1}$$

To assure the above optimization problem has a unique solution, we require that $J_\gamma(\bar{u}_{1k}, u_{2k})$ is strictly concave in u_{2k} for every fixed u_{1k}, $k \in [1, K]$. The following lemma translates the above requirement.

Lemma 3.2. *Consider the dynamic game (2.7,2.11). For every open-loop strategy u_{1k} of the first player, its cost function is strictly concave in u_{2k} if and only if*

$$\gamma I + \bar{B}^T S_{k+1} \bar{B} > 0, \;\; k \in [1, K] \tag{3.2}$$

where the sequence of matrices S_{K+1}, $k \in [1, K]$, is constructed by the Riccati equation

$$\begin{aligned} &\bar{A}_{12} S_k = \tilde{Q}_{12} - A_1^T S_{k+1} \tilde{A}_{12} - A_1^T S_{k+1} \bar{B} \left(\gamma I + \bar{B}^T S_{k+1} \bar{B}\right) \bar{B}^T S_{k+1} \tilde{A}_{12}; \\ &S_{K+1} = Q_{2K}, \end{aligned} \tag{3.3}$$

where $\bar{A}_{12} = I + Y_1 B_2 R_{22\gamma}^{-1} W^T$, $\tilde{A}_{12} = A_1 - Y_1 B_2 R_{22\gamma}^{-1} W^T$, $\tilde{Q}_{12} = \bar{Q} - W R_{22\gamma}^{-1} W^T$, $\bar{B}\bar{B}^T = Y_1 B_2 R_{22\gamma}^{-1} B_2^T Y_1^T$.

Proof. Since the game (2.7,2.11) is zero-sum game and the cost function $J_\gamma(\bar{u}_{1k}, u_{2k})$ is a function in quadratic form of u_{2k}, then the condition that (3.1) must be strictly concave is equivalent to finding the unique solution of the optimal control problem

$$\min_{u_{2k}} \left[-J_\gamma(\bar{u}_{1k}, u_{2k})\right] \tag{3.4}$$

subject to (2.7) and for each $\bar{u}_{1k}$, $k \in [1, K]$. Without any loss of generality, we can take $\bar{u}_{1k} = 0$ and then the optimal control problem (3.4) is in the form of standard dynamic one-person optimization problem

$$\min_{u_{2k}} \sum_{k=1}^{K} \left(u_{2k}^T \Theta_1 u_{2k} - \xi_k^T \Theta_2 \xi_k\right)$$

subject to the dynamical system

$$\xi_{k+1} = A_1 \xi_k + Y_1 B_2 u_{2k}$$

for some matrices Θ_1, Θ_2, and state ξ_k. From this, the result follows from the dynamic programming procedure for discrete-time systems (see e.g. [30], sec 5.5.1) which admits the unique solution if and only if (3.2) and (3.3) hold. □

So, Lemma 1 assures the strict convexity and concavity of the open-loop soft-constrained game (2.7,2.11). From Theorem 1, than the game has a unique OLSP solution. Before we present the solution of the game (2.7,2.11), here we recall from [30] about some conditions when the game has a saddle-point solution.

Theorem 3.3. *Consider the discrete-time two-person zero-sum dynamic game*

$$x_{k+1} = f_k(x_k, u_{1k}, u_{2k}) \, k \in [1, K]$$

with the cost function

$$J_i(u_1, u_2) = \sum_{k=1}^{K} g_{ik}(x_{k+1}, u_{1k}, u_{2k}),$$

let

(1) *$f_k(\cdot, u_{1k}, u_{2k})$ be continuously differentiable on $\mathbb{R}^n, k \in [1, K]$,*

(2) $g_{ik}(\cdot, u_{1k}, u_{2k})$ *be continuously differentiable on* $\mathbb{R}^n \times \mathbb{R}^n, k \in [1, K]$,
(3) $f_k(\cdot,\cdot,\cdot)$ *be convex on* $\mathbb{R}^n \times U_{1s} \times U_{2s}, k \in [1, K]$.

Then, if the open-loop saddle-point solution of the game is $\{u^*_{ik}, i = 1, 2\}$ *with the corresponding state trajectory is* $\left\{x^*_{k+1}; k \in [1, K]\right\}$, *then a finite sequence of* n *dimensional (costate) vectors* $\{\lambda_2, ..., \lambda_{k+1}\}$ *exists such that it satisfies the following relations:*

$$x^*_{k+1} = f_k\left(x^*_k, u^*_{1k}, u^*_{2k}\right), x^*_1 = x_1, \tag{3.5}$$

$$\begin{aligned} &H_{ik}\left(\lambda_{ik+1}, u^*_{1k}, u_{2k}, x^*_{1k}\right) \leq H_{ik}\left(\lambda_{ik+1}, u^*_{1k}, u^*_{2k}, x^*_{1k}\right) \leq \left(\lambda_{ik+1}, u_{1k}, u^*_{2k}, x^*_{1k}\right), \\ &\forall u_{ik} \in U_{si} \end{aligned} \tag{3.6}$$

$$\begin{aligned} \lambda_k = &\tfrac{\partial}{\partial x_k} f_k\left(x^*_k, u^*_{1k}, u^*_{2k}\right)^T \left[\lambda_{k+1} + \left(\tfrac{\partial}{\partial x_{k+1}} g_{ik}\left(x_{k+1}, u_{1k}, u_{2k}, x_k\right)\right)^T\right] \\ &+ \left[\tfrac{\partial}{\partial x_k} g_{ik}\left(x_{k+1}, u_{1k}, u_{2k}, x_k\right)\right] \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} &H_{ik}\left(\lambda_{ik+1}, u_{1k}, u_{2k}, x_{1k}\right) = g_{ik}\left(f_k\left(x_k, u_{1k}, u_{2k}\right), u_{1k}, u_{2k}, x_k\right) \\ &+\lambda^T_{ik+1} f_k\left(x_k, u_{1k}, u_{2k}\right),\ k \in [1, K]. \end{aligned}$$

Now, we ready to state the solution of the soft-constrained game (2.7,2.11) as the main result in this paper. The following notation is used in the next theorem $G = \begin{bmatrix} R_{11} & N \\ N^T & R_{22\gamma} \end{bmatrix} := \bar{I}\bar{G}$, where $\bar{G} = \begin{bmatrix} R_{11} & N \\ -N^T & -R_{22\gamma} \end{bmatrix}$, $\bar{I} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, $Z = \begin{bmatrix} V & W \end{bmatrix}$, $\bar{Z} = \begin{bmatrix} V & -W \end{bmatrix}$, $B = \begin{bmatrix} Y_1B_1 & Y_2B_2 \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} Y_1B_1 & -Y_2B_2 \end{bmatrix}$, $\hat{Q} = \bar{Q} - ZG^{-1}Z^T$, and $\hat{A}_1 = A_1 - BG^{-1}Z^T$.

Theorem 3.4. *Consider the open-loop zero-sum soft-constrained linear quadratic discrete-time dynamic game described by (2.7), with the quadratic cost function for the first player is (2.11) and the second player has the opposite objective function* $-J_\gamma(u_{1k}, u_{2k})$. *Assume* $R_i,\ i = 1, 2$ *are positive definite and* $R_{22\gamma}$ *is negative definite, and* $M_k,\ k \in 1, K$, *is a sequence of matrices generated by*

$$M_k = \hat{Q} + \hat{A}^T_1 M_{k+1} \Delta^{-1}_k \hat{A}_1 \tag{3.8}$$

where

$$\Delta_k = I + BG^{-1}B^T M_{k+1}.$$

Then,

(1) *The matrices* $\Delta_k,\ k \in [1, K]$ *are invertible.*
(2) *The dynamic game (2.7,2.11) has a unique OLSP solution, given by*

$$\begin{bmatrix} u^*_{1k} \\ u^*_{2k} \end{bmatrix} = -\bar{I}\bar{G}^{-1}\left(\bar{Z}^T x^*_{1k} + \tilde{B}^T M_{k+1} x^*_{1k+1}\right), k \in [1, K] \tag{3.9}$$

where $\left\{x^*_{1k+1}, k \in [1, K]\right\}$ *is the corresponding state trajectory, generated by dynamical system*

$$x^*_{1k+1} = \Delta^{-1}_k \hat{A}_1 x^*_{1k}. \tag{3.10}$$

(3) *The saddle-point value of the game is*

$$J_\gamma^*\left(u_{1k}^*, u_{2k}^*\right) = \left(\left[\begin{array}{cc} I & 0 \end{array}\right] X^{-1} x_{1,1}\right)^T M_1 \left(\left[\begin{array}{cc} I & 0 \end{array}\right] X^{-1} x_{1,1}\right).$$

(4) *The upper value of the dynamic game (2.7,2.11) becomes unbounded if the matrix in (3.2) has at least one negative eigenvalue.*

Proof. Firstly, let us notice that under condition (3.2) and with the open-loop information structure, the linear quadratic zero-sum game (2.7,2.11) is strictly convex-concave which admits a unique OLSP solution by Theorem 3.1. Second, from Theorem 3.1 we infer that this unique saddle-point solution should meet equations (3.5) - (3.7), which for the game (2.7,2.11) it can be rewritten as follows:

$$x_{1(k+1)} = A_1 x_{1k} + Y_1 B_1 u_{1k} + Y_1 B_2 u_{2k},\ x_{1(k=1)}^* = x_{1(k=1)},$$

$$H_{ik}\left(\lambda_{ik+1}, u_{1k}^*, u_{2k}, x_{1k}^*\right) \leq H_{ik}\left(\lambda_{ik+1}, u_{1k}^*, u_{2k}^*, x_{1k}^*\right) \leq \left(\lambda_{ik+1}, u_{1k}, u_{2k}^*, x_{1k}^*\right), \\ \forall u_{ik} \in U_{si},$$

and

$$\lambda_{ik} = \bar{Q} x_{1k} + V u_{1k} + W u_{2k} + A_1^T \lambda_{ik+1},$$

where

$$H_{ik}\left(\lambda_{ik+1}, u_{1k}, u_{2k}, x_{1k}\right) = z_k^T M z_k + \lambda_{ik+1}\left(A_1 x_{1k} + Y_1 B_1 u_{1k} + Y_1 B_2 u_{2k}\right), \\ k \in [1, K].$$

An inductive argument indicates that the above relations have the solution

$$\left[\begin{array}{c} u_{1k}^* \\ u_{2k}^* \end{array}\right] = -\bar{I}\bar{G}^{-1}\left(\bar{Z}^T x_{1k}^* + \tilde{B}^T M_{k+1} x_{1k+1}^*\right)$$

which verifies (3.9). The corresponding value of the state vector, x_{1k+1}^*, in the equation above satisfies

$$\Delta_k x_{1k+1}^* = \hat{A}_1 x_{1k}^*.$$

Since the game has a unique saddle-point solution, it causes a one-to-one correspondence between x_{1k}^* and x_{1k+1}^* that implies the matrix Δ_k should be invertible for each $k \in [1, K]$. Hence

$$x_{1k+1}^* = \Delta_k^{-1} \hat{A}_1 x_{1k}^*$$

which verifies (3.10). □

4. Concluding Remark

In this paper we have constructed the theorem regarding the existence of an open-loop saddle point (OLSP) equilibrium for open-loop discrete-time zero-sum linear quadratic soft-constrained dynamic games. We have shown how the Riccati equation (15) plays an important role for the games. Nevertheless, we only consider an open-loop information structure in this paper. So, another information structure such as feedback and delayed information is an open problem left for further research.

Appendix

The following are shorthand notation used in this paper:
$\tilde{Q} := X_1^T Q X_1$, $V := -X_1^T Q X_2 Y_2 B_1$,
$W := -X_1^T Q X_2 Y_2 B_2$, $N := B_1^T Y_2^T X_2^T Q X_2 Y_2 B_2$,
$R_{11} := B_1^T Y_2^T X_2^T Q X_2 Y_2 B_1 + R_1$, $R_{22\gamma} := B_2^T Y_2^T X_2^T Q X_2 Y_2 B_2 - \gamma R_2$.

References

1. Dockner, E., Jørgensen, S., Long, N. van, Sorger, G.: *Differential Game in Economic and Management Science*, Cambridge Univesity Press, Cambridge, 2000.
2. Haurie, A., Krawczyk, J., Zaccour, G.: *Games and Dynamic Games, vol. 1*, World Scientific Publishing Company, Singapore, 2012.
3. Katayama, T., Minamino, K.: Linear Quadratic Regulator and Spectral Factorization for Continuous-time Descriptor Systems, *Proceedings IEEE CDC* **31** (1992), 967–972.
4. Mahmoud, M. S., Boukas, E. K.: Delay-Dependent Robust Stability and Control of Uncertain Discrete Singular Systems with State-Delay, *Journal of Mathematical Control Science and Applications* **1:1** (2015), 11–27.
5. Mansouri, M., Baddas, L.B., Darouach, M., Messaoud, H.: Decentralized Observers for a Class of Largescale sSingular Systems via LMI , *International Journal of Intelligent Computing and Cybernetics* **6:2** (2013), 158–181.
6. Li, J., Zhang, Q., Ren, J., Zhang, Y.: Robust Decentralised Stabilisation of Uncertain Large-scale Interconnected Nonlinear Descriptor Systems via Proportional Plus Derivative Feedback, *International Journal of Systems Science* **48:14** (2017), 1–10.
7. Zhang, Y., Shi, Z.: Sliding Mode Control for Uncertain t-s Fuzzy Singular Biological Economic System, *IEEE Access* **7** (2019), 14387–14395.
8. Newcomb, R. W., Dziurla, B.: Some Circuits and Systems Applications of Semistate Theory, *Circuits Systems Signal Processes* **8** (1989), 235–260.
9. Newcomb, R. W.: The Semistate Description of Nonlinear Time-variable Circuits, *IEEE Transactions on Circuits Systems* **28** (1981), 62–71.
10. Kumar, A., Daoutidis, P.: State-space Realizations of Linear Differential Algebraic-equation Systems with Control-dependent State Space, *IEEE Transactions on Automatic Control* **41** (1996), 269–274.
11. Aranguiz, E., Barbosa, K.A., Verdejo, H.: D-Stability Analysis of Descriptor Systems and Application to Power Systems, *IEEE Latin America Transactions* **13:2** (2015), 1888–1892.
12. Hossain, M.S. and Uddin, M.M.: Reduce Order Modelling of Power System Models using Interpolatory Projections Technique, *International Journal of Modeling and Optimization* **5:3** (2015), 228–233.
13. Muller, P.C: Linear Mechanical Descriptor Systems: Identifcation, Analysis and Design, *IFAC Proceedings Volumes* **30:6** (1997) 375-380.
14. Muller, P.C: Optimal Control of Mechanical Descriptor Systems, in: *Van Campen D.H. (eds) IUTAM Symposium on Interaction between Dynamics and Control in Advanced Mechanical Systems. Solid Mechanics and Its Applications* , (1997) 247–254, IUTAM, Dordrecht.
15. Buzurovic, I.M.: Dynamic Model of Medical Robot Represented as Descriptor System, *International Journal of Information and Systems Sciences* **2:2** (2007), 316-333.
16. Buzurovic, I.M., Debeljkovic, D.L.: Contact Problem and Controllability for Singular Systems in Biomedical Robotics, *International Journal of Information and Systems Sciences* **6:2** (2010), 128–141.
17. Musthofa, M. W. and Salmah and Engwerda, J. C. and Suparwanto, A.: Robust Optimal Control Design with Differential Game Approach for Open-loop Linear Quadratic Descriptor Systems, *Journal of Optimization Theory and Applications (JOTA)* **163** (2016), 1046–1064.
18. Başar, T., Bernhard, P.: *H_∞ Optimal Control and Related Minimax Design Problem*, Modern Birkhäuser Classics, Boston, 1995.
19. Xu, H.: A Dynamic Game Approach to Disturbance Attenuation Control of Discrete-time Descriptor Systems *IFAC Proceedings Volumes* **32:2** (1999), 3171–3176.

20. Engwerda, J.C.: *Linear Quadratic Dynamic Optimization and Differential Games*, John Wiley & Sons West Sussex, 2005.
21. Mizukami, K., Xu, H., Suzumura, F.: Open Loop Linear Quadratic Nash Game for Discrete Descriptor Systems *Trans. of the Society of Instrument and Control Engineers* **27** (1991), 169–176.
22. Salmah: Feedback Zero-sum Linear Quadratic Dynamic Game for Descriptor System *WSEAS TRANSACTIONS on SYSTEMS and CONTROL* **10** (2015), 476–483.
23. Xu, H., Mukaidani, H.: The Linear Quadratic Dynamic Game for Discrete-time Descriptor Systems *International Game Theory Review* **5** (2003), 361–374.
24. Jungers, M.: Feedback Strategies for Discrete-time Linear-quadratic Two-player Descriptor Games *Linear Algebra and its Applications* **440** (2014), 1–23.
25. Chaves-Fuentes, J. R., Costa, E. F., Mayta, J. E., Terra, M. H.: Regularity and Stability Analysis of Discrete-time Markov Jump Linear Singular Systems *Automatica* **76** (2017), 32–40.
26. Apostolopoulos, N., Ortega, F., Kalogeropoulos, G.: Causality of Singular Linear Discrete Time Systems *arXiv preprint arXiv:1512.04740* (2015), 1–8.
27. Nocholas, N. K., Chu, D.: *Regularization of Descriptor Systems*, Springer, Switzerland, 2015.
28. Mehrmann, V. L., Thoma, M. and Wyner, A.: *The Autonomous Linear Quadratic Control Problems*, Springer-Verlag, Berlin, 1991.
29. Engwerda, J.C., Salmah: The Open Loop Linear Quadratic Differential Game for Index One Descriptor Systems *Automatica* **45** (2009), 585–592.
30. Başar, T., Olsder, G.J.: *Dynamic Noncooperative Game Theory*, Academic Press SIAM, New York, 1999.

Muhammad Wakhid Musthofa: Mathematics Department, UIN Sunan Kalijaga Yogyakarta, Indonesia
E-mail address: muhammad.musthofa@uin-suka.ac.id