Journal of Mathematical Control Science & Applications, Vol. 8 No. 1 (January-June, 2022)
ISSN : 0974-0570

FOURTH-ORDER COMPACT METHOD FOR FISHER'S EQUATION

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Abstract. This work consists of the study of fourth-order compact method for solving the well-known Fishers equation. The Fishers equation is a second-order nonlinear partial differential equation. Numerical calculations can be furnished in the formation of accurate results [5]. The proposed problems that seem to be very difficult at a first sight can be solved by supposing the unknown functions as x(t) and y(t). The proposed problems considered may be turned into the more complex form, the standard second-order methods become less suitable for use due to increase in the number of grid points necessary for accuracy. The compact method procedure requires only three nodes to yield a fourth-order accuracy as compared to the five nodes that are required to get the similar precision. Mainly two different schemes or approximations methods are used naming forward difference approximation and backward difference approximation. Fourth-order compact method is a useful system as it provides us a higher order approximation consuming fewer mesh points than the five mesh points that a standard fourth-order approximation requires. These are constructed by difference techniques that consider the function and all necessary derivatives as unknowns. The relation of these functions and derivative leads to simple tri-diagonal equations that can easily be solved.

1. Introduction

To recognize the existence of PDEs in a mathematical study of real life, we must be familiar with many of the real word problems that are explained by choosing a function having two or more two variables which are independent. Therefore, in this way, we can construct a function leading its ordinary derivatives to partial derivatives in the formation of PDE. In mathematical physics, PDEs are mostly used to classify the equations [1, 2]. PDEs play a vital role in many other subjects like electrical engineering, mechanical engineering, computer science, modeling of biological and chemical structures, and many more. They are also very useful in diffusion equation, heat conduction, and viscous fluid flow. We can generalize the DE having ordinary as well as partial derivatives. The solutions obtained from these equations will have one or more than one variable. These obtained equations are natural, for example, if someone is dealing with the situation having position in space over time and he wants to model this situation, he will be in need of a

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2000 *Mathematics Subject Classification.* Primary 35A09; Secondary 65M06, 65M32.
Key words and phrases. Partial differential equations; Compact method; Fishers equation; Finite difference method.

function of several variables for spatial dimensions with respect to time. Some useful partial differential equations are given below:

1-D wave equation is given by

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} \tag{1.1}$$

1-D heat equation is given by

$$\frac{\partial f}{\partial t} = c^2 \frac{\partial^2 f}{\partial x^2} \tag{1.2}$$

2-D Laplace equation is given by

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{1.3}$$

2-D Poisson's equation is shown below

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = g(x,\ y) \tag{1.4}$$

2. Materials and methods

In partial differential equations, we use the letter f instead of y, which is commonly used in multivariable functions. The same methodology can be applied to explain partial differential equations as for ordinary differential equations. We introduce any equation that involves one or more than one partial derivative of a function having multivariable [1]. In equation (4), g(x, y) is just a function consists of variables x and y and has no effect on f(x, y). All partial differential equations used in (1), (2), (3), and (4) are linear and of second order. The equations (1), (2), and (3) are homogenous but (4) is non-homogenous as g(x, y) does not involve f or any of its derivatives. Homogeneity and linearity of the partial differential equations mentioned above are discussed because of the principle of superposition. In this principle, we consider the solution of linear homogenous partial differential equations. Consider f_1 and f_2 be two functions satisfying a linear homogenous DE. We know that the derivative of the sum of two functions will be equal to the sum of their derivatives. And the sum of f_1 and f_2 will also be the solution of a particular DE. Lets consider an example, in which f_1 = Cos(xy) & f_2 = Sin(xy) are the solutions of any partial differential equations (5) and (6), respectively. The PDEs defined by (5) and (6) are of first order, linear, and homogenous.

$$xf_{1_x} - yf_{1_y} = 0 \tag{2.1}$$

$$xf_{2_x} - yf_{2_y} = 0 \tag{2.2}$$

This principle is very important and essential, because it supports us to construct a particular solution out of infinite solutions with the help of Fourier series. We can categorize the partial differential equations as parabolic, elliptic or hyperbolic [4]. This classification depends upon the sign of b^2 4ac, which is mentioned as follows:

i) If b^2 4ac is greater than zero, then the given equation will be hyperbolic.

ii) If b^2 4ac is smaller than zero, then the given equation will be elliptic.
iii) If b^2 4ac is equal to zero, then the given equation will be parabolic.
Such systems have various distinctive behaviors and can be categorized as follows:

Name of Equation	***Wave***	***Diffusion***	***Poisson's***
Symbolization	$f_x = a^2 f_{xx}$	$f_t = a^2 f_{xx}$	$f_{xx} + f_{yy} = g\,(x,\ y)$
Coefficients of a, b, c	1, 0, -a^2	0, 0, -a^2	1, 0, 1
Sign of discriminant	+Ve	0	-Ve
PDE Class Name	Hyperbolic	Parabolic	Elliptic

Partial differential equations can be solved by different methods and can be explained by examples.

3. Fourth-order compact method for Fishers equation

The fourth-order compact method [9, 10] has been constructed for the one-dimensional gas dynamics equations. The compact method is mainly used to give the accuracy of fourth-order with the help of only three nodes, but in other methods, we need at least five nodes to get the same accurate results. It is treated by adopting a differencing technique that studies the function of its all essential derivatives as unknowns. Simple tridiagonal system of equations can be correlated with the relations of above mentioned derivatives and then can easily be solved. The results accomplished by fourth-order compact method [3] are more accurate using fewer calculations. We see that the precision attained by fourth-order compact method is considerably better than the others.

4. Introduction to Fishers equation

Fisher's equation is a 2^{nd} order nonlinear PDE [8]. It is very useful in many other subjects such as problems created in physics. $U_t = U_{xx} + U - U^2$
It admits traveling wave solutions followed as:

$$f\,(x,\ y) = \frac{1}{\left[1 + c^{\frac{-5t}{6}} \pm \frac{x}{\sqrt{6}}\right]^2}$$

Where c is an arbitrary constant.

5. Numerical scheme of fourth-order compact method for Fishers equation

Consider the second-order nonlinear Fisher's equation

$$U_t = U_{xx} + U\,(1 - U) \tag{5.1}$$

With the following boundary and initial conditions
Left boundary condition is

$$U\,(-0.2,\ t) = 1 \tag{5.2}$$

Right boundary condition is

$$U\,(0.8,\ t) = 0 \tag{5.3}$$

Initial condition is

$$U(x,\ 0) = U_0(x) = \frac{1}{[1 + e^{\frac{x}{\sqrt{6}}}]^2} \tag{5.4}$$

We now discuss a fourth-order approximation consisting of three grid points only. It is a very useful scheme as it provides us a higher order approximation involving fewer grid points than the five grid points that a standard fourth-order approximation involves. This method estimates (7) by two difference equations of fourth-order using only three grid points x_{i-1},x_i and x_{i+1}. To develop the suitable finite difference equations [6, 7], firstly, new variables for the derivatives are introduced. Suppose the first and second derivatives of U with respect to x are F and S, respectively.

That is,

$$U_x = F \;\&\; U_{xx} = S \tag{5.5}$$

Integrating both sides of equation (5.5) from " -1 to $i+1$ " we get

$$U_{i+1}^n - U_{i-1}^n = \int_{x_{i-1}}^{x_{i+1}} F(\xi,\ 1)\ d\xi$$

$$U_{i+1}^n = U_{i-1}^n + \int_{x_{i-1}}^{x_{i+1}} F(\xi,\ 1)\ d\xi \tag{5.6}$$

Approximating the integral by Simpson's Rule:

$$U_{i+1}^n = U_{i-1}^n + \frac{h}{3}\left[F_{i-1}^n + 4F_i^n + F_{i+1}^n\right] + \frac{1}{90}\left[h^5 F^{(4)}(\xi,\ 1)\right]$$

$$F_{i-1}^n + 4F_i^n + F_{i+1}^n + \frac{3}{h}\left[U_{i-1}^n - U_{i+1}^n\right] = \frac{-1}{30}\left[h^4 F^{(4)}(\xi,\ 1)\right]$$

Thus to the fourth-order we have

$$F_{i-1}^n + 4F_i^n + F_{i+1}^n + \frac{3}{h}\left[U_{i-1}^n - U_{i+1}^n\right] = 0 \tag{5.7}$$

To get the second equation, we evaluate (5.1) at the midpoint i.

$$U_t\big|_i^n = S_i^n + U_i^n\left(1 - U_i^n\right) \tag{5.8}$$

We need an expression for S_i^n .If we express U_{i+1}^n,U_{i-1}^n in the Taylor's expansion about the point i. We get

(5.9)

$$U_{i+1}^n = U_i^n + h\, U_x\big|_i^n + \frac{h^2}{2!} U_{xx}\big|_i^n + \frac{h^3}{3!} U_{xxx}\big|_i^n + \frac{h^4}{4!} U_{xxxx}\big|_i^n + \frac{h^5}{5!} U_{xxxxx}\big|_i^n + \ldots$$

(5.10)

$$U_{i-1}^n = U_i^n - h\, U_x\big|_i^n + \frac{h^2}{2!} U_{xx}\big|_i^n - \frac{h^3}{3!} U_{xxx}\big|_i^n + \frac{h^4}{4!} U_{xxxx}\big|_i^n - \frac{h^5}{5!} U_{xxxxx}\big|_i^n + \ldots$$

Now adding (5.9) and (5.10).

$$U_{i+1}^n + U_{i-1}^n = 2U_i^n + h^2\, U_{xx}\big|_i^n + \frac{h^4}{12} U_i^{n(4)} + \frac{h^6}{360} U_i^{n(6)}(\xi,\ 1)$$

Because $U_{xx} = S$

$$U_{i+1}^{n} + U_{i-1}^{n} = 2U_{i}^{n} + h^{2}S_{i}^{n} + \frac{h^{4}}{12}U_{i}^{n(4)} + \frac{h^{6}}{360}U_{i}^{n(6)}(\xi,\ 1) \tag{5.11}$$

(5.12)

$$F_{i+1}^{n} = F_{i}^{n} + h\ F_{x}\Big|_{i}^{n} + \frac{h^{2}}{2!}\ F_{xx}\Big|_{i}^{n} + \frac{h^{3}}{3!}\ F_{xxx}\Big|_{i}^{n} + \frac{h^{4}}{4!}\ F_{xxxx}\Big|_{i}^{n} + \frac{h^{5}}{5!}\ F_{xxxxx}\Big|_{i}^{n} + \ldots$$

(5.13)

$$F_{i-1}^{n} = F_{i}^{n} - h\ F_{x}\Big|_{i}^{n} + \frac{h^{2}}{2!}\ F_{xx}\Big|_{i}^{n} - \frac{h^{3}}{3!}\ F_{xxx}\Big|_{i}^{n} + \frac{h^{4}}{4!}\ F_{xxxx}\Big|_{i}^{n} - \frac{h^{5}}{5!}\ F_{xxxxx}\Big|_{i}^{n} + \ldots$$

Subtracting (5.12) and (5.13), we get

$$F_{i+1}^{n} - F_{i-1}^{n} = 2h\ F_{x}\Big|_{i}^{n} + \frac{2h^{3}}{3!}\ F_{xxx}\Big|_{i}^{n} + \frac{2h^{5}}{5!}F_{i}^{n(5)}(\xi,\ 1)$$

$F_{x} = S\ ,\ F_{xxx}\Big|_{i}^{n} = U_{i}^{n(1.4)}\ ,\ F_{i}^{n(2.1)} = U_{i}^{n(2.2)}$

$$F_{i+1}^{n} - F_{i-1}^{n} = 2hS_{i}^{n} + \frac{h^{3}}{3}U_{i}^{n(4)} + \frac{h^{5}}{60}U_{i}^{n(6)}(\xi,\ 1) \tag{5.14}$$

Now we eliminate $U_{i}^{n(4)}$ from equations (5.11) and (5.14).
Multiplying the equation (5.14) by $\frac{h}{4}$, we get

$$\frac{h}{4}\left[F_{i+1}^{n} - F_{i-1}^{n}\right] = \frac{h^{2}}{2}S_{i}^{n} + \frac{h^{4}}{12}U_{i}^{n(4)} + \frac{h^{6}}{240}U_{i}^{n(6)}(\xi,\ 1) \tag{5.15}$$

Subtracting equation (5.15) from equation (5.11), we get

$$U_{i+1}^{n} + U_{i-1}^{n} - \frac{h}{4}\left[F_{i+1}^{n} - F_{i-1}^{n}\right] = 2U_{i}^{n} + \frac{h^{2}}{2}S_{i}^{n} - \frac{h^{6}}{720}U_{i}^{n(6)}(\xi,\ 1)$$

$$\frac{h^{2}}{2}S_{i}^{n} = U_{i+1}^{n} + U_{i-1}^{n} - \frac{h}{4}\left[F_{i+1}^{n} - F_{i-1}^{n}\right] - 2U_{i}^{n} + \frac{h^{6}}{720}U_{i}^{n(6)}(\xi,\ 1)$$

$$S_{i}^{n} = \frac{2}{h^{2}}[U_{i+1}^{n} + U_{i-1}^{n} - \frac{h}{4}\left[F_{i+1}^{n} - F_{i-1}^{n}\right] - 2U_{i}^{n} + \frac{h^{6}}{720}U_{i}^{n(6)}(\xi,\ 1)]$$

$$S_{i}^{n} = \frac{2}{h^{2}}[U_{i+1}^{n} - 2U_{i}^{n} + U_{i-1}^{n}] - \frac{1}{2h}\left[F_{i+1}^{n} - F_{i-1}^{n}\right] + \frac{h^{4}}{360}U_{i}^{n(6)}(\xi,\ 1) \tag{5.16}$$

Therefore, the equation (5.1) becomes

$$\frac{dU_{i}^{n}}{dt} = S_{i}^{n} + U_{i}^{n}(1 - U_{i}^{n})$$

From the equation (5.16), putting the value of S_{i}^{n} in equation (5.8), we get

$$\frac{dU_{i}^{n}}{dt} = \frac{2}{h^{2}}[U_{i+1}^{n} - 2U_{i}^{n} + U_{i-1}^{n}] - \frac{1}{2h}\left[F_{i+1}^{n} - F_{i-1}^{n}\right] + U_{i}^{n}[1 - U_{i}^{n}]$$

$$U_{t}\Big|_{i}^{n} = \frac{2}{h^{2}}[U_{i+1}^{n} - 2U_{i}^{n} + U_{i-1}^{n}] - \frac{1}{2h}\left[F_{i+1}^{n} - F_{i-1}^{n}\right] + U_{i}^{n}[1 - U_{i}^{n}] \tag{5.17}$$

Equation (5.1) is replaced by two equations (5.7) and (5.17).
For i = 1

$F_0 + 4F_1 + F_2 + \frac{3}{h}[U_0 - U_2] = 0$ and

$$\frac{dU_1}{dt} = \frac{2}{h^2}[U_2 - 2U_1 + U_0] - \frac{1}{2h}[F_2 - F_0] + U_1[1 - U_1]$$

By letting $q = \frac{2}{h^2}$, $r_2 = \frac{1}{2h}$ & $r_1 = \frac{1}{h}$
The above two equations become

$$F_0 + 4F_1 + F_2 + 3r_1[U_0 - U_2] = 0 \tag{5.18}$$

$$\frac{dU_1}{dt} = q[U_2 - 2U_1 + U_0] - r_2[F_2 - F_0] + U_1[1 - U_1] \tag{5.19}$$

By using forward difference approximation

$$F_i = \frac{U_{i+1} - U_i}{h} \quad \& \quad U_0 = 1$$

And For $i = 0$, the above expression becomes

$$F_0 = \frac{U_1 - U_0}{h} = r_1(U_1 - 1)$$

Therefore, equation (5.18) & equation (5.19) become
$r_1[U_1 - 1] + 4F_1 + b_2(2, 1) + 3r_1[1 - U_2] = 0$, where $F_2 = b_2(2, 1)$
$r_1[U_1 - 3U_2 + 2] + 4F_1 + b_2(2, 1) = 0$ and

$$\frac{dU_1}{dt} = q[U_2 - 2U_1 + 1] - r_2[b_2(2, 1) - r_1(U_1 - 1)] + U_1[1 - U_1]$$

Similarly, we will have the expressions
for $\boldsymbol{i = 2, 3, \ldots, n-2}$ from the equations (5.7) & (5.17).
For $\boldsymbol{i = n - 1}$
By using backward difference approximation

$$F_i = \frac{U_i - U_{i-1}}{h} \quad \& \quad U_n = 0$$

And for $i = n$, the above expression becomes

$$F_n = \frac{U_n - U_{n-1}}{h} = \frac{0 - U_{n-1}}{h} = -r_1U_{n-1}$$

Therefore, equation (5.7) & equation (5.17) become

$$F_{n-2} + 4F_{n-1} + F_n + 3r_1[U_{n-2} - U_n] = 0$$

$$F_{n-2} + 4F_{n-1} + F_n + 3r_1U_{n-2} = 0 \tag{5.20}$$

$$\frac{dU_{n-1}}{dt} = q[U_n - 2U_{n-1} + U_{n-2}] - r_2[F_n - F_{n-2}] + U_{n-1}[1 - U_{n-1}],$$

where $F_{n-2} = b_{n-2}(n-2, 1)$

$$\frac{dU_{n-1}}{dt} = q[0 - 2U_{n-1} + U_{n-2}] - r_2[-r_1U_{n-1} - b_{n-2}(n-2, 1)] + U_{n-1}[1 - U_{n-1}]$$

$$\frac{dU_{n-1}}{dt} = q[U_{n-2} - 2U_{n-1}] \mp [r_1U_{n-1} + b_{n-2}(n-2, 1)] + U_{n-1}[1 - U_{n-1}] \tag{5.21}$$

By equation (5.7)

$$F_{i-1}^n + 4F_i^n + F_{i+1}^n + \frac{3}{h}\left[U_{i-1}^n - U_{i+1}^n\right] \;=\; 0$$

This equation shows a set of linear algebraic equations.
Equation (5.7) becomes

$$\frac{dU_i^n}{dt} = \frac{2}{h^2}[U_{i+1}^n - 2U_i^n + U_{i-1}^n] - \frac{1}{2h}\left[F_{i+1}^n - F_{i-1}^n\right] + U_i^n[1 - U_i^n].$$

This equation shows a system of differential equations and the superscript n is utilized to show the grid line of time. We have this compact scheme for the Fishers equation. We solve this system of differential equations using ODEs solver LSODE. Therefore, for each time we call LSODE to solve ODEs, we have to solve a set of linear algebraic equations for F. Now there is a comparison between the numerical solutions and the exact solutions of the Fishers equation using the fourth-order compact method by taking the points n = 70 and for different times t, i.e.

6. Results and discussion

In the figures 1, 2, 3, and 4, blue line shows the numerical solution of Fishers equation and the green line shows the exact solution using fourth-order compact method by taking n = 70 points at time t = 10^8 , t = 10^7 , t = 10^6 , and t = 10^5 , respectively. Therefore, it is very clear from the diagrams that the fourth-order compact method yields equally accurate results.

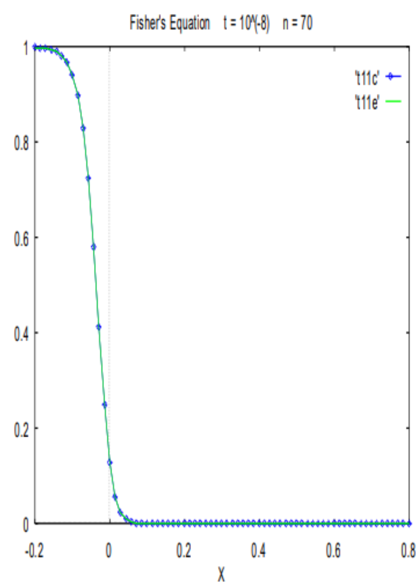

FIGURE 1. t=10^{-8}

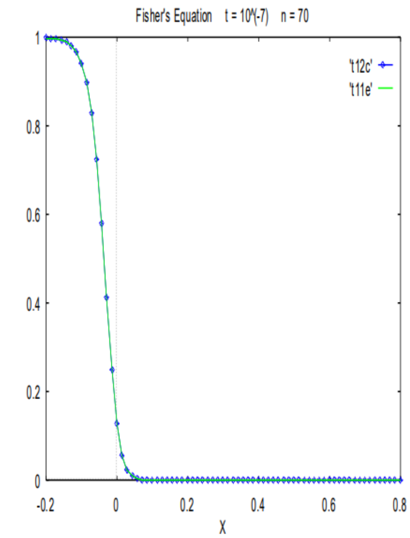

FIGURE 2. t=10^{-7}

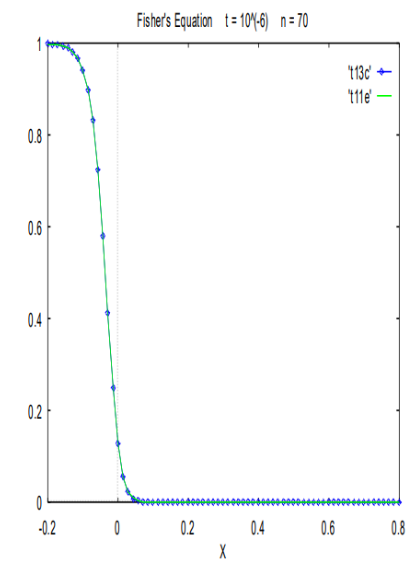

FIGURE 3. t=10^{-6}

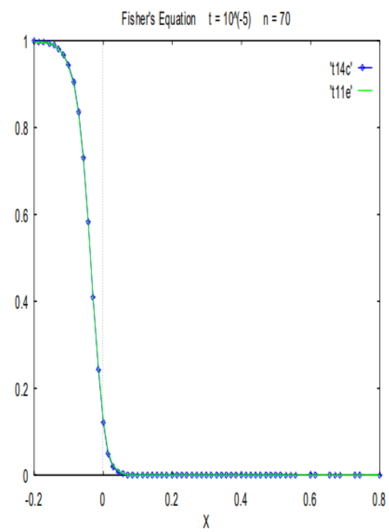

FIGURE 4. t=10^{-5}

7. Conclusion

In this study, we have discussed fourth-order compact method for solving the Fishers equation. The results that are obtained numerically can be compared with the exact solution. We observe that the results obtained from this method, using three nodes only are more accurate and precise.

Acknowledgment. All authors are grateful to those who encouraged and helped in producing such research work.

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