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A NEW DIVIDED DIFFERENCE INTERPOLATION METHOD FOR TWO-VARIABLE FUNCTIONS

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Abstract. Two-variable interpolation by polynomials is investigated for the given $f : \mathbb{R}^2 \to \mathbb{R}$. The new idea is to compute for the points on the two sides of a rectangle. In this paper, we present a generalization of the Newton divided interpolation polynomials in two dimension. The only bet is that in $(2n + 1)$ distinct points of function have similar quantities.

1. Introduction

Interpolation for one variable functions has been investigate extensively in the literature [1-15], but for two-variable functions less have been said because of the existence of the problems. So we suggest a special interpolation points which guarantee the existence of polynomial as sum of two polynomials corresponding to each variable. The existence and uniqueness of the method are inherent by one variable polynomial counterpart. It is worth to no to the interpolation is obtained by divided differences. The idea of this article can be extended for several variable functions. There is no need to mention its application, since wherever we need to compute double integration or even more, the integrand is not accessible or the antiderivative couldn't be found or the computations are very complicated. In section 2 the main idea is explained and finally numerical experiments are quoted.

Various kinds of interpolation formulas of one variable have long since been proposed, such as Round interpolation [21], Newton's, Stirling's, Bessel's, Gaus's, Everett's, etc [16-35]. They have served to the practical numerical analysis, that is, to the interpolation itself, to the mechanical quadrature, to the difference equation for ordinary differential equation, etc. But little has been worked out, with regard to the interpolation of the function of two or more independent variables. Though a few interpolation formulas of two variables were given by Shekhtman [20], Uchimura [33], Vasil'ev [34], Wulbert [35] and etc., the methods of their derivation, from the practical stand-point, is very tedious and troublesome and almost no further developments of the work have been made.

Interpolation of the function of two variables is important in nonlinear equations, geophysics, process of numeral image and drawing with computer, at this application, we must make a formula than can solve them simplicity. In other

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words, the important role of polynomials is that, the work with them is very simple.

2. Definition And Preliminaries

Convention 2.1. *For simplicity in double variable functions, we write f_{ij} instead of $f(x_i, y_j)$.*

Convention 2.2. *The points (x_i, y_j, f_{ij}) and (x_m, y_n, f_{mn}) are distinct if*

$$\forall i \neq m, \quad \forall j \neq n: \quad x_i \neq x_m \quad \& \quad y_j \neq y_n.$$

Definition 2.3. Two changing polynomial $P(x, y)$ is (m, n)-degree if its largest degree of x in this polynomial is m and it's largest degree of y is n.

Lemma 2.4. *The determinant of an lower (upper) triangular matrix is the product of its diagonal entries, i.e.*

$$| L |= \prod_{i=1}^{n} l_{ii}, \qquad | U |= \prod_{i=1}^{n} u_{ii}.$$

Lemma 2.5. *Let A be an $n \times n$ matrix. Then A is invertible if and only if its determinant is nonzero.*

3. Structure of interpol polynomial

Suppose we have $(2n + 1)$ distinct points (x_0, y_j, f_{0j}), $j = 0, 1, 2, \ldots, n$ and (x_i, y_0, f_{i0}), $i = 1, 2, \ldots, n$ from continuous function $f(x, y)$. Now, we are going to calculate the value of $f(x, y)$ in other points. We should remind that $f(x, y)$ might not be accessible or $f(x, y)$ is so complicated so we would interpolate the function, using a polynomial with two variables. Like

$$f(x, y) = \int_{-\infty}^{\infty} e^{-y^2x^2-z-xe^{-z}} dz,$$

that its value is $f(0.5, 0.03)$ is needful. So for replying this, we can interpolating $f(x, y)$ continuous function by two changing polynomial $P(x, y)$ and accept $P(0.5, 0.03)$ as a approximate of this example. We consider two-variable polynomial $P(x, y)$ as follows:

$$P(x, y) = a_0 + \sum_{i=1}^{n} a_i \prod_{j=0}^{i-1} (x - x_j) + b_0 + \sum_{i=1}^{n} b_i \prod_{j=0}^{i-1} (y - y_j). \tag{3.1}$$

To find a unique interpolation polynomial $P(x, y)$, we assume (a_0+b_0) is a constant. So, we have $(2n + 1)$ unknown coefficients $(a_0 + b_0)$, a_i and b_i, $i = 1, 2, \ldots, n$ that should be calculated. Therefore, we define

$$P(x, y) = (a_0 + b_0) + \sum_{i=1}^{n} \left(a_i \prod_{j=0}^{i-1} (x - x_j) + b_i \prod_{j=0}^{i-1} (y - y_j) \right). \tag{3.2}$$

Now, $P(x,y)$ and $f(x,y)$ have the same values at $2n+1$ distinct interpolation points, i.e.

$$\begin{aligned} p(x_0,y_j) &= f(x_0,y_j) = f_{0j}, \qquad j=0,1,2,\cdots,n, \\ P(x_i,y_0) &= f(x_i,y_0) = f_{i0}, \qquad i=1,2,\cdots,n. \end{aligned} \tag{3.3}$$

To find the interpolation polynomial $P(x,y)$, two steps are necessary. In the first step, we will show a_k and b_k, $(k=1,2,\ldots,n)$ are uniquely exist. So, $P(x,y)$ exist and it is unique. In the second step, we have to calculate the unknown coefficients.

Theorem 3.1. *If* (x_0,y_j), $j=0,1,2,\cdots,n$ *and* (x_i,y_0), $i=1,2,\cdots,n$ *are* $(2n+1)$ *distinct points and* $f:\mathbb{R}^2\to\mathbb{R}$ *is a function whose values are given at this points, then there exists a unique polynomial* $P:\mathbb{R}^2\to\mathbb{R}$ *of degree at most* (n,n) *with the property of* (3.3)*, and this polynomial is given by* (3.2)*.*

Proof. By substituting interpolation conditions (3.3) into the (3.2), we have

$$\begin{aligned}
P(x_0,y_0) &= a_0+b_0 = f_{00}, \\
P(x_1,y_0) &= (a_0+b_0)+a_1(x_1-x_0) = f_{10}, \\
P(x_2,y_0) &= (a_0+b_0)+a_1(x_2-x_0)+a_2(x_2-x_0)(x_2-x_1) = f_{20}, \\
&\vdots \\
P(x_n,y_0) &= (a_0+b_0)+a_1(x_n-x_0)+a_2(x_n-x_0)(x_n-x_1)+\cdots \\
&\quad + a_n(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1}) = f_{n0}, \\
P(x_0,y_1) &= (a_0+b_0)+b_1(y_1-y_0) = f_{01}, \\
P(x_0,y_2) &= (a_0+b_0)+b_1(y_2-y_0)+b_2(y_2-y_0)(y_2-y_1) = f_{02}, \\
&\vdots \\
P(x_0,y_n) &= (a_0+b_0)+b_1(y_n-y_0)+\cdots+b_n(y_n-y_0)\ldots(y_n-y_{n-1}) = f_{0n}.
\end{aligned}$$

Now, it is easy to see that the above system is a system of $(2n+1)$ linear equations in $(2n+1)$ unknowns. Hence, we can solve it. Then for simplicity, we write the matrix form of this system. So, we define

$$b = \begin{bmatrix} f_{00} \\ f_{10} \\ \vdots \\ f_{n0} \\ f_{01} \\ f_{02} \\ \vdots \\ f_{0n} \end{bmatrix}_{(2n+1)\times 1}, X = \begin{bmatrix} a_0+b_0 \\ a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{bmatrix}_{(2n+1)\times 1}, \tag{3.4}$$

and

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right], \tag{3.5}$$

where A_{11}, A_{12}, A_{21} and A_{22} are of orders of $(n+1)\times(n+1)$, $(n+1)\times n$, $n\times(n+1)$ and $n\times n$, respectively, such that

$$A_{11} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & (x_1-x_0) & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (x_n-x_0) & \dots & (x_n-x_0)\cdots(x_n-x_{n-1}) \end{bmatrix}_{(n+1)\times(n+1)}$$

$$A_{12} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{(n+1)\times n}, \quad A_{21} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}_{n\times(n+1)},$$

and

$$A_{22} = \begin{bmatrix} (y_1-y_0) & \dots & 0 \\ \vdots & \ddots & \vdots \\ (y_n-y_0) & \dots & (y_n-y_0)\cdots(y_n-y_{n-1}) \end{bmatrix}_{n\times n}.$$

Now, we known that if A is an invertible matrix, then the system of linear equations given by $Ax=b$ has the unique solution $X=A^{-1}b$. Since A is a lower triangular matrix, using Lemma 2.5 we have

$$\det(A) = \det[A_{11}]\det[A_{22}] = \prod_{\substack{1\le i\le n \\ 0\le j\le n-1 \\ i\neq j}} (x_i-x_j)(y_i-y_j).$$

Because all of our points are distinct then by the convention 2.2, $\det(A)\neq 0$ and by use the lemma 2.5, A is an invertible matrix and so our system has a unique solution, i.e. tthe coefficients $(a_0+b_0), a_1, \dots, a_n, b_1, \dots, b_n$ and then also $P(x,y)$ are uniquely exist. □

Lemma 3.2. *In the interpolation polynomial $P(x,y)$, the coefficients (a_0+b_0), a_k and b_k, $i=1,2,\cdots,n$ are giving as*

$$(a_0+b_0) = f_{00},$$

$$b_k = (-1)^k \left[\sum_{i=0}^{k} \frac{f_{0i}}{\prod_{\substack{j=0\\ j\neq i}}^{k} (y_i-y_j)}\right], \qquad k=1,2,\dots,n,$$

$$a_k = \sum_{i=0}^{k} \frac{f_{i0}}{\prod_{\substack{j=0\\ j\neq i}}^{k} (x_i-x_j)}, \qquad k=1,2,\dots,n.$$

Proof. By using (3.2), we have

$$P(x,y) = (a_0+b_0) + a_1(x-x_0) + \dots + a_n(x-x_0)\cdots(x-x_{n-1}) + b_1(y-y_0) + \dots + b_n(y-y_0)(y-y_n).$$

Then by substituting the point (x_0,y_0) in $P(x,y)$, we have

$$P(x_0,y_0) = a_0+b_0 = f_{00}.$$

Now, by use (x_0, y_1) we have

$$P(x_0, y_1) = a_0 + b_0 + b_1(y_1 - y_0) = f_{01} \Rightarrow f_{00} + b_1(y_1 - y_0) = f_{01}.$$

Then

$$b_1 = (-1)\left[\frac{f_{00}}{y_1 - y_0} + \frac{f_{01}}{y_0 - y_1}\right], \tag{3.6}$$

and also for (x_0, y_2), we can write

$$b_2 = \frac{f_{00}}{(y_1 - y_0)(y_2 - y_0)} + \frac{f_{01}}{(y_0 - y_1)(y_2 - y_1)} + \frac{f_{02}}{(y_0 - y_2)(y_1 - y_2)}. \tag{3.7}$$

Similarly, by repeating this procedure we have

$$b_k = (-1)^k\left[\sum_{i=0}^{k} \frac{f_{0i}}{\prod_{\substack{j=0 \\ j\neq i}}^{k}(y_i - y_j)}\right], \quad k = 1, 2, \ldots, n.$$

Now, we are going to calculate a_k. For this purpose by use (x_1, y_0), we have

$$P(x_1, y_0) = (a_0 + b_0) + a_1(x_1 - x_0) = f_{10}.$$

Hence, $a_1(x_1 - x_0) = f_{10} - f_{00}$. Then, we have

$$a_1 = \frac{f_{00}}{(x_0 - x_1)} + \frac{f_{10}}{(x_1 - x_0)}.$$

Similarly for (x_2, y_0), we can write

$$P(x_2, y_0) = (a_0 + b_0) + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f_{20}.$$

Then

$$a_2 = \frac{f_{00}}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_{10}}{(x_1 - x_0)(x_1 - x_2)} + \frac{f_{20}}{(x_2 - x_0)(x_2 - x_1)},$$

and so, we can write

$$a_k \quad = \quad \sum_{i=0}^{k} \frac{f_{i0}}{\prod_{j=0, j\neq i}^{k}(x_i - x_j)}.$$

So, to calculate the coefficients $(a_0 + b_0), a_1, \ldots, a_n, b_1, \ldots, b_n$, we must use formulae (3.6), (3.7) and (3.8) and make the interpolation polynomial (3.2). □

4. Numerical Examples

Example 4.1. Let $f(x, y) = x^2 + y^2$ and we have seven distinct points of $f(x, y)$ in the table 1. We want to interpolate $f(x, y)$ by one polynomial like $P(x, y)$ and then approximate $f(0.5, 0.79)$.

$y\backslash x$	0	1	1.15	1.8
0.25	0.0625	1.0625	1.385	3.3025
0.27	0.0729			
1.4	1.96			
2	4			

table 1

Solution: By Lemma 3.2:
$a_0 + b_0 = f_{00} = 0.0625$ then $(a_0 + b_0) = 0.0625$ and now by use (6) and (9) for $k = 1, 2, 3$ we can write

$$a_1 = 1, \qquad a_2 = 1, \qquad a_3 = 0.$$

Similarly

$$b_1 = .52, \qquad b_2 = 1, \qquad b_3 = 0.$$

Then $P(x, y) = x^2 + y^2$, so $f(x, y) = P(x, y)$ then $f(.5, .79) = P(.5, .79) = .8125$.

Lemma 4.2. *We have exact methods for polynomial functions from degree (m, n) that have not "xy" sentences.*

Example 4.3. Let $f(x, y) = \int_{-\infty}^{\infty} e^{-y^2x^2-z-xe^{-z}} dz$. We want to approximate $f(0.5, 0.03)$. (attenuation to the below table).

$y\backslash x$	0.4	0.7	1
0	2.500	1.429	1
0.05	2.487		
0.1	2.456		

table 2

Solution: By Lemma 3.2:
$a_0 + a_1 = 2.5$ and similarly we can write

$$a_1 = -3.57, \qquad a_2 = 3.566666667$$

And also

$$b_1 = -.26\ ,\ b_2 = -3.6$$

Then

$$\begin{aligned} P(x, y) &= 2.5 - 3.57(x - 0.4) + 3.566666667(x - 0.4)(x - 0.7) \\ &- .26y - 3.6y(y - 0.05) \end{aligned}$$

Then the approximate value of $f(0.5, 0.03)$ is $P(0.5, 0.03) = 2.04802$.

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