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INVARIANT MANIFOLDS OF THE STOCHASTIC BENJAMIN-BONA-MAHONY EQUATION

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Abstract. This paper studies the stochastic analogue of the Benjamin–Bona–Mahony equation, which is a model of long waves in nonlinear dispersive media with dissipation, as well as the solvability and solution stability of this equation. To this end, the stochastic Benjamin–Bona–Mahony equation is considered as a special case of a stochastic semilinear Sobolev-type equation. The paper also establishes the type of phase space defined as the set of admissible initial data and proves the existence of infinite-dimensional stable and finite-dimensional unstable invariant manifolds of the stochastic Benjamin–Bona–Mahony equation.

Introduction

When studying mathematical models of long waves propagating in one direction in dissipative and dispersive media, the following equation was obtained [1]:

$$\lambda z_t - z_{xxt} = \nu z_{xx} - z z_x. \tag{0.1}$$

Equation (0.1) is commonly referred to as the Benjamin–Bona–Mahony equation. This equation was studied as an improvement of the Korteweg-de Vries equation when constructing a model of small-amplitude surface gravity waves. [2] and [3] considered equation (0.1) as a one-dimensional case of the Oskolkov equation system. They investigated the solvability of the initial-boundary value problem for the Benjamin–Bona–Mahony equation at $\lambda \in \mathbb{R}_+$. [4] studied this problem at an arbitrary value of $\lambda \in \mathbb{R}$. They showed that its phase space is the union of two connected components. The review paper [5] presents results on the stability of the Benjamin–Bona–Mahony equation. The purpose of this paper is to study a stochastic analogue of the Benjamin–Bona–Mahony equation. To this end, we consider equation (0.1) as a special case of a semilinear stochastic Sobolev-type equation

$$L \overset{o}{\eta} = M\eta + N(\eta). \tag{0.2}$$

Here, the stochastic process acts as the desired quantity, its derivative is considered as a Nelson–Glicklikh derivative [6], L and M are linear and bounded operators, and N is a nonlinear operator. The number of articles studying stochastic Sobolev-type equations has been constantly increasing, [7] considered a linear equation of

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form (0.2) (i.e., at $N \equiv \mathbb{O}$) in the case of a relatively bounded operator, [8] a relatively sectorial operator, and [9] a relatively radial operator. [10] investigates multipoint problems for equation (0.2), while [11] studies high-order stochastic Sobolev-type equations. [12] considers the nonlinear equation

$$L \overset{o}{\eta} = M(\eta). \tag{0.3}$$

where L is the linear operator and M is the nonlinear operator demonstrating the conditions for the existence of the solutions to equation (0.2).

In this paper, we follow the results of [12] on the solvability of nonlinear equations of form (0.2), as well as [13] and [14] on the stability of semilinear Sobolev-type equations. The rest of the paper consists of three sections. The first section defines the concepts of a random variable, a stochastic process, the Nelson–Glicklikh derivative, random **K**-variables, and **K**-"noise" spaces. The second section considers the solvability of equations of form (0.2) and establishes the existence of stable and unstable invariant manifolds. The third section applies the results of the second section to the stochastic Benjamin–Bona–Mahony equation and shows the existence of solutions to this equation and their stability in terms of invariant manifolds.

1. Spaces of random K-variables and K-"noises"

Let $\mathbf{CL}_2$ be the space of continuous stochastic processes $\eta = \eta(t,\omega)$ with the norm

$$\|\eta\|^2 = \sup_{t\in\mathfrak{I}}(\mathbf{D}\eta(t,\omega)),$$

with the interval $\mathfrak{I} \subset \mathfrak{R}$. Let us fix $\eta \in \mathbf{CL_2}$ and $t \in \mathcal{J}$. The σ-algebra generated by the random value $\eta(t)$ will be denoted by $\mathcal{N}_t^\eta$. $\mathbf{E}_t^\eta$ denotes the conditional expectation function $\mathbf{E}(\cdot|\mathcal{N}_t^\eta)$, where $\mathcal{N}_t^\eta$ is the σ-algebra generated by the random value $\eta(t)$.

Definition 1.1. [6] Let $\eta \in \mathbf{CL_2}$. *The Nelson–Glicklikh derivative* $\overset{o}{\eta}$ *of the stochastic process* η *at the point* $t \in \mathcal{J}$ is the random value

$$\overset{o}{\eta} = \frac{1}{2}\left(\lim_{\Delta t\to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t+\Delta t,\cdot)-\eta(t,\cdot)}{\Delta t}\right) + \lim_{\Delta t\to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t,\cdot)-\eta(t-\Delta t,\cdot)}{\Delta t}\right)\right),$$

if there is a limit in the sense of a uniform metric on $\mathbb{R}$.

The Nelson–Glicklikh derivative $\overset{o}{\eta}(\cdot,\omega)$ exists in the interval $\mathfrak{I}$ (almost in the interval $\mathfrak{I}$), if it exists at all points (at almost all points) of the interval $\mathfrak{I}$. Let us consider a set of stochastic processes, where the trajectories are almost certainly Nelson–Glicklikh differentiable by $\mathcal{J}$ to the l-th order, and introduce the norm:

$$\|\eta\|^2_{\mathbf{C}^1\mathbf{L}_2} = \sup_{t\in\mathfrak{I}} \sum_{k=1}^{l} \mathbf{D} \overset{o}{\eta}{}^l (t,\omega),$$

where $\overset{o}{\eta}{}^0 \equiv \eta$. We let $\mathbf{C}^l\mathbf{L}_2$ denote the obtained Banach space, $l \in \mathbb{N}$, and call it the space of differentiable "noises" (see [6]–[9]).

Let the spaces $\mathfrak{U}$ and $\mathfrak{F}$ be separable Hilbert spaces. We use $\{\varphi_k\}$ and $\{\psi_k\}$ to denote the orthonormal basis with respect to the corresponding scalar product in $\mathfrak{U}$

and $\mathfrak{F}$. We chose a sequence of real numbers $\mathbf{K} = \{\lambda_k \subset \mathbb{R}_+\}$, where $\sum\limits_{k=1}^{\infty} \lambda_k^2 < \infty$, and a sequence of uniformly bounded random values $\{\xi_k\} \subset \mathbf{L_2}$ $(\{\zeta_k\} \subset \mathbf{L_2})$. Let us construct a $\mathfrak{U}$-*digit random* $\mathbf{K}$-*value* $\xi = \sum\limits_{k=1}^{\infty} \lambda_k \xi_k \varphi_k$ and a $\mathfrak{F}$-*digit random* $\mathbf{K}$-*value* $\zeta = \sum\limits_{k=1}^{\infty} \lambda_k \zeta_k \psi_k$. We complete the linear envelope of the random $\mathfrak{U}$-digit $\mathbf{K}$-values in the norm $\|\xi\|^2_{\mathbf{U_K L_2}} = \sum\limits_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k$, and the linear envelope of the random $\mathfrak{F}$-digit $\mathbf{K}$-values in the norm $\|\zeta\|^2_{\mathbf{F_K L_2}} = \sum\limits_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\zeta_k$. We obtain the Hilbert spaces of $\mathfrak{U}$-*digit random* $\mathbf{K}$-*values* $\mathbf{U_K L}_2$ and $\mathfrak{F}$-*digit random* $\mathbf{K}$-*values* $\mathbf{F_K L}_2$.

Notably, different monotonic positive sequences $\mathbf{K}$ can be chosen for the spaces $\mathfrak{U}$ and $\mathfrak{F}$. For example, the sequences $\mathbf{K} = \{\lambda'_k\}$ and $\mathbf{K} = \{\lambda''_k\}$ are such that $\lim\limits_{k\to\infty} \frac{\lambda'_k}{\lambda''_k} = \text{const}$. Then, the random $\mathbf{K}$-values will be as follows $\xi = \sum\limits_{k=1}^{\infty} \lambda'_k \xi_k \varphi_k$, $\zeta = \sum\limits_{k=1}^{\infty} \lambda''_k \zeta_k \psi_k$.

Let the interval $(\varepsilon, \tau) \subset \mathbb{R}$ and $\{\xi_k\}$ be a sequence from $\mathbf{CL}_2$. A *continuous stochastic* $\mathbf{K}$-*process* is the expression

$$\eta(t) = \sum_{k=1}^{\infty} \lambda_k \xi_k(t) \varphi_k, \tag{1.1}$$

if the series converges uniformly in the norm $\|\cdot\|_{\mathbf{U_K L_2}}$ on any compact set in (ε, τ), and the trajectories of the process $\eta = \eta(t)$ are almost certainly continuous. $\mathbf{C}(\mathcal{J}, \mathbf{U_K L}_2)$ denotes the space of continuous stochastic $\mathbf{K}$-processes. The continuous stochastic $\mathbf{K}$-process $\eta = \eta(t)$ is called *continuously Nelson–Glicklikh differentiable* on (ε, τ) if the series

$$\overset{o}{\eta}(t) = \sum_{k=1}^{\infty} \lambda_k \overset{o}{\xi}_k(t) \varphi_k \tag{1.2}$$

converges uniformly on any compact set in (ε, τ) in the norm $\|\cdot\|_{\mathbf{U_K L_2}}$, and the trajectories of the process $\overset{o}{\eta} = \overset{o}{\eta}(t)$ are almost certainly continuous. $\mathbf{C}^l(\mathcal{J}, \mathbf{U_K L}_2)$ denotes the space of stochastic $\mathbf{K}$-processes continuously differentiable up to the order $l \in \mathbb{N}$.

2. Stochastic Sobolev type equations

Let $\mathfrak{U}$ and $\mathfrak{F}$ be separable Hilbert spaces, and the operators $L,\ M : \mathbf{U_K L}_2 \to \mathbf{F_K L}_2$ be linear and continuous. The following lemma is correct.

Lemma 2.1. *The operator A acting from the space $\mathfrak{U}$ to the space $\mathfrak{F}$ is linear and continuous if and only if the same operator A acting from the space of $\mathfrak{U}$-digit random $\mathbf{K}$-values $\mathbf{U_K K}_2$ to the space of $\mathfrak{F}$-digit random $\mathbf{K}$-values $\mathbf{F_K K}_2$ is linear and continuous.*

Based on lemma 2.1 we transfer the results [15, ch. 3] to the space of random **K**-values. The set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathbf{F_K L_2}; \mathbf{U_K L_2})\}$ is called the *L-resolvent set*, and the set the *L-spectrum of the operator M*. If the L-spectrum of the operator M is bounded, the operator M is called the (L, σ)-*bounded operator*.

Remark 2.2. The concept of the (L, σ)-boundedness of the operator M with the operators $L,\ M : \mathfrak{U} \to \mathfrak{F}$ coincides with the concept of the (L, σ)-boundedness of the operator M with the operators $L,\ M : \mathbf{U_K L_2} \to \mathbf{F_K L_2}$.

For the (L, σ)-boundedness of the operator M, we can construct the following projections

$$P = \frac{1}{2\pi i} \int_\gamma (\mu L - M)^{-1} L d\mu \in \mathcal{L}(\mathbf{U_K L_2}), \tag{2.1}$$

$$Q = \frac{1}{2\pi i} \int_\gamma L(\mu L - M)^{-1} d\mu \in \mathcal{L}(\mathbf{F_K L_2}). \tag{2.2}$$

Here, $\sigma^L(M)$ lies in an area bounded by the contour $\gamma \subset \mathbb{C}$. Projections (2.1), (2.2) split the spaces $\mathbf{U_K L_2} = \mathbf{U^0_K L_2} \oplus \mathbf{U^1_K L_2}$ and $\mathbf{F_K L_2} = \mathbf{F^0_K L_2} \oplus \mathbf{F^1_K L_2}$, where $\mathbf{U^0_K L_2}$ $(\mathbf{U^1_K L_2}) = \ker P$ $(\mathrm{im} P)$, $\mathbf{F^0_K L_2}$ $(\mathbf{F^1_K L_2}) = \ker Q$ $(\mathrm{im} Q)$. L_k (M_k) denotes the restriction of the operator L (M) on $\mathbf{U^k_K L_2}$, $k = 0, 1$.

If the operator M is (L, σ)-bounded and the operators $L_k(M_k) : \mathbf{U^k_K L_2} \to \mathbf{F^k_K L_2}$, $k = 0, 1$, are linear continuous operators, there exist linear continuous operators $M_0^{-1} : \mathbf{F^0_K L_2} \to \mathbf{U^0_K L_2}$, $L_1^{-1} : \mathbf{F^1_K L_2} \to \mathbf{U^1_K L_2}$. Let us consider the operators $H = L_0^{-1} M_0$ and $S = L_1^{-1} M_1$. If the operator M is (L, σ)-bounded and $H \equiv \mathbb{O}$, $p = 0$ or $H^p \neq \mathbb{O}$, $H^{p+1} \equiv \mathbb{O}$, it is called an (L, p)-*bounded operator*. Let us turn to the existence of solutions to equation (0.2).

Let $\mathcal{J} = \{0\} \cup \mathbb{R}_+$. The stochastic **K**-process $\eta \in \mathbf{C}^1(\mathcal{J}; \mathbf{U_K L_2})$ is called a *solution to equation* (2) if all its trajectories almost certainly satisfy equation (0.2) for all $t \in \mathcal{J}$. The solution $\eta = \eta(t)$ to equation (0.2) is called the *solution to the Cauchy problem*

$$\lim_{t \to +\infty} (\eta(t) - \eta_0) = 0, \tag{2.3}$$

if equality (2.3) is satisfied for a random **K**-value $\eta_0 \in \mathbf{U_L L_2}$.

Definition 2.3. The set $\mathbf{P_L L_2} \subset \mathbf{U_L L_2}$ is called the *stochastic phase space* of equation (0.2) if

(i) every trajectory of the solution $\eta = \eta(t)$ to equation (0.2) almost certainly lies in $\mathbf{P_L L_2}$, i.e. $\eta(t) \in \mathbf{P_L L_2}, t \in \mathbb{R}$, for almost all trajectories;

(ii) for $\eta_0 \in \mathbf{P_L L_2}$ there almost certainly exists a solution to problems (0.2), (2.3).

At a fixed $\omega \in \Omega$ the solution to equation (0.2) is the trajectory $\eta = \eta(t)$, it almost certainly lies in the set

$$\mathbf{M} = \begin{cases} \{\eta \in \mathbf{U_K L_2} : (\mathbb{I} - Q)(M\eta + N(\eta)) = 0\}, & \ker L \neq \{0\}; \\ \mathbf{U_K L_2}, \quad \ker L = \{0\}. \end{cases}$$

As mentioned in the introduction, [12] studied the solvability of problems (0.3), (2.3) and showed, with some assumptions, the existence of a resolving semigroup defined on the set $\mathbf{M}$.

Theorem 2.4. [12] *Let the operator M be (L,p)-bounded, the operator $N \in C^1(\mathbf{U_K L}_2, \mathbf{F_K L}_2)$, and the set $\mathbf{M}$ be a simple Banach C^1-manifold at the point $\eta_0 \in \mathbf{U_K L}_2$. Then the set $\mathbf{M}$ is the phase space of equation (0.2).*

Remark 2.5. According to theorem2.4, the phase space of the linear equation

$$L \overset{o}{\eta} = M\eta \tag{2.4}$$

is the space $\mathbf{U^1_K L}_2$. Moreover, if there is the operator $L^{-1} \in \mathcal{L}(\mathbf{F_K L}_2; \mathbf{U_K L}_2)$, then $\mathbf{U^1_K L}_2 = \mathbf{U_K L}_2$.

Now let us turn to the stability of equation (0.2). Let the L-spectrum of the operator M be such that:

$$\left.\begin{array}{c} \sigma^L(M) = \sigma^L_r(M) \bigcup \sigma^L_l(M), \\ \sigma^L_{r(l)}(M) = \{\mu \in \sigma^L(M) : \mathrm{Re}\mu > 0 \ (\mathrm{Re}\mu > 0)\}, \\ \sigma^L_{r(l)}(M) \neq \oslash. \end{array}\right\} \tag{2.5}$$

Then there exist projections

$$P_{l(r)} = \frac{1}{2\pi i} \int\limits_{\gamma_{r(l)}} (\mu L - M)^{-1} L d\mu,$$

where the contour $\gamma_{r(l)}$ bounds the area containing $\sigma^L_r(M)$ $(\sigma^L_l(M))$ and lies on the right (left) half-plane.

Definition 2.6. The subspace $\mathbf{I}^{s(u)}$ of the phase space $\mathbf{P_K L}_2$ of equation (2.4) is called a *stable (unstable) invariant space* of equation (2.4) if

(i) at any $\eta_0 \in \mathbf{I}^{s(u)}$ the solution to problem (0.2), (2.3) $\eta(0) = \eta_0$ for equation (2.4) $\eta \in \mathbf{C}^1(\mathbb{R}; \mathbf{I}^{s(u)})$.

(ii) there exist constants $N_k \in \mathbb{R}_+, \nu_k \in \mathbb{R}_+, k = 1, 2$, such that

$$\|\eta^1(t)\|_{\mathbf{U_K L_2}} \leq N_1 e^{-\nu_1(s-t)} \|\eta^1(s)\|_{\mathbf{U_K L_2}} \text{ for } s \geq t$$
$$\left(\|\eta^2(t)\|_{\mathbf{U_K L_2}} \leq N_2 e^{-\nu_2(t-s)} \|\eta^2(s)\|_{\mathbf{U_K L_2}} \text{ for } t \geq s\right)$$

where $\eta^1 = \eta^1(t) \in \mathbf{I}^s$ $(\eta^2 = \eta^2(t) \in \mathbf{I}^u)$ for all $t \in \{0\} \cup \mathbb{R}_+$.

Definition 2.7. The set

$$\mathbf{M}^{s(u)} = \{\eta_0 \in \mathbf{U_K L}_2 : \|P_{l(r)}\eta_0\|_{\mathbf{U_K L_2}} \leq R_1, \ \|\eta(t,\eta_0)\|_{\mathbf{U_K L_2}} \leq R_2, \ t \in \mathbb{R}_+\},$$

meeting the following conditions:

(i) $\mathbf{M}^{s(u)}$ is diffeomorphic to the closed sphere in $\mathbf{I}^{s(u)}$ and touches $\mathbf{I}^{s(u)}$ at the zero point;

(ii) $\|\eta(t,\eta_0)\|_{\mathbf{U_K L_2}} \to 0$ $(+\infty)$ at $t \to +\infty$, $\eta_0 \in \mathbf{M}^{s(u)}$

is called a *stable (unstable) invariant manifold* of equation (0.2).

Theorem 2.8. *Let the operator M be (L,p)-bounded, condition (2.5) be met, and the operator $N \in C^1(\mathbf{U_K L}_2, \mathbf{F_K L}_2)$ be such that $N(0) = 0$, $N'_0 = \mathbb{O}$. Then, there exists stable $\mathbf{M}^s$ and unstable $\mathbf{M}^u$ invariant manifolds of equation (0.2) modeled by the stable $\mathbf{I}^s = \mathrm{im}P_l$ and unstable $\mathbf{I}^u = \mathrm{im}P_r$ invariant spaces of equation (2.4).*

Now we will outline the proof. Let the operator M be (L,p)-bounded, and the set $\mathbf{M}$ be a smooth simple manifold. We denote $\delta : \mathbf{U^1_K L}_2 \to \mathbf{M}$ — C^1–diffeomorphism, where $\delta^{-1} = P$. We fix $\omega \in \Omega$ and assume that $\mathfrak{I} = \{0\} \cup \mathbb{R}_+$, then the solution $\eta = \eta(t)$, $t \in \Im$, of equation (0.2) can be presented as $\eta = \eta^1 + \delta(\eta^1)$. Here $\eta^1 = \eta^1(t)$, $t \in \Im$, is a solution to the equation

$$\overset{o}{\eta^1} = S\eta^1 + F(\eta^1), \tag{2.6}$$

with the operator $F = L_1^{-1}QN(\mathbb{I} + \delta)$. $S^{s(u)}$ denotes the restriction S on $\mathbf{I}^{s(u)}$. It follows from condition (2.5) that

$$\|e^{tS_{s(u)}}\|_{\mathcal{L}(\mathbf{U_K L_2})} \leq Ce^{-\alpha t},\ \|Se^{tS_{s(u)}}\|_{\mathcal{L}(\mathbf{U_K L_2})} \leq Ce^{-\alpha t},\ t \in \mathbb{R}_{+(-)},$$

where $C \in [1, +\infty)$ and $\alpha \in \mathbb{R}_+$.

Let us consider the set

$$\mathbf{M}^s = \{\eta_0 \in \mathbf{M} : \eta_0 = \mathbb{I} + \delta(\mathbb{I} + \sigma)(\eta_0^s),\ \|\eta_0^s\|_{\mathbf{U_K L_2}} \leq \rho(2C)^{-1}\},$$

where $\sigma(\eta_0^s) = -\int\limits_0^\infty e^{sS_u} P_r F(\eta(s, \eta_0^s))ds$ and $\eta_0^s = P_l \eta_0^1$. The formula $D = \mathbb{I} + \delta(\mathbb{I} + \sigma)$ is used to set the C^1-diffeomorphism of the sphere in $\mathbf{U}^l$ and $\mathbf{M}^s$, wherein $(\mathbb{I} - D)(\eta) = o(\eta)$ at $\|\eta\|_{\mathbf{U_K L_2}} \to 0$. The solution $\eta(t, \eta_0)$ to equation (0.2) is as follows: $\eta(t, \eta_0) = (\mathbb{I} + \delta)\eta^1(t, P\eta_0)$ and $\delta(\eta) = o(\eta)$ at $\|\eta\|_{\mathbf{U_K L_2}} \to 0$. Then, $\|\eta(t, \eta_0)\|_{\mathbf{U_K L_2}} = \|(\mathbb{I} + \delta)\eta^1(t, P\eta_0)\|_{\mathbf{U_K L_2}} \to 0$ at $t \to +\infty$. Thus, the set $\mathbf{M}^s$ is a stable invariant manifold of equation (0.2). The existence of the unstable invariant manifold is shown similarly.

3. The Stochastic Benjamin-Bona-Mahony equation

Let us consider a stochastic analog of equation (0.1). To this end, we set the spaces

$$\mathfrak{U} = \{u \in W_p^{l+2}(-\pi, \pi) : u(-\pi) = u(\pi) = 0\},\ \mathfrak{F} = W_p^l(-\pi, \pi),$$

where $l \in \{0\} \cup \mathbb{N}$, $p \in [2, +\infty)$. The space $\mathfrak{U}$ is continuously and densely embedded in the space $\mathfrak{F}$ at $n \geq 3$ and $2 \leq p \leq 4/(n-2) + 2$. The basis in the Hilbert space $W_p^{l+2}(-\pi, \pi)$ is the sequence $\{\sin kx\}$ of the eigenfunctions of the Laplace operator Δ with the spectrum $\sigma(\Delta) = -k^2$. Let the sequence $\{\chi_k\} \subset \mathbf{L_2}$ ($\{\zeta_k\} \subset \mathbf{L_2}$) be uniformly bounded. The elements of the space of $\mathbf{K}$-values $\mathbf{U_K L_2}$ ($\mathbf{F_K L_2}$) will be the vectors

$$\chi = \sum_{k=1}^{\infty} \lambda_k \chi_k \sin kx \ \left(\zeta = \sum_{k=1}^{\infty} \lambda_k \zeta_k \sin kx\right),$$

where the positive monotonic sequence $\{\lambda_k\}$ is such that $\sum\limits_{k=1}^{\infty} \lambda_k < +\infty$. We can choose $\lambda_k = |-k^2|^{-m} = k^{-2m}$, $m \in \mathbb{N}$, as the sequence $\mathbf{K} = \{\lambda_k\}$, wherein the series $\sum\limits_{k=1}^{\infty} k^{-2m}$ converges at any $m \in \mathbb{N}$.

Stochastic equation (0.1) will be considered as a special case of the

$$L \overset{o}{\chi} = M\chi + N(\chi), \tag{3.1}$$

with the operators

$$L = \lambda - \frac{\partial^2}{\partial^2},\ M = \nu \frac{\partial^2}{\partial^2},\ N : \chi \to -\chi_x \chi.$$

The operators $L,\ M : \mathfrak{U} \to \mathfrak{F}$ are constructed as linear and continuous. According to lemma 2.1, the operators $L,\ M : \mathbf{U_K L}_2 \to \mathbf{F_K L}_2$ are linear and continuous.

The L-spectrum of the operator M is as follows:

$$\sigma^L(M) = \left\{ -\frac{\nu n^2}{n^2+\lambda} : n \in \mathbb{N} \setminus \{l : \lambda = l^2\} \right\},$$

and, hence, bounded. Since the Frechet derivative of the operator N at the point χ is

$$N'_\chi \psi : \psi \to \chi\psi_x + \chi_x \psi,$$

and $N(0) = 0,\ N'_0 \equiv \mathbb{O}$. The following lemma is correct.

Lemma 3.1. *(i) For $n \geq 3$ and $2 \leq p \leq 4/(n-2)+2$, the operator M is $(L,0)$-bounded;*

(ii) the operator $N \in C^1(\mathfrak{U};\mathfrak{F})$, $N(0) = 0,\ N'_0 \equiv \mathbb{O}$.

Let us construct the set $\mathbf{M}$ and the space $\mathbf{U}^1_\mathbf{K}\mathbf{L}_2$ having the following form

$$\mathbf{M} = \begin{cases} \mathbf{U_K L}_2,\ \lambda \neq -n^2; \\ \{\chi \in \mathbf{U}^1_\mathbf{K}\mathbf{L}_2 : \int\limits_{-\pi}^{\pi} (\nu\chi_{xx} - \chi_x\chi)\sin lx dx = 0,\ \lambda = l^2\}; \end{cases}$$

$$\mathbf{U}^1_\mathbf{K}\mathbf{L}_2 = \begin{cases} \mathbf{U_K L}_2,\ \lambda \neq -n^2; \\ \{\chi \in \mathbf{U_K L}_2 : \int\limits_{-\pi}^{\pi} \chi \sin lx dx = 0,\ \lambda = l^2\}. \end{cases}$$

[4] showed that the phase space of equation (0.1) consists of two parts, each of which is a simple Banach C^∞-manifold. Accordingly, let us assume that $\chi \in \mathbf{U_K L}_2$ is the operator kernel $\ker L = \operatorname{span}\{\sin lx\}$.

Then $\chi = a\sin lx + \upsilon$, where $\upsilon \in \mathbf{U}^1_\mathbf{K}\mathbf{L}_2$, $a = \lambda_l \chi_l$. The point $\chi \in \mathfrak{M}$ is correct when

$$\int\limits_{-\pi}^{\pi} (\nu(a\sin lx + \upsilon)_{xx} - (a\sin lx+\upsilon)_x(a\sin lx + \upsilon))\sin lx dx = 0$$

or

$$-a(l^2\nu + \frac{1}{4}\int\limits_{-\pi}^{\pi} \upsilon(\sin lx)^2_x dx) = \int\limits_{-\pi}^{\pi} \upsilon\upsilon_x \sin lx dx. \tag{3.2}$$

Equation (3.2) has an unambiguous solution at

$$\int\limits_{-\pi}^{\pi} \upsilon(\sin lx)^2_x dx = -4l^2\nu,\ \int\limits_{-\pi}^{\pi} \upsilon \sin 2lx = -8l\nu.$$

Then $\mathbf{U}^1_\mathbf{K}\mathbf{L}_2 = \mathbf{U}^{11}_\mathbf{K}\mathbf{L}_2 \bigcup \mathbf{U}^{12}_\mathbf{K}\mathbf{L}_2$, where

$$\mathbf{U}^{11}_\mathbf{K}\mathbf{L}_2 = \begin{cases} \mathbf{U_K L}_2,\ \lambda \neq -n^2; \\ \{\chi \in \mathbf{U}^1_\mathbf{K}\mathbf{L}_2 : \int\limits_{-\pi}^{\pi} \upsilon \sin 2lx dx > -8l\nu,\ \lambda = l^2\}, \end{cases}$$

$$\mathbf{U}^{12}_{\mathbf{K}}\mathbf{L}_2 = \begin{cases} \mathbf{U}_{\mathbf{K}}\mathbf{L}_2, \ \lambda \neq -n^2; \\ \{\chi \in \mathbf{U}^1_{\mathbf{K}}\mathbf{L}_2 : \int\limits_{-\pi}^{\pi} \upsilon \sin 2lxdx < -8l\nu, \ \lambda = l^2\}. \end{cases}$$

The expression $\delta : \mathbf{U}^{11}_{\mathbf{K}}\mathbf{L}_2 \bigcup \mathbf{U}^{12}_{\mathbf{K}}\mathbf{L}_2 \to \mathbf{M}$ is as follows

$$\delta(\upsilon) = \upsilon + \frac{8 \int\limits_{-\pi}^{\pi} \upsilon\upsilon_x \sin lxdx}{-8al^2\nu - \int\limits_{-\pi}^{\pi} \upsilon \sin 2lxdx}.$$

δ is C^∞-diffeomorphic, wherein $\delta^{-1} = P$ with the projection

$$P = \begin{cases} \mathbb{I}, \ \lambda \neq -n^2; \\ \mathbb{I} - < \cdot, \sin lx > \sin lx, \ \lambda = l^2\}. \end{cases}$$

Then, according to theorem 2.4, the following theorem in correct.

Theorem 3.2. *For all $\lambda, \ \nu \in \mathbb{R} \setminus \{0\}$ the phase space of equation* (3.1) *is the union of two simple Banach C^∞-manifolds modeled by the space $\mathfrak{U}^1$.*

We hereinafter use $\mathbf{M}$ to denote the component of this set that contains the zero point. When studying stability, we confine ourselves to the case of the positive parameter ν. The L-spectrum of the operator M consists of two members

$$\sigma^L_l(M) = \left\{-\frac{\nu n^2}{n^2 + \lambda} : \ n^2 > -\lambda\right\}, \ \sigma^L_r(M) = \left\{-\frac{\nu n^2}{n^2 + \lambda} : \ n^2 < -\lambda\right\}.$$

Then the finite-dimensional space $\mathbf{U}^s = \{\chi \in \mathbf{U}_{\mathbf{K}}\mathbf{L}_2 : \int\limits_{-\pi}^{\pi} \chi \sin mxdx = 0, \ m^2 > -\lambda\}$, $\dim \mathbf{U}^s = \max\{m^2 > -\lambda\}$, and the infinite-dimensional space $\mathbf{U}^u = \{\chi \in \mathbf{U}_{\mathbf{K}}\mathbf{L}_2 : \int\limits_{-\pi}^{\pi} \chi \sin mxdx = 0, \ m^2 < -\lambda\}$, $\dim \mathbf{U}^u = \dim \mathbf{U}^s + \dim \ker L$, will be stable and unstable invariant spaces of the linear member of equation (3.1), respectively.

Theorem 3.3. *For any $\lambda \in \mathbb{R} \setminus \{0\}, \ \nu \in \mathbb{R}_+$ equation* (3.1) *in the vicinity of the zero point has finite-dimensional unstable $\mathbf{M}^s$ and infinite-dimensional stable $\mathbf{M}^u$ nvariant manifolds modeled by the spaces $\mathbf{U}^s$ and $\mathbf{U}^u$, respectively.*

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