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MARSHALL-OLKIN EXPONENTIAL LOMAX DISTRIBUTION: PROPERTIES AND ITS APPLICATION

VASILI.B.V.NAGARJUNA AND R.VISHNU VARDHAN*

Abstract. In handling heavy tailed data sets, there are wide variety of distributions in the literature. The generalized family of distributions are always advantageous in fitting and dealing with the shape of the data. In this paper, we deal with one such generalized family, namely Marshall-Olkin to address the issues relating to hazard shapes and better fit of data. Recently, El-Bassiouny et.al. came out with Exponential Lomax distribution and show its application to income and wealth. It is observed that certain hazard shapes i.e. inverted bathtub curves and bathtub curves have been presented properly. However, to address this also to show the role and importance of having additional shape parameter, we made an attempt to propose a new distribution namely *Marshall-Olkin Exponential Lomax distribution*. The behaviour of each parameter is well studied, along with this the mgf, cgf, moments and other statistical properties are derived. The practical application of the proposed distribution is demonstrated using a real data set as well as through simulation studies. Results depicts that the proposed distribution fits well for highly skewed data and it is noticed that, this is due to additional shape parameter.

1. Introduction

Over the six decades, one form of Pareto, namely Lomax distribution (Type II) [16] has gained attention by many researchers broadly branching the fields of Science and Engineering. Further, the theoretical developments and practical applications of Lomax distribution were widely noticed in medicine and life sciences [11, 13, 18].
The functional forms of Lomax distribution with $(\alpha,\ \beta)$ [shape, scale] parameters (both too are positive) are given by

$$F(x) = 1 - \left(\frac{\beta}{x+\beta}\right)^{\alpha} \tag{1.1}$$

$$f(x) = \frac{\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{\alpha+1}, x > 0,\ (\alpha, \beta) > 0 \tag{1.2}$$

Bryson [5] has used Lomax distribution as an alternative to the exponential, gamma and weibull distributions for heavy tailed data. Lingappaiah [15] proposed various procedures of estimation for the Lomax distribution. Chahkandi and Ganjali [6] considered Lomax distribution an important model for lifetime data to capture the advantage of explaining the behaviour of decreasing rate.

To address and handle the heavy tails which usually gets influenced by skewness adding an additional shape/scale parameters will resolve the issue and possess at most

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* Corresponding Author.

best fit of such data. The proposal of new distribution of their kind dates back to several years and wide variety of life testing problems were and are being addressed. A few generalized family of distributions to mention are Malshal-Olkin-G [17], Kumaraswamy-G [10], Exponentiated generalized -G [9], Transformed- Transformer(T-X) family [3], Lomax-G [8], Odd Lomax-G [7].

Among these families of distributions, the form proposed by Marshall and Olkin [17] is based on survival function. The survival function, $\bar{F}(x) = 1 - F(x)$ where $F(x)$ is the cdf. The functional forms of $\bar{F}(x)$ and $f(x)$ of Marshall-Olkin (MO-G) family are

$$\bar{F}(x) = \frac{\theta\bar{G}(x)}{1 - \bar{\theta}\bar{G}(x)} \tag{1.3}$$

$$f(x) = \frac{\theta g(x)}{\left[1 - \bar{\theta}\bar{G}(x)\right]^2} \tag{1.4}$$

In similar lines, researchers have continued to come out with new distributions such as Exponentiated- Lomax [1], Exponential-Lomax [12], weibull-Lomax [23], Power Lomax [20].

Recently, El-Bassiouny et.al. [12] proposed a distribution namely Exponential Lomax (ExL) distribution which is an extension of Lomax distribution. It is shown as an best alternative and performs better than exponential, weibull, gamma distributions in fitting the heavy tailed data.

One of the advantage of MO-G family is with the additional parameter 'θ', which helps in understanding the variant forms of the distribution. Apart from this, it is shown that MO-G works better in addressing the skewely distributed patterns. The main aspect of the paper is to make use of these advantages of MO-G and the ExL to generate and exhibit different density forms as well as the bathtub and up-side down bathtub curves, which were not showed by [12]. The cdf and pdf of ExL are

$$G(x) = 1 - e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}, \tag{1.5}$$

$$g(x) = \frac{\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1} e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}, x \geq -\beta, \alpha, \beta > 0 \tag{1.6}$$

where α is shape parameter, β is scale parameter and λ is the location parameter.
In next section, we detailed out the mathematical framework of the new proposed distribution and in subsequent sections its characterizations are presented.

2. Marshall-Olkin Exponential Lomax distribution

Let 'X' be a random variable that is distributed as Exponential Lomax with α(shape), β(scale), λ(location) as parameters respectively. The expressions (1.5) and (1.6) are substituted in (1.3) and (1.4) respectively and resulting the distributional forms of it's *cdf* and *pdf* for the proposed distribution named it as *Marshall-Olkin Exponential Lomax (MOEL) distribution*

$$\bar{F}(x) = \frac{\theta e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}}{1 - \bar{\theta}e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}} \tag{2.1}$$

$$f(x) = \frac{\frac{\theta\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1} e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}}{\left[1-\bar{\theta}e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]^2} \quad x > -\beta, \ \ (\alpha, \beta, \theta, \lambda) > 0 \tag{2.2}$$

where θ is an additional shape parameter along with α, β and λ. For different values of the parameters θ, α, β and λ the density curves and distribution function curves are depicted in Figures (1) and (2) respectively.

Figure $1(a)$, (b), (c) and (d) depict different forms of density curves where the parameters θ, α, β and λ have been varied to understand the symmetric and asymmetric forms. In figure $1(a)$, θ values were changed by fixing α, β and λ, similarly in $1(b)$, α is varied, in $1(c)$, λ is varied and in $1(d)$, β is varied. By varying values of each parameter, in figure $1(a)$, it is witnessed that for $\theta \leq 1.5$, curves attained right skewed forms are possessed and at $\theta = 3$, symmetric form is observed and for $\theta \geq 3$, slightly left skewed behaviour is observed. With respect to varying α, for $\alpha < 2.7$, symmetrical/ skew symmetrical curves are noticed with elongated tails, where as for $\alpha > 2.7$ the curves are obtained to be flat and also with elongated tails. In terms of λ, for $\lambda \leq 0.11$ right skewed patterns are nicely witnessed, further $\lambda > 0.11$ flats curves are observed.

In figure $1(d)$, with $\beta \leq 0.4$ skewed forms are noticed and for $\beta > 0.4$, flat shaped curves have been exhibited. So, in total, with kind of boundary values for $\alpha \leq 2.7$, $\beta \leq 0.4$ and $\lambda \geq 0.11$ skewed forms have been witnessed. Clearly, however, θ values depicted in both symmetric as well as asymmetric forms. Here, one point to be emphasized that 'θ' provides support to visualize both asymmetric (θ <(>) 3) and symmetric ($\theta = 3$) forms. We took the help of 'θ' to show the mathematical tractability in the proposed distribution and advantage in visualizing density curves and hazard shapes.

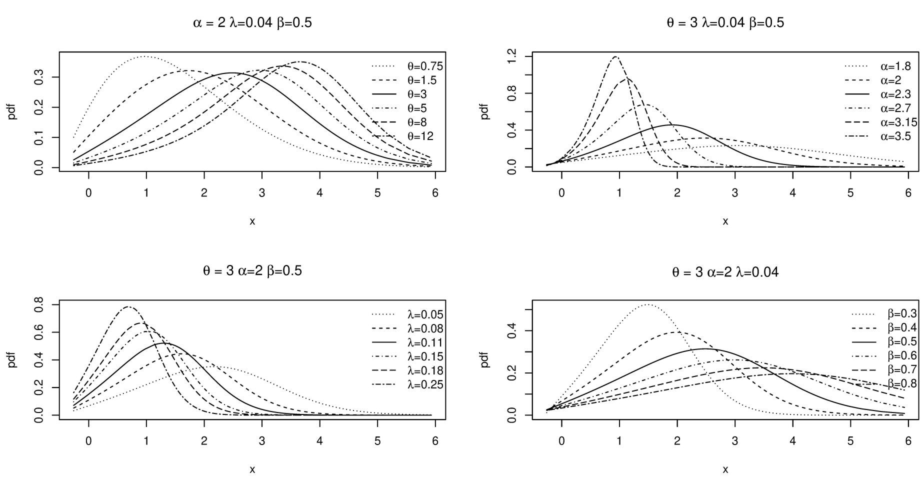

FIGURE 1. Density curves at different parameter values of the MOEL distribution

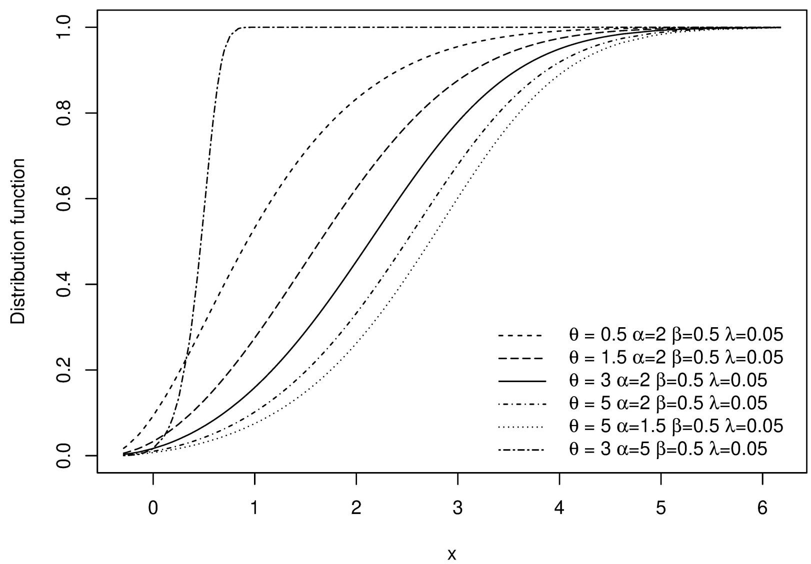

FIGURE 2. Nature of distribution function at different parameter values of MOEL distribution

3. Properties of MOEL Distribution

3.1. *Quantiles*. Let $u = F(x)$, where $u \sim U(0,1)$, then the quantile function, $Q(u)$ and median, (Md) are given by $x = F^{-1}(u) = Q(u)$

$$Q(u) = x_q = \beta\left\{\left[\left(-\frac{1}{\lambda}\right) log\left(\frac{1-u}{1-\bar{\theta}u}\right)\right]^{\frac{1}{\alpha}} - 1\right\} \tag{3.1}$$

$$\text{for q=0.5,}\ \ x_{Md} = \beta\left\{\left[\left(-\frac{1}{\lambda}\right) log\left(\frac{0.5}{1-\bar{\theta}(0.5)}\right)\right]^{\frac{1}{\alpha}} - 1\right\} \tag{3.2}$$

3.2. *Hazard Rate Function*. The hazard rate $h(x)$, and the reverse hazard rate $\tau(x)$, of the MOEL distribution are

$$h(x) = \frac{f(x)}{1-F(x)}$$

$$h(x) = \frac{\theta\frac{\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1} e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}}{\left[1-\bar{\theta}e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]^2}\,\frac{1-\bar{\theta}e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}}{\theta e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}}$$

$$h(x) = \frac{\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}\left[1-\bar{\theta}e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]^{-1} \tag{3.3}$$

3.3. *Reverse hazard Rate*.

$$\tau(x) = \frac{f(x)}{F(x)}$$

$$\tau(x) = \frac{\theta\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1}\left[e^{\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}} - 1\right]^{-1}\left[1 - \bar{\theta}e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]^{-1} \tag{3.4}$$

Using the expressions (3.3) and with varying parameter values, different hazard forms were depicted in figure (3). Here, we made an attempt to obtain the bathtub curves and upside down bathtub curves which are not presented/shown in the work of Bassiouny et.al. [12]. Another way to say is that with 'θ', a additional shape parameter in the proposed distribution, it was supportive enough to overcome the limitation of not presenting the bathtubs and inverted bathtub curves. Along with the above two forms, decreasing, increasing, increasing-decreasing-increasing forms were also obtained and depicted in figure (3).

4. Moments of MOEL Distribution

Let 'X' be a random variable and it's probability density function be $f(x)$ given in (1.6) then its r^{th} moment about mean (μ) is expressed as

$$\mu_r = E(X-\mu)^r = \int_{-\beta}^{\infty}(x-\mu)^r f(x)\,dx$$

By using generalized Binomial exapnsion,

$$(1-Z)^{-s} = \sum_{i=0}^{\infty}\binom{s+i-1}{i}Z^i$$

the *pdf* can be rewritten in the following form

$$f_{MOEL}(x) = \sum_{i=0}^{\infty}(i+1)\theta(\bar{\theta})^i\frac{\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha} e^{-(i+1)\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}} \tag{4.1}$$

Now, considering the r^{th} moment can be rewritten as

$$\mu_r = E(X-\mu)^r = \int_{-\beta}^{\infty}(x-\mu)^r \sum_{i=0}^{\infty}(i+1)\theta(\bar{\theta})^i\frac{\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha} e^{-(i+1)\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\,dx \tag{4.2}$$

Let $u = \lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}$ then $dx = \frac{\beta}{\alpha(\sqrt[\alpha]{\lambda})}u^{(\frac{1}{\alpha}-1)}du$, $u(-\beta) = 0$, and $u(\infty) = \infty$ also $x = \frac{\beta}{\sqrt[\alpha]{\lambda}}u^{(\frac{1}{\alpha})} - \beta$ then

$$E(X-\mu)^r = \sum_{i=0}^{\infty}(i+1)\theta(\bar{\theta})^i\int_{0}^{\infty}\left(\frac{\beta}{\sqrt[\alpha]{\lambda}}u^{(\frac{1}{\alpha})} - \beta - \mu\right)^r e^{-(i+1)u}\,du$$

on simplification, we get the r^{th} moment about mean is

$$E(X-\mu)^r = \sum_{i=0}^{\infty}\sum_{k=0}^{r}\binom{r}{k}(i+1)\theta(\bar{\theta})^i(-\beta-\mu)^{r-k}\left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^k\frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}} \quad r = 1,2,3,\ldots \tag{4.3}$$

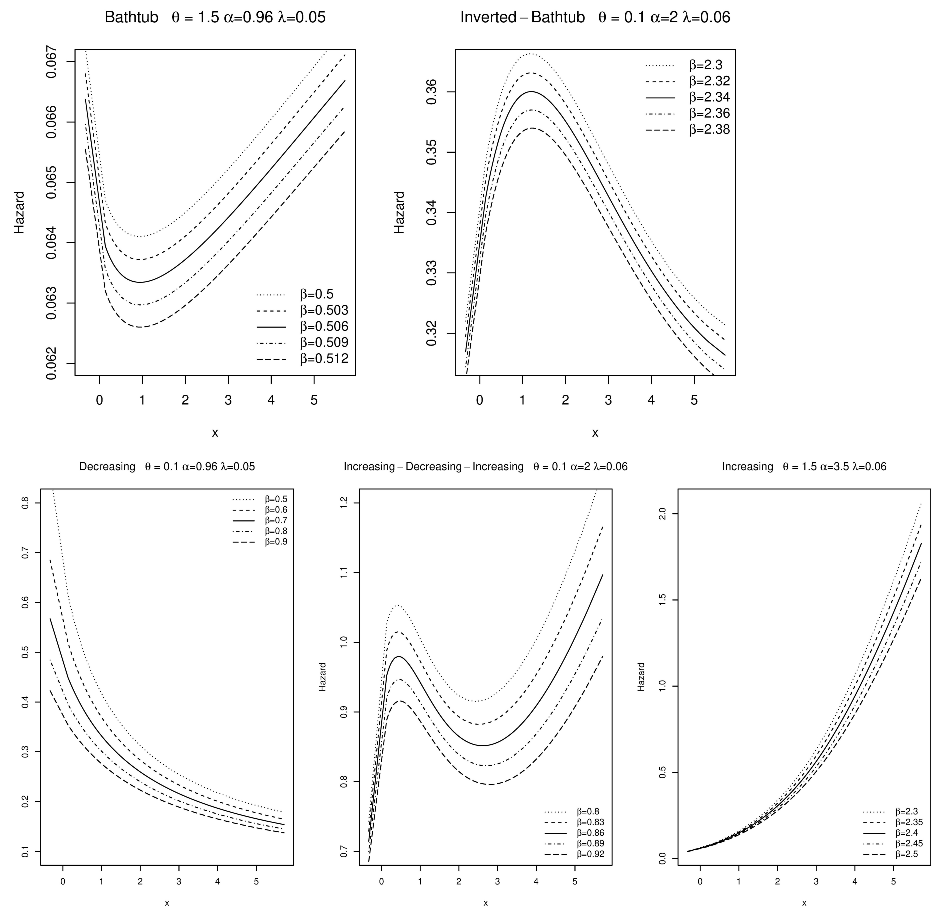

Figure 3. Shapes of hazard function at different parameter values of MOEL distribution

if $\mu = 0$, then the r^{th} moment about origin is

$$\mu'_r = E(X)^r = \sum_{i=0}^{\infty} \sum_{k=0}^{r} \binom{r}{k} (i+1)\theta(\bar{\theta})^i (-\beta)^{r-k} \left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^k \frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}} \quad r = 1, 2, 3, \ldots \tag{4.4}$$

taking $r = 1$, the expression for Mean of the MOEL distribution is obtained

$$\mu'_1 = E(X) = \sum_{i=0}^{\infty} \sum_{k=0}^{1} \binom{1}{k} (i+1)\theta(\bar{\theta})^i (-\beta)^{1-k} \left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^k \frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}} \tag{4.5}$$

again taking $r = 2$, the second raw moment is obtained

$$\mu_2' = E(X^2) = \sum_{i=0}^{\infty}\sum_{k=0}^{2}\binom{2}{k}(i+1)\theta(\bar{\theta})^i(-\beta)^{2-k}\left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^k \frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}}$$

then, the variance of MOEL distribution is

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = \left[\sum_{i=0}^{\infty}\sum_{k=0}^{2}\binom{2}{k}(i+1)\theta(\bar{\theta})^i(-\beta)^{2-k}\left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^k \frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}}\right] - \left[\sum_{i=0}^{\infty}\sum_{k=0}^{1}\binom{1}{k}(i+1)\theta(\bar{\theta})^i(-\beta)^{1-k}\left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^k \frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}}\right]^2 \quad (4.6)$$

On imputing the different combinations of the parameter set, the moments such as μ_1', μ_2', μ_3', μ_4' and μ_2 values are computed and are presented in Table (1).

TABLE 1. Moments with different parameter values of MOEL distribution

Parameters	θ	E(X)	$E(X^2)$	$E(X^3)$	$E(X^4)$	Variance
$\alpha = 2$ $\lambda = 0.5$ $\beta = 1.5$	1	0.3799712	1.110086	1.754611	4.794055	0.9657083
	2	0.7743841	1.665172	3.323766	8.680723	1.0655018
	3	1.0133957	2.125446	4.610775	12.180833	1.0984752
	4	1.1847160	2.513618	5.726648	15.376371	1.1100662
	5	1.3178959	2.849401	6.721139	18.330367	1.1125509
$\alpha = 2$ $\lambda = 0.5$ $\beta = 2$	1	0.5066283	1.973487	4.159079	15.15158	1.716815
	2	1.0325121	2.960307	7.878556	27.43537	1.894225
	3	1.3511942	3.778571	10.929243	38.49745	1.952845
	4	1.5796213	4.468655	13.574278	48.59693	1.973451
	5	1.7571945	5.065601	15.931589	57.93301	1.977868
$\alpha = 1.5$ $\lambda = 0.5$ $\beta = 1.5$	1	0.6495282	2.551918	7.757056	34.14734	2.130031
	2	1.2349640	4.074620	14.366423	63.97131	2.549484
	3	1.6042887	5.325369	20.020119	91.19338	2.751626
	4	1.8754383	6.387358	25.040195	116.41630	2.870090
	5	2.0898274	7.313904	29.591563	140.03837	2.946526
$\alpha = 1.5$ $\lambda = 0.5$ $\beta = 1.5$	3	1.604289	5.325369	20.02012	91.19338	2.751626
	7	2.418130	8.882986	37.66236	183.51014	3.035631
	10	2.770500	10.772013	47.96998	241.66872	3.096342
	14	3.104861	12.763941	59.51202	309.92188	3.123778
	17	3.297938	14.003982	67.03331	356.01779	3.127587

5. Mean Deviation of MOEL Distribution

This section deals with the derivations of mean deviation about the mean and the median and are given in (5.1) and (5.2) respectively. The mean deviation about mean, D(μ) is

$$D(\mu) = E(|X-\mu|) = \int_{d}^{\infty} |x-\mu|\ f(x)\ dx = 2\int_{d}^{\mu} F(x)\ dx$$

on simplification, we have

$$D(\mu) = 2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{\theta}^{i}(-1)^{j}\frac{1}{(j\alpha+1)}\left\{\left(\frac{i\lambda}{\beta^{\alpha}}\right)^{j} - \left(\frac{(i+1)\lambda}{\beta^{\alpha}}\right)^{j}\right\}\left[(\mu+\beta)^{j\alpha+1} - (d+\beta)^{j\alpha+1}\right] \quad (5.1)$$

The mean deviation about median, D(m) is

$$D(m) = E(|X-m|) = \int_{d}^{\infty}|x-m|\ f(x)\,dx = \mu - m + 2\int_{d}^{m}F(x)\ dx$$

on simplification, we have

$$D(m) = \mu - m + 2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{\theta}^{i}(-1)^{j}\frac{1}{(j\alpha+1)}\left\{\left(\frac{i\lambda}{\beta^{\alpha}}\right)^{j} - \left(\frac{(i+1)\lambda}{\beta^{\alpha}}\right)^{j}\right\}\left[(m+\beta)^{j\alpha+1} - (d+\beta)^{j\alpha+1}\right] \quad (5.2)$$

6. Generating Functions

For any distribution, generating functions plays a prominent role in obtaining the moments which helps in understanding the behaviour and tendency of the data. So, accordingly the mgf, cgf and characteristic function are derived and presented below.

Theorem 6.1. *If $X \sim MOEL(\theta,\alpha,\beta,\lambda)$, then it's mgf and cgf are*

$$(a)M_X^{(t)} = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{r}\frac{t^r}{r!}\binom{r}{k}(i{+}1)\theta(\bar{\theta})^{i}(-\beta)^{-r-k}\left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^{k}\frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}}$$

$$(b)K_X^{(t)} = log\left[\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{r}\frac{t^r}{r!}\binom{r}{k}(i+1)\theta(\bar{\theta})^{i}(-\beta)^{-r-k}\left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^{k}\frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}}\right]$$

Proof. (a) *Let* $X \sim MOEL(\theta,\alpha,\beta,\lambda)$, then by definition of *mgf*

$$M_X^{(t)} = E(e^{tX}) = \int_{0}^{\infty} e^{tx}f(x;\theta,\alpha,\beta,\lambda)\,dx$$

substituting (4.1) in the above expression we get

$$M_X^{(t)} = \int_{-\beta}^{\infty} e^{tx}\ \sum_{i=0}^{\infty}(i+1)\theta(\bar{\theta})^{i}\frac{\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha} e^{-(i+1)\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}$$

on simplification, we have

$$M_X^{(t)} = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_{-\beta}^{\infty}x^{r}f(x;\theta,\alpha,\beta,\lambda)\,dx$$

again, the integral part can be written as $E(X^r)$ and considering (4.4) in the above expression, we get

$$M_X^{(t)} = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{r} \frac{t^r}{r!}\binom{r}{k}(i+1)\theta(\bar{\theta})^i(-\beta)^{-r-k}\left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^k \frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}} \tag{6.1}$$

(b) Consider, by definition of *cgf*

$$K_X^{(t)} = log(M_X^{(t)})$$

$$K_X^{(t)} = log\left[\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{r} \frac{t^r}{r!}\binom{r}{k}(i+1)\theta(\bar{\theta})^i(-\beta)^{-r-k}\left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^k \frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}}\right] \tag{6.2}$$

□

Theorem 6.2. *If $X \sim MOEL(\theta,\alpha,\beta,\lambda)$, then it's characteristic function (cf) is*

$$\phi_X^{(t)} = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{r} \frac{(it)^r}{r!}\binom{r}{k}(i+1)\theta(\bar{\theta})^i(-\beta)^{-r-k}\left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^k \frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}}$$

Proof. *Let $X \sim MOEL(\theta,\alpha,\beta,\lambda)$,* then by definition of *cf* is defined as

$$\phi_X^{(t)} = E(e^{itX}) = \int_0^{\infty} e^{itx} f(x;\theta,\alpha,\beta,\lambda)\,dx$$

taking (4.1) in the above expression we have

$$\phi_X^{(t)} = \int_{-\beta}^{\infty} e^{itx} \sum_{i=0}^{\infty}(i+1)\theta(\bar{\theta})^i \frac{\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha} e^{-(i+1)\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}$$

on simplification, we have

$$\phi_X^{(t)} = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\int_0^{\infty} x^r f(x;\theta,\alpha,\beta,\lambda)\,dx$$

again, the integral part can be written as $E(X^r)$ and taking (4.4) in the above expression, we get

$$\phi_X^{(t)} = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{r} \frac{(it)^r}{r!}\binom{r}{k}(i+1)\theta(\bar{\theta})^i(-\beta)^{-r-k}\left(\frac{\beta}{\sqrt[\alpha]{\lambda}}\right)^k \frac{\Gamma(\frac{k}{\alpha}+1)}{(i+1)^{(\frac{k}{\alpha}+1)}} \tag{6.3}$$

□

7. Information Measures

In this section the information measures like Renyi entropy and β - entropy measures are discussed. An entropy is measure of variation or uncertainty of a random variable X. The two different measures are can be discussed as follows

7.1. Renyi Entropy. Renyi [21] provided an extension of the Shannon entropy. The Renyi entropy of a random variable X with *pdf* (1.6) can be defined as

$$I_R^{(\rho)} = \frac{1}{1-\rho}\, log \int_{-\beta}^{\infty} (f(x))^{\rho}\, dx \qquad where\ \rho > 0\ and\ \rho \neq 1 \tag{7.1}$$

consider,

$$\rho(x) = \int_{-\beta}^{\infty} (f(x))^{\rho}\, dx = \int_{-\beta}^{\infty} \left[\sum_{i=0}^{\infty}(i+1)\theta(\bar{\theta})^{i}\frac{\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha} e^{-(i+1)\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]^{\rho} dx$$

put $u = \lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}$ then $dx = \frac{\beta}{\alpha(\sqrt[\alpha]{\lambda})}u^{(\frac{1}{\alpha}-1)}du,\ \ u(-\beta) = 0$, and $u(\infty) = \infty$

then we get

$$\rho(x) = \omega_i^{\rho}\left(\frac{\alpha}{\beta}\right)^{\rho-1}(\lambda)^{\left(\frac{\rho-1}{\alpha}\right)}\frac{\Gamma\left[(\frac{1}{\alpha}-1)(1-\rho)+1\right]}{[(i+1)\rho]^{\left((\frac{1}{\alpha}-1)(1-\rho)+1\right)}}$$

after simple algebraic manipulations, we obtain an expression for $I_R^{(\rho)}$ as

$$I_R^{(\rho)} = \frac{1}{1-\rho}log\left[\omega_i^{\rho}\left(\frac{\alpha}{\beta}\right)^{\rho-1}(\lambda)^{\left(\frac{\rho-1}{\alpha}\right)}\frac{\Gamma\left[(\frac{1}{\alpha}-1)(1-\rho)+1\right]}{[(i+1)\rho]^{\left((\frac{1}{\alpha}-1)(1-\rho)+1\right)}}\right] \tag{7.2}$$

$$\text{where } \omega_i = \sum_{i=0}^{\infty}(i+1)\theta(\bar{\theta})^{i}$$

when $\rho \to 1$ then Renyi entropy measure converges into Shannon entropy and which was studied by Song [22].

7.2. q - Entropy. The q or β - entropy was discovered by Havrda and Charvat [14] and later it was developed by Tsallis [24] which is applied it to physical problems. For a random variable X the q - entropy can be defined as

$$I_H^{(q)} = \frac{1}{1-q}\left[1 - \int_{-\beta}^{\infty}(f(x))^{q}\,dx\right] \qquad where\ q > 0\ and\ q \neq 1$$

$$I_H^{(q)} = \frac{1}{1-q}\left[1 - \omega_i^{q}\left(\frac{\alpha}{\beta}\right)^{q-1}(\lambda)^{\left(\frac{q-1}{\alpha}\right)}\frac{\Gamma\left[(\frac{1}{\alpha}-1)(1-q)+1\right]}{[(i+1)q]^{\left((\frac{1}{\alpha}-1)(1-q)+1\right)}}\right] \tag{7.3}$$

8. Order Statistics

Let $X_1, X_2, \ldots X_n$ be a random samples drawn from population with *cdf* as $F(x)$ and *pdf* as $f(x)$ then the *pdf* of the *r*th order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is given as

$$f_r(x) = r\binom{n}{r}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) \ \ for\ \ 1 \leq r \leq n$$

substituting the equations (1.5) and (1.6) in $f_r(x)$ we get the *pdf* of the rth order statistics of MOEL distribution with parameters $\theta, \lambda, \alpha, \beta$ as follows

$$f_r(x) = r\binom{n}{r}\theta^{n-r+1}\frac{\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1} e^{-(n-r+1)\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}} \frac{\left[1-e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]^{r-1}}{\left[1-\bar{\theta}e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]^{n+1}}$$

for $r = 1$, we get the *pdf* of the first order statistics $X_{(1)} = min\,(X_1, X_2, \ldots, X_n)$ are as follows

$$f_1(x) = n\,[1-F(x)]^{n-1}f(x)$$

$$f_1(x) = n\theta^n\frac{\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1} e^{-n\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\left[1-\bar{\theta}e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]^{-(n+1)}$$

similarly, for $r = n$, we get the *pdf* of the nth order statistics $X_{(n)} = max\,(X_1, X_2, \ldots, X_n)$ as follows

$$f_n(x) = n[F(x)]^{n-1}f(x)$$

$$f_n(x) = n\theta\frac{\lambda\alpha}{\beta}\left(\frac{\beta}{x+\beta}\right)^{-\alpha+1} e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}} \frac{\left[1-e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]^{n-1}}{\left[1-\bar{\theta}e^{-\lambda\left(\frac{\beta}{x+\beta}\right)^{-\alpha}}\right]^{n+1}}$$

The joint *pdf* of (x_i, x_j) for $1 \le i \le j \le n$ is given by

$$f_{i:j:n}^{(x_i,x_j)} = C[F(x_i)]^{i-1}[F(x_j)-F(x_i)]^{j-i-1}[1-F(x_j)]^{n-j}f(x_i)f(x_j)$$

where $C = \dfrac{n!}{(i-1)!(j-i-1)!(n-j)!}$

Then, the joint distribution function of the i^{th} and j^{th} order statistics of MOEL distribution as

$$f_{i:j:n}^{(x_i,x_j)} = C\left[\frac{1-\psi(x_i)}{1-\bar{\theta}\psi(x_i)}\right]^{i-1}\left[\frac{1-\psi(x_j)}{1-\bar{\theta}\psi(x_j)}-\frac{1-\psi(x_i)}{1-\bar{\theta}\psi(x_i)}\right]^{j-i-1}\left[\frac{\theta\psi(x_j)}{1-\bar{\theta}\psi(x_j)}\right]^{n-j} \frac{\frac{\theta\lambda\alpha}{\beta}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha+1}\psi(x_i)}{\left[1-\bar{\theta}\psi(x_i)\right]^2}\frac{\frac{\theta\lambda\alpha}{\beta}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha+1}\psi(x_i)}{\left[1-\bar{\theta}\psi(x_i)\right]^2} \quad (8.1)$$

where $\psi(x_i) = e^{-\lambda\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}$ and $\psi(x_j) = e^{-\lambda\left(\frac{\beta}{x_j+\beta}\right)^{-\alpha}}$

In particular, $i^* = 1$ and $j^* = n$ the joint density function of minimum and maximum order statistics are

$$f_{1n}(x) = n(n-1)[F(x_n)-F(x_1)]^{n-2}f(x_1)\,f(x_n)$$

$$f_{1n}(x) = n(n-1)\left[\frac{1-\psi(x_n)}{1-\bar{\theta}\psi(x_n)} - \frac{1-\psi(x_1)}{1-\bar{\theta}\psi(x_1)}\right]^{n-2} \left[\frac{\frac{\theta\lambda\alpha}{\beta}\left(\frac{\beta}{x_1+\beta}\right)^{-\alpha+1}\psi(x_1)}{1-\bar{\theta}\psi(x_1)}\right]\left[\frac{\frac{\theta\lambda\alpha}{\beta}\left(\frac{\beta}{x_n+\beta}\right)^{-\alpha+1}\psi(x_n)}{1-\bar{\theta}\psi(x_n)}\right] \tag{8.2}$$

where $\psi(x_1) = e^{-\lambda\left(\frac{\beta}{x_1+\beta}\right)^{-\alpha}}$ and $\psi(x_n) = e^{-\lambda\left(\frac{\beta}{x_n+\beta}\right)^{-\alpha}}$

9. Parameter estimation

In this section, the method of estimation procedure for the parameters of MOEL distribution are discussed. The Maximum likelihood estimation (MLE) method is considered and discussed.

9.1. Maximum Likelihood Estimation. Let $x_1, x_2, \ldots x_n$ be random samples drawn from MOEL distribution, then it's likelihood function and log-likelihood function corresponding to the equation (1.6) are

$$L(x) = \left(\frac{\theta\lambda\alpha}{\beta}\right)^n \prod_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha+1} e^{-\lambda\sum\limits_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}} \prod_{i=1}^{n}\left[1-\bar{\theta}e^{-\lambda\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}\right]^{-2} \tag{9.1}$$

Let $Z_i = e^{\lambda\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}}$, then $L(x)$ can be rewritten as

$$L(x) = \left(\frac{\theta\lambda\alpha}{\beta}\right)^n \prod_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha+1} e^{-\lambda\sum\limits_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}} \prod_{i=1}^{n}\left[1-\bar{\theta}Z_i^{-1}\right]^{-2}$$

$$logL(x) = n\,log\left(\frac{\theta\lambda\alpha}{\beta}\right) + (1-\alpha)\sum_{i=1}^{n} log\left(\frac{\beta}{x_i+\beta}\right) - \lambda\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha} - 2\sum_{i=1}^{n} log(1-\bar{\theta}Z_i^{-1}) \tag{9.2}$$

Now partial differentiating the log-likelihood function with respect to the parameters θ, λ, and α are

$$\frac{\partial}{\partial\theta}logL(x) = \frac{n}{\theta} - 2\sum_{i=1}^{n}\left[Z_i - \bar{\theta}\right]^{-1} \tag{9.3}$$

$$\frac{\partial}{\partial\lambda}logL(x) = \frac{n}{\lambda} - \sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha} + 2\sum_{i=0}^{n}\bar{\theta}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha}\left[Z_i - \bar{\theta}\right]^{-1} \tag{9.4}$$

$$\frac{\partial}{\partial\alpha}logL(x) = \frac{n}{\alpha} - \sum_{i=1}^{n} log\left(\frac{\beta}{x_i+\beta}\right) + \lambda\sum_{i=1}^{n}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha} log\left(\frac{\beta}{x_i+\beta}\right) + 2\lambda\sum_{i=1}^{n}\bar{\theta}\left(\frac{\beta}{x_i+\beta}\right)^{-\alpha} log\left(\frac{\beta}{x_i+\beta}\right)\left[Z_i - \bar{\theta}\right]^{-1} \tag{9.5}$$

Closed form solutions for the above expressions cannot be obtained easily, hence, these are to be solved numerically by iterative methods and the MLE of β is obtained by taking its an estimate on the first order statistics of the given data.

10. Simulation Study

Monte Carlo simulations are carried out to assess the finite sample behavior of the Maximum Likelihood Estimators $\hat{\theta}$, $\hat{\alpha}, \hat{\lambda}$ and $\hat{\beta}$. The random variables are generated by using quantile function technique and the parameters are estimated by maximum likelihood method with random sample of size n. The Maximum Likelihood Estimations of the four model parameters along with the respective bias and mean square errors (MSE) for different sample sizes of n are $n = \{20, 100, 150, 200\}$ are shown in Table (2). As mentioned initially about the advantage of having additional shape parameters, the same is witnessed in terms of bias and MSE.

Table 2. Bias in parenthesis and MSE for different sample sizes of MOEL distribution

$(\theta, \alpha, \lambda, \beta)$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$
(0.75,2.7,0.15,0.3)				
n=20	(0.83766) 0.70167	(-1.69387) 2.86919	(0.28565) 0.08160	(0.27642) 0.07641
n=100	(3.06236) 18.91015	(-0.76348) 0.79403	(0.29325) 0.16916	(0.45823) 0.21075
n=150	(2.86985) 11.78414	(-0.94617) 1.10216	(0.43394) 0.25041	(0.50090) 0.25152
n=200	(1.61837) 3.18842	(-0.94563) 0.92130	(0.23180) 0.06332	(0.40062) 0.16051
(1.25,2.7,0.15,0.5)				
n=20	(2.25086) 7.30153	(1.11625) 8.19760	(0.07267) 0.03690	(0.83073) 0.99891
n=100	(1.57778) 8.90054	(0.17721) 2.81461	(0.117315) 0.04797	(0.85294) 0.81237
n=150	(1.84614) 7.32271	(-0.11295) 0.92425	(0.11711) 0.03731	(0.78058) 0.63355
n=200	(1.03586) 1.84255	(-0.48649) 0.38594	(0.08894) 0.01316	(0.80432) 0.65562
(3,1.5,0.05,1.5)				
n=20	(2.25086) 18.39052	(0.10606) 0.38165	(0.06841) 0.01571	(3.78156) 17.1479
n=100	(1.52195) 11.43951	(0.01458) 0.20319	(0.04583) 0.00848	(3.15180) 11.28272
n=150	(1.30870) 2.37345	(-0.21889) 0.06226	(0.04138) 0.00205	(2.46409) 6.12007
n=200	(0.28224) 1.87167	(0.00551) 0.06155	(-0.01098) 0.00062	(2.29950) 5.66002

11. Measures of Goodness of Fit and Model Adequacy

11.1. ***Goodness of fit.*** The measures of goodness of fit is used to verify whether the set of ordered random samples $x_1, x_2, \ldots, x_n$ follows MOEL distribution. Cramer Von

Mises (W*), Anderson Darling (A*) and Kolmogorov-Smirnov (K-S) statistic (D_n) are considered to observe the fit of distribution.

$$W_n^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_i,\hat{\theta}) - \frac{2i-1}{n}\right)^2$$

$$A_n^2 = -n - \sum_{i=1}^{n}\frac{2i-1}{n}\left[\log(F(x_i,\hat{\theta})) + \log(1-F(x_i,\hat{\theta}))\right]$$

$$D_n = \max_i\left(\frac{i}{n} - F(x_i,\hat{\theta}), F(x_i,\hat{\theta}) - \frac{i-1}{n}\right)$$

The above test statistics W_n^2, A_n^2 and D_n are then compared with p-value to have a decision about the fit of the data.

11.2. ***Model Adequacy*.** The following adequacy measures are widely used to know which distribution suits in a better manner.

$$AIC = -2L(\hat{\theta}) + 2k$$

$$BIC = -2L(\hat{\theta}) + k\,log(n)$$

Here $L(\hat{\theta})$ denotes the log-likelihood function value evaluated at the ML estimates of the parameters, k is the number of parameters, n is the sample size and θ denotes the parameters of the distribution i.e., $\theta = (a, b, \alpha, \beta, \lambda)$. The estimated parameter values $\hat{\theta}$ can obtained by iterative procedures from ML estimation method.

12. Application of the MOEL Model

To illustrate the practical application of the proposed distribution, gauge length of 41 samples of single fiber with 20 and 101 mm [4] is considered. The data is as follows: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3,5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0. The histogram shows the skewed nature of the data set and the density curves depict that MOEL is observed to have pattern of better closeness to the histogram plot.

TABLE 3. Goodness of Fit tests and AIC and BIC values for Gauge length data

Distribution	W*	A*	D_n	p-Value	AIC	BIC
MOEL	**0.0416**	**0.2986**	**0.0706**	**0.9885**	**169.9192**	**176.6747**
WL	0.0567	0.4332	0.0958	0.8561	170.8006	177.5562
EEL	0.0614	0.4615	0.1007	0.8122	172.0181	178.7736
EXL	0.0742	0.5519	0.1064	0.7557	173.5595	178.6262
TLGL	0.3141	1.9706	0.1625	0.2413	188.8122	193.8788
ELx	0.3160	1.9807	0.1717	0.1890	189.3240	194.3906
PL	0.2806	1.7867	0.3079	0.0010	212.3536	217.4203
Lx	0.2360	1.5377	0.3758	2.48e-05	238.4607	241.8385

From Table (3), it is witnessed that the data set have a better fit for the proposed MOEL distribution than the other lomax based distributions lomax (Lx) [16], Exponentiated lomax (ELx) [1], Exponential lomax (EXL) [12], Weibull lomax (WL) [23], Power lomax

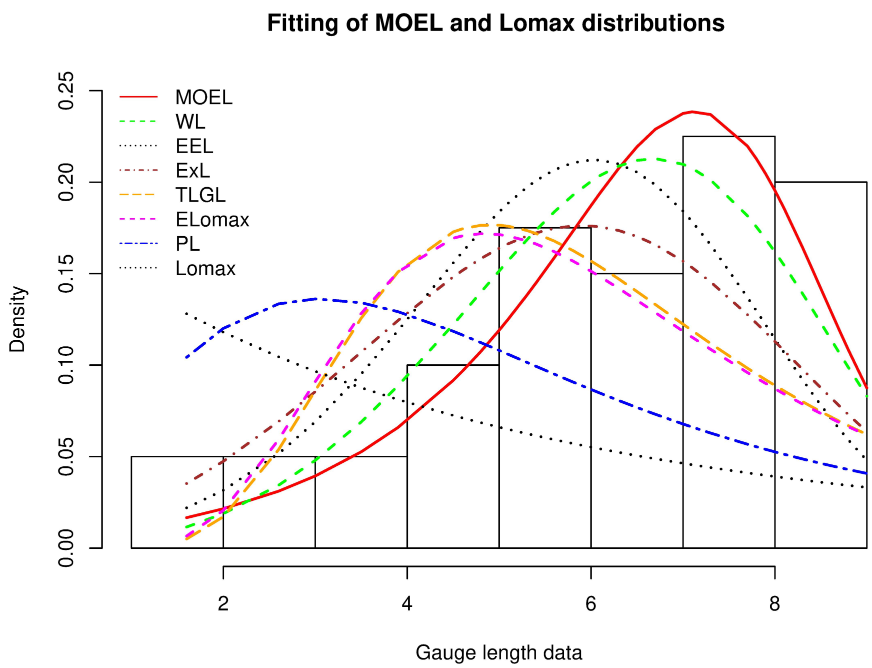

FIGURE 4. Fitted density curve to the data set

(PL) [20], Topp-Leone generalized lomax (TLGL) [19] and Exponentiated Exponential lomax (EEL) [2] distributions. On observing the AIC and BIC values that are reported in table (3), it is more evident that the parameter estimates that are obtained through MOEL are reliable and adequate enough to speak about the nature of data. The MLE's of the MOEL distribution are reported in table (4). Further, this result can also be related to the histogram plot of data set.

TABLE 4. MLEs for the Gauge length data

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$
MOEL	**4.8570**	**4.9180**	**0.0244**	**4.7907**
WL	0.1020	3.3344	2.7799	16.3824
EEL	1.6795	5.6281	0.0246	6.5579
EXLx	-	3.8835	0.0025	1.8760
TLGL	11.9484	8.4137	-	31.5969
ELx	11.3256	13.3683	-	25.0414
PL	-	1.1966	32.6029	2.0484
Lx	-	4.5528	-	25.2683

13. Summary

The present work focused on proposing a new distributional form to handled skew nature data . Exponential Lomax distribution is considered as a parent distribution and is substituted in the family of Marshall-Olkin-G distribution given by [17]. With the support of additional shape parameter θ of the proposed *Marshall-Olkin Exponential Lomax (MOEL) distribution*, better fit of the skewed data is witnessed. The structural properties and characterizations like moments, mgf, cgf, hazard function, quantile functions are derived. Further, the practicability of the MOEL is supported and illustrated using a real data set. From the results of goodness of fit and model adequacy, it is understood that summary estimates of the MOEL can be treated as the better estimates, since the considered data sets observe to have least AIC and BIC values. It can be summarized that any data which has extended can be fitted in a better manner using the proposed MOEL distribution.

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Department of Statistics, Pondicherry University ,Pondicherry - 605 014, India.
Email address: arjun.vasili@gmail.com

Department of Statistics, Pondicherry University, Pondicherry - 605 014, India.
Email address: vrstatsguru@gmail.com