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# KHOVANOV HOMOLOGY AS A CATEGORIFICATION OF THE GENERALIZED KAUFFMAN BRACKET FOR KNOTS IN THE THICKENED TORUS

ALYONA A. AKIMOVA

Abstract. The goal of this paper is to generalise the invariant called Khovanov homology in the sense of a categorification of the generalized Kauffman bracket for knots in the thickened torus. Namely, passage to an appropriate Euler characteristic gives the corresponding polynomial invariant. In this paper, we deal with the generalization of Kauffman bracket taking values only in polynomials in the variables $a$ and $x$, since we take into account only the numbers of trivial and nontrivial curves, i.e. we do not take into consideration the homotopy types of nontrivial curves, and associate each nontrivial curve with the same variable $x$. As a result of taking into account the type of curve (trivial or nontrivial), in order to construct Khovanov homology, we use an additional grading. The paper describes a construction of a chain complex for a knot diagram on the torus, and proves its correctness and invariance.

## Introduction

One of the main problems of the knot theory is to distinguish the objects under study. This approach involves the problem to construct and compute knot invariants and see if some of them are helpful in the considered particular situation. For example, the generalized Kauffman bracket turned out to be enough to distinguish all knots in the thickened torus having diagrams with at most 4 crossings [1].

Despite the fact that the generalized Kauffman bracket allows to distinguish all the constructed knots in the thickened torus $T \times I$ [1], we consider the construction of a stronger invariant, the need for which may arise during further tabulation or to solve other problems connected with knots in the thickened torus $T \times I$. In computation of Khovanov homology, we take into account not only the number of curves obtained by resolving the diagram according to the given state, as in the Kauffman bracket, but also their interaction during the transition between neighboring states. Obviously, Khovanov homology retains more information about a knot diagram than the Kauffman bracket, and therefore is a stronger invariant. The considered homology theory is called a categorification of polynomial invariant, because passage to an appropriate Euler characteristic gives the corresponding polynomial invariant. Namely, we consider a categorification of the generalized

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Kauffman bracket [1], i.e. the transition from abstract polynomials to a category whose objects are Abelian groups, while morphisms are homomorphisms.

For the first time, Khovanov homology was proposed in [2] for classical knots, i.e. knots in the 3-dimensional sphere. Note also the papers [3], [4], and [5], which give perfect description of Khovanov homology for classical knots.

Note the papers [6] and [7], where Khovanov homology are constructed for some other generalizations of the Kauffman bracket of links, which include links in the thickened torus as a particular case.

Namely, the paper [6] constructs Khovanov homology as a categorification of the generalized Kauffman bracket [8] for twisted links, i.e. oriented links in orientable three-manifolds that are orientated $I$-bundles over closed but not necessarily orientable surfaces. The generalization of the Kauffman bracket [8] can be considered for knots in the thickened torus (as a particular case of twisted links) and takes into account the numbers of orienting and non-orienting curves. In this paper, we deal with the generalized Kauffman bracket [1] taking into account the numbers of trivial and nontrivial curves.

In its turn, the paper [7] constructs Khovanov homology as a categorification of another generalized Kauffman bracket for links in $I$-bundles over surfaces. Namely, the proposed generalized Kauffman bracket can be considered as the set of nontrivial curves endowed with coefficients taking values in polynomials in the variable $A$. Therefore, such a generalization can be identified with a polynomial only for knots in the thickened 2-dimensional disk or the thickened 2-dimensional sphere. In this paper, we deal with the generalization of Kauffman bracket [1] taking values only in polynomials in the variables $a$ and $x$. Therefore, we always have a polynomial instead of the set of curves endowed with coefficients, since we do not take into account the homotopy types of nontrivial curves, and associate each nontrivial curve with the same variable $x$. As a result, in order to construct Khovanov homology, we use an additional grading.

The paper is organized as follows. Section 1 gives the necessary definitions of knots in the thickened torus and the generalized Kauffman bracket [1]. In Section 2, we describe a construction of a chain complex for a knot diagram on the torus. To this end, we construct Abelian groups $\mathcal{C}^n$ and homomorphisms $d^n$. In Section 3, we show the correctness of the constructed chain complex. Then, in Section 4, we prove the invariance of the proposed construction. In Section 5, we formulate and prove the main theorem that the proposed construction is a categorification of the generalized Kauffman bracket [1].

## 1. Definitions

**1.1. Knots in the Thickened Torus.** Consider a two-dimensional torus $T = S^1 \times S^1$ and an interval $I = [0, 1]$. By *a thickened torus* we mean a 3-dimensional manifold homeomorphic to the direct product $T \times I$. A smooth embedding of a curve in $Int(T \times I)$ is called a *knot in* $T \times I$ and denoted by $K \subset T \times I$.

As in the classical case, knots in the thickened torus can be given by their diagrams. *A diagram* $D \subset T$ *of a knot* $K \subset T \times I$ is defined by analogy with the diagram of the classical knot except that a knot is projected into the torus $T$ instead of the plane.

**1.2. The Generalized Kauffman Bracket.** Let us recall the definition of the generalized Kauffman bracket [1]. In contrast to the usual Kauffman bracket of classical knots [9] (see also [10] for the original version of the Kauffman bracket called the Jones polynomial), the considered generalized version takes into account types of curves on the torus (trivial, i.e. bounded a 2-disk, and nontrivial).

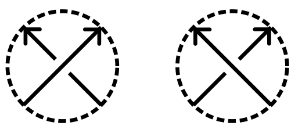
Let $D$ be a diagram of a knot in the thickened torus. Endow each angle of each crossing of $D$ with the marker $A$ or $B$ according to the rule given in the center of Fig. 1.2. Each *state* $s$ of the diagram $D$ is defined by a combination of ways to smooth each crossing of $D$ such as to join together either two angles endowed with the marker $A$, or two angles endowed with the marker $B$, see Fig. 1.2 on the left and right, respectively. Obviously, if the diagram $D$ has $\mathbf{n}$ crossings, then there exist exactly $2^{\mathbf{n}}$ states of $D$.


FIGURE 1. $A$- and $B$-smoothings of a crossing

By the writhe of an oriented classical knot diagram $D$ with $\mathbf{n}$ crossings we mean the sum over all crossings of $D$

$$\omega(D) = \sum_{r=1}^{\mathbf{n}} \varepsilon(r), \tag{1.1}$$

where $\varepsilon(r)$ is the sign of the $r$-th crossing of $D$ defined by the rules given in Fig. 1.2.



$\varepsilon(r) = 1$ $\qquad$ $\varepsilon(r) = -1$

FIGURE 2. Rules to define the sign $\varepsilon(r)$ of the $r$-th crossing

The exact formula of the generalized Kauffman bracket [1] is as follows:

$$\mathcal{X}\,(a,x)_D = (-a)^{-3\omega(D)} \sum_s a^{\alpha(s)-\beta(s)} (-a^2 - a^{-2})^{\gamma(s)} x^{\delta(s)}. \tag{1.2}$$

Here $\alpha(s)$ and $\beta(s)$ are the numbers of markers $A$ and $B$ in the given state $s$, and $\gamma(s)$, $\delta(s)$ are the numbers of trivial and nontrivial curves in the torus obtained by smoothing of all crossings according to the state $s$, and $\omega(D)$ is the writhe of $D$. The sum is taken over all $2^{\mathbf{n}}$ states of $D$.

## 2. Construction of the chain complex $\mathcal{C}$

According to [3], we consider a chain complex $\mathcal{C}$ to be an infinity sequence of Abelian groups $\mathcal{C}^n$ and homomorphisms $d^n$

$$\ldots \to \mathcal{C}^{n-1} \xrightarrow{d^{n-1}} \mathcal{C}^n \xrightarrow{d^n} \mathcal{C}^{n+1} \to \ldots \tag{2.1}$$

under the condition $d^n d^{n+1} = 0$ for any $n$.

Associate each knot diagram on the torus $T$ with a chain complex of graded vector spaces such that the cohomology of the chain complex is the discussed invariant. Nevertheless, following historically established terminology, when constructing Khovanov construction, we talk about homology, not cohomology.

**2.1. Construction of the Abelian groups $\mathcal{C}^n$.** Consider two vector spaces $V = \{v^+, v^-\}$ and $W = \{w^+, w^-\}$ over the field $\mathbb{R}$, which are generated by the elements with degrees

$$\deg(w^+) = \deg(v^+) = +1,$$
$$\deg(w^-) = \deg(v^-) = -1.$$

Let

$$V^{\otimes\gamma} = \underbrace{V \otimes V \otimes ... \otimes V}_{\gamma}$$

be the vector space, which is the tensor product of $\gamma$ copies of the vector space $V$. Denote the basic elements of the vector space $V^{\otimes\gamma}$ by

$$v^{\otimes\gamma} = v_1 \otimes ... \otimes v_\gamma,$$

where $v_p \in \{v^+, v^-\}$ for all $p \in \{1, ..., \gamma\}$.

Define the degree of the basic element $v^{\otimes\gamma} \in V^{\otimes\gamma}$ as the sum of the degrees of the tensor factors in this element. In other words,

$$deg(v^{\otimes\gamma}) = \sharp v^+ - \sharp v^-,$$

where $\sharp v^+$ (respectively, $\sharp v^-$) is the number of elements $v^+$ (respectively, $v^-$) among all elements $v_1, ..., v_\gamma$ included in the tensor product that forms the element $v^{\otimes\gamma}$.

Recall that the choice of division of a vector space into a direct sum of its subspaces is called a grading on the space. Define the grading on the vector space $V^{\otimes\gamma}$ as the values of the degrees of the basic elements, i.e.

$$V^{\otimes\gamma} = V^1 \oplus V^2 \oplus ... \oplus V^{\gamma-1} \oplus V^\gamma = \oplus_p V^p,$$

where $V^p$ is the subspace generated by the basic elements $v^{\otimes\gamma}$ of the degree $p$:

$$V^p = \langle v^{\otimes\gamma} | deg(v^{\otimes\gamma}) = p \rangle.$$

Similarly, the grading on the vector space $W^{\otimes\delta}$ can be defined as

$$W^{\otimes\delta} = W^1 \oplus W^2 \oplus ... \oplus W^{\delta-1} \oplus W^\delta = \oplus_q W^q,$$

where $W^q$ is the subspace generated by the basic elements $w^{\otimes\delta}$ of the degree $q$:

$$W^q = \langle w^{\otimes\delta} | deg(w^{\otimes\delta}) = q \rangle,$$

where $w^{\otimes\delta} = w_1 \otimes ... \otimes w_\delta$, $w_q \in \{w^+, w^-\}$ for all $q \in \{1, ..., \delta\}$, and

$$deg(w^{\otimes\delta}) = \sharp w^+ - \sharp w^-.$$

A vector space endowed with a grading is called a graded vector space.

Let $s$ be a state of the given diagram $D$, and $\mathcal{D}_s$ be the set of curves on the torus $T$ obtained by resolving the diagram $D$ according to the state $s$. Associate each trivial and nontrivial curve of $\mathcal{D}_s$ with a vector space $V$ and $W$, respectively. As above, $\gamma(s)$ and $\delta(s)$ are the numbers of trivial and nontrivial curves on the torus in $\mathcal{D}_s$. Then associate each state $s$ of the diagram $D$ with the vector space

$$\mathcal{V}_s = V^{\otimes\gamma(s)} \otimes W^{\otimes\delta(s)},$$

which is the tensor product of $\gamma(s)$ copies of the vector space $V$ and $\delta(s)$ copies of the vector space $W$. The basic elements $\nu_s$ of the space $\mathcal{V}_s$ have the form

$$\nu_s = v^{\otimes\gamma(s)} \otimes w^{\otimes\delta(s)}.$$

On the constructed vector space $\mathcal{V}_s$, introduce the three gradings (homological $i(\nu_s)$, quantum $j(\nu_s)$, and additional $k(\nu_s)$) as follows:

$$\begin{aligned} &i(\nu_s) = \tfrac{\omega(D)-\sigma(s)}{2} \text{ (homological)}, & \sigma(s) &= \alpha(s) - \beta(s),\\ &j(\nu_s) = \tfrac{3\omega(D)-\sigma(s)+2\tau(\nu_s)}{2} \text{ (quantum)}, & \tau(\nu_s) &= \sharp v^+ - \sharp v^-,\\ &k(\nu_s) = \phi(\nu_s) \text{ (additional), where} & \phi(\nu_s) &= \sharp w^+ - \sharp w^-. \end{aligned} \tag{2.2}$$

Recall that the smoothing numbers of types $A$ and $B$ in the given state $s$ are denoted by $\alpha(s)$ and $\beta(s)$, respectively.

*Remark* 2.1. It follows from (2.2) that the value of the homological grading $i(\nu_s)$ depends only on the state $s$ and does not depend on the tensor factors by which the element $\nu_s = v^{\otimes\gamma(s)} \otimes w^{\otimes\delta(s)}$ is generated. Therefore, it is more correct to talk about the homological grading $i(s)$. Nevertheless, for reasons of uniformity of the notation for gradings, we write $i(\nu_s)$.

Divide the space $\mathcal{V}_s$ into the direct sum of the subspaces: $\mathcal{V}_s = \oplus_{i,j,k}\mathcal{V}^{i,j,k}$, where $\mathcal{V}^{i,j,k}$ is the subspace generated by the basic elements $\nu_s \in \mathcal{V}_s$, for which the values of each of the gradings are $i, j$, and $k$, respectively.

Define the Abelian group $\mathcal{C}^n$ of the chain complex as the direct sum of the vector spaces $\mathcal{V}_s$ such that $i(\nu_s) = n$ for all $\nu_s \in \mathcal{V}_s$. Therefore,

$$\mathcal{C}^n = \oplus_{s:i(\nu_s)=n}\mathcal{V}_s.$$

**2.2. Construction of the Homomorphisms $d^n$.** Two states, $s_A$ and $s_B$, of the given diagram $D$ are called neighboring states, if all the corresponding crossings are smoothed in the same way with the exclusion of the unique crossing, which is smoothed by type $A$ in the state $s_A$ and by type $B$ in the state $s_B$.

As above, $\mathcal{D}_s$ is a set of curves on the torus $T$ obtained by resolving the diagram $D$ with $\mathbf{n}$ crossings according to the state $s$, and $\mathcal{V}_s$ is a vector space corresponding to the state $s$. Associate each vertex of the unit $\mathbf{n}-$dimensional cube with a triple

$(s,\, \mathcal{D}_s,\, \mathcal{V}_s)$ such that each edge of the cube connects the vertices corresponding to triples of neighboring states: $(s_A,\, \mathcal{D}_{s_A},\, \mathcal{V}_{s_A})$ and $(s_B,\, \mathcal{D}_{s_B},\, \mathcal{V}_{s_B})$. Note that the edges of the cube are oriented: the beginning of each edge is the vertex corresponding to the triple $(s_A,\, \mathcal{D}_{s_A},\, \mathcal{V}_{s_A})$.

Let us consider the passage along the edge of the unit $\mathbf{n}-$dimensional cube as a linear map of vector spaces corresponding to neighboring states:

$$f : \mathcal{V}_{s_A} \to \mathcal{V}_{s_B}.$$

This map $f$ acts identically on all curves in the set $\mathcal{D}_{s_A}$, with the exception of one or two curves. Namely, as a result of switching the type of smoothing of a crossing from $A$ to $B$, we have one of the following possibilities:
a curve maps to a curve
$\eta : W \mapsto W;$

two curves combine into a curve
$m_1 : V \otimes V \mapsto V, \qquad m_2 : V \otimes W \mapsto W, \qquad m_3 : W \otimes W \mapsto V;$

a curve splits into two curves
$\Delta_1 : V \mapsto V \otimes V, \qquad \Delta_2 : W \mapsto V \otimes W, \qquad \Delta_3 : V \mapsto W \otimes W.$

It is easy to see that there are no other combinations of the map of trivial and nontrivial curves on the torus, i.e. $f \in \{\eta, m_t, \Delta_t\}$ and $t = 1, 2, 3$. Note the linear map $\eta$, which appears only in the case of knots in the thickened torus $T \times I$ and has no analogies in the case of classical knots, where only two linear maps, $m_1$ and $\Delta_1$, are possible.

In order to define these maps, we use the following conditions. As a result of applying any map, the homological grading $i(\nu_s)$ increases by 1, while the other two, the quantum grading $j(\nu_s)$ and the additional grading $k(\nu_s)$, remain the same:

$$i(\nu_{s_B}) = i(\nu_{s_A}) + 1, \quad j(\nu_{s_B}) = j(\nu_{s_A}), \quad k(\nu_{s_B}) = k(\nu_{s_A}). \tag{2.3}$$

Conditions (2.3) are used to prove Theorem 5.1, since under these conditions the degree of the constructed differential $d^i$ is 0, see [4]: $d^i(\mathcal{C}^{i,j,k}) \subset \mathcal{C}^{i+1,j,k}\ \forall i, j, k \in \mathbb{Z}$ .

According to formulas (2.2), condition (2.3) implies the following conditions on the ratio of the degrees of generators in the elements $\nu_{s_A} \in \mathcal{V}_{s_A}$ and $\nu_{s_B} \in \mathcal{V}_{s_B}$, which correspond to the trivial

$$\tau(\nu_{s_B}) = \tau(\nu_{s_A}) - 1 \tag{2.4}$$

and non-trivial

$$\phi(\nu_{s_B}) = \phi(\nu_{s_A}) \tag{2.5}$$

curves. Indeed, $j(\nu_{s_B}) = j(\nu_{s_A})$ implies (2.4), and $k(\nu_{s_B}) = k(\nu_{s_A})$ implies (2.5). Using conditions (2.4) and (2.5), it is easy to see that the actions of the linear maps on basic elements of the spaces are defined by the following rules.

$\eta(w^-) = \eta(w^+) = 0$

$m_1(v^-\otimes v^-)=0$ $\quad m_2(v^-\otimes w^-)=0$ $\quad m_3(w^-\otimes w^-)=0$
$m_1(v^-\otimes v^+)=v^-$ $\quad m_2(v^-\otimes w^+)=0$ $\quad m_3(w^-\otimes w^+)=v^-$
$m_1(v^+\otimes v^-)=v^-$ $\quad m_2(v^+\otimes w^-)=w^-$ $\quad m_3(w^+\otimes w^-)=v^-$
$m_1(v^+\otimes v^+)=v^+$ $\quad m_2(v^+\otimes w^+)=w^+$ $\quad m_3(w^+\otimes w^+)=0$

$\Delta_1(v^-)=v^-\otimes v^-$ $\quad \Delta_1(v^+)=v^+\otimes v^-+v^-\otimes v^+$
$\Delta_2(w^-)=v^-\otimes w^-$ $\quad \Delta_2(w^+)=v^-\otimes w^+$
$\Delta_3(v^-)=0$ $\quad \Delta_3(v^+)=w^+\otimes w^-+w^-\otimes w^+$

Therefore, the linear maps $\eta$, $\Delta_t$, and $m_t$, $t=1,2,3$, are given by the following matrices, where columns and rows correspond to the basic elements of preimage and image, respectively.

| $\eta$ | $w^-$ | $w^+$ |
|---|---|---|
| $w^-$ | 0 | 0 |
| $w^+$ | 0 | 0 |

| $\mathbf{m_1}$ | $v^-\otimes v^-$ | $v^-\otimes v^+$ | $v^+\otimes v^-$ | $v^+\otimes v^+$ |
|---|---|---|---|---|
| $v^-$ | 0 | 1 | 1 | 0 |
| $v^+$ | 0 | 0 | 0 | 1 |

| $\mathbf{m_2}$ | $v^-\otimes w^-$ | $v^-\otimes w^+$ | $v^+\otimes w^-$ | $v^+\otimes w^+$ |
|---|---|---|---|---|
| $w^-$ | 0 | 0 | 1 | 0 |
| $w^+$ | 0 | 0 | 0 | 1 |

(2.6)

| $\mathbf{m_3}$ | $w^-\otimes w^-$ | $w^-\otimes w^+$ | $w^+\otimes w^-$ | $w^+\otimes w^+$ |
|---|---|---|---|---|
| $v^-$ | 0 | 1 | 1 | 0 |
| $v^+$ | 0 | 0 | 0 | 0 |

| $\mathbf{\Delta_1}$ | $v^-$ | $v^+$ |
|---|---|---|
| $v^-\otimes v^-$ | 1 | 0 |
| $v^-\otimes v^+$ | 0 | 1 |
| $v^+\otimes v^-$ | 0 | 1 |
| $v^+\otimes v^+$ | 0 | 0 |

| $\mathbf{\Delta_2}$ | $w^-$ | $w^+$ |
|---|---|---|
| $v^-\otimes w^-$ | 1 | 0 |
| $v^-\otimes w^+$ | 0 | 1 |
| $v^+\otimes w^-$ | 0 | 0 |
| $v^+\otimes w^+$ | 0 | 0 |

| $\mathbf{\Delta_3}$ | $v^-$ | $v^+$ |
|---|---|---|
| $w^-\otimes w^-$ | 0 | 0 |
| $w^-\otimes w^+$ | 0 | 1 |
| $w^+\otimes w^-$ | 0 | 1 |
| $w^+\otimes w^+$ | 0 | 0 |

Associate each edge of the unit $\mathbf{n}-$dimensional cube with the linear map $\eta$, $m_t$ or $\Delta_t$, $t=1,2,3$, taking with a sign determined by the following rule. Let $r_1$ be a number of the crossing, where the type of smoothing is switched from $A$ to $B$ when we go along the edge. Associate each crossing having the number $r_2<r_1$ with a number

$$\xi_{r_2}=\begin{cases}1, & \text{if the crossing with the number } r_2 \text{ is smoothed by the type } B,\\ 0, & \text{if the crossing with the number } r_2 \text{ is smoothed by the type } A.\end{cases}$$

Then we endow the map corresponding to this edge with the sign

$$(-1)^{\sum_{r_2<r_1}\xi_{r_2}}. \tag{2.7}$$

Recall that Abelian group of the constructed chain complex is $\mathcal{C}^n = \oplus_{s:i(\nu_s)=n}\mathcal{V}_s$. Fix $n$ and consider all states $s$ such that $i(\nu_s) = n$, where $\nu_s \in \mathcal{V}_s$. Then for each state $\widetilde{s}$, which is neighboring with some state $s$, we have $i(\nu_{\widetilde{s}}) = n+1$, where $\nu_{\widetilde{s}} \in \mathcal{V}_{\widetilde{s}}$. Define the deferential $d^n : \mathcal{C}^n \to \mathcal{C}^{n+1}$ as a sum of all linear maps corresponding to edges that connect the vertices $(s,\, \mathcal{D}_s,\, \mathcal{V}_s)$ and $(\widetilde{s},\, \mathcal{D}_{\widetilde{s}},\, \mathcal{V}_{\widetilde{s}})$.

Then, for each element $\nu_s \in \mathcal{V}_s$, the element $d^n(\nu_s) = \oplus\nu_{\widetilde{s}} \in \oplus\mathcal{V}_{\widetilde{s}}$ belongs to the direct sum of the vector spaces $\mathcal{V}_{\widetilde{s}}$, where the sum is taken over all the vector spaces $\mathcal{V}_{\widetilde{s}}$ corresponding to the vertices $(\widetilde{s},\, \mathcal{D}_{\widetilde{s}},\, \mathcal{V}_{\widetilde{s}})$, which are the endpoints of the edges with the begining at the vertex $(s,\, \mathcal{D}_s,\, \mathcal{V}_s)$.

## 3. Correctness of the constructed chain complex

**Lemma 3.1.** *For any $n$, the condition $d^{n+1}d^n = 0$ is fulfilled.*

*Proof.* STEP 1. Construct an arbitrary two-dimensional face $F$ of the unit $\mathbf{n}-$dimensional cube, see Fig.3, as follows.

Recall that if two vertices are connected by an edge, then the states that correspond to these vertices, for example, $s_1$ and $s_2$, are neighboring (i.e., the states are such that smoothing types are the same in all the corresponding crossing, except one). For the same reason, in the states $s_1$ and $s_4$ connected by two edges consecutively, the smoothing types are the same in all the corresponding crossings, except some pair of crossings denoted by $c_1$ and $c_2$ such that both these crossings are smoothed by the type $A$ in the state $s_1$, and are smoothed by the type $B$ in the state $s_4$. Therefore, there exist exactly two crossings, $c_1$ and $c_2$, of the diagram $D$ that switch the type of smoothing within the face $F$.

Hereinafter, without loss of generality, instead of the sets $\mathcal{D}_s$, we consider the simplified sets $\mathcal{D}'_s$ that contain only one or two curves, which are involved in the considered linear maps. In other words, we take into account only the curves that appear or disappear within the considered combination of the linear maps. Then each of the spaces $\mathcal{V}'_s$ is a tensor product

$$\mathcal{V}'_s = V^{\otimes\gamma'(s)} \otimes W^{\otimes\delta'(s)}, \tag{3.1}$$

where $\gamma'(s)$ and $\delta'(s)$ are the numbers of trivial and nontrivial curves, respectively, in the simplified set $\mathcal{D}'_s$.

Without loss of generality, hereinafter we take into account only these two crossings $c_1$ and $c_2$, and consider the simplified sets $\mathcal{D}'_{s_r}$, $r = 1, 2, 3, 4$, that contain only the curves, each of which is involved in at least one of the linear maps associated with the edges of the face $F$. In other words, we take into account only the curves that appear or disappear within the face $F$.

Each vertex of the face $F$ is endowed with a triple of the form $(s_r,\, \mathcal{D}'_{s_r},\, \mathcal{V}'_{s_r})$, $r = 1, 2, 3, 4$. Here the set $\mathcal{D}'_{s_r}$ includes only the curves on the torus $T$ obtained by smoothing the crossings $c_1$ and $c_2$, and $\mathcal{V}'_{s_r}$ is a vector space corresponding to the set $\mathcal{D}_{s_r}$. In the sets $\mathcal{D}_{s_r}$, in order to save information about crossings, the curves are connected by dashed arcs at the points of smoothing crossings. At the same time, each of the simplified sets $\mathcal{D}'_s$ contains only two dashed arcs indicating the points of smoothing the two crossings $c_1$ and $c_2$.

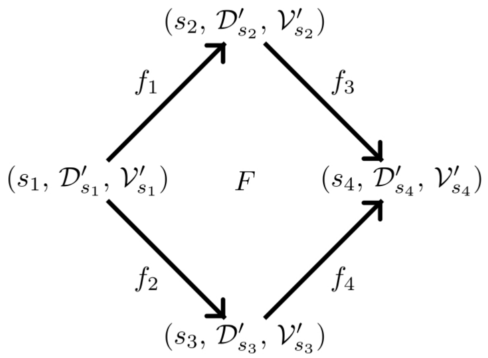



FIGURE 3. A two-dimensional face $F$ of the unite $\mathbf{n}-$ dimensional cube

Each edge of the face $F$ is endowed with a linear map $f_r \in \{\eta, m_t, \Delta_t\}$, where $r = 1, 2, 3, 4$, and $t = 1, 2, 3$. Here all the maps $f_r$ are positive, i.e. are considered without regard to the signs determined by formula (2.7) when constructing a chain complex.

STEP 2.

Let us show that the constructed face $F$ is commutative, i.e. $f_3 \circ f_1 = f_4 \circ f_2$. Here by $f_{p+2} \circ f_p$ we mean a superposition of the linear maps $f_p$ and $f_{p+2}$ such that the map $f_p$ is applied first of all, $p = 1, 2$.

In the general case, the commutativity of the face $F$ is verified by multiplication of the matrices corresponding to the linear maps $f_r$, i.e. $\eta$, $m_t$ or $\Delta_t$, where $r = 1, 2, 3, 4$, and $t = 1, 2, 3$. Enumeration of superpositions is significantly reduced by the following conditions.

(1) The maps $f_p$ (i.e., $f_1$ and $f_2$) have the same preimage space $\mathcal{V}_{s_1}$,
(2) The maps $f_{p+2}$ (i.e., $f_4$ and $f_3$) have the same image space $\mathcal{V}_{s_4}$.

Let $\rho(s_r)$ be the number of all (both trivial and nontrivial) curves in the set $\mathcal{D}'_{s_r}$, and $\rho = \max\limits_r \rho(s_r)$. Obviously, $\rho \leq 4$, since on a torus there exist no more than 4 curves connected by two dashed arcs such that at each curve there is an endpoint of at least one arc.

STEP 2. 1. Let $\rho = 1$.

Obviously, the face $F$ is commutative. Indeed, $f_1 = f_2 = f_3 = f_4 = \eta$, since $\mathcal{V}'_{s_r} = W$ for all $r = 1, 2, 3, 4$.

STEP 2. 2. Let $\rho = 2$ or $\rho = 3$.

Enumeration of all possible cases to switch the type and number of curves, as well as multiplication of matrices of the corresponding maps, is a routine check, which can be carried out as follows.

(1) Fix the values of $\rho(s_1)$ and $\rho(s_4)$.
(2) Enumerate all pairs of the corresponding spaces $\mathcal{V}'_{s_1}$ and $\mathcal{V}'_{s_4}$.
(3) For each fixed pair of the spaces $\mathcal{V}'_{s_1}$ and $\mathcal{V}'_{s_4}$, consider all pairs of the maps $f_p : \mathcal{V}'_{s_1} \to \mathcal{V}'_{s_{p+1}}$ and $f_{p+2} : \mathcal{V}'_{s_{p+1}} \to \mathcal{V}'_{s_4}$, for which the superposition $f_{p+2} \circ f_p : \mathcal{V}'_{s_1} \to \mathcal{V}'_{s_4}$ is defined, where $p = 1, 2$.
(4) For fixed spaces $\mathcal{V}'_{s_1}$ and $\mathcal{V}'_{s_4}$, show that the results of all the constructed superpositions $f_{p+2} \circ f_p : \mathcal{V}'_{s_1} \to \mathcal{V}'_{s_4}$ are the same, i.e. the result is

independent of the chosen maps $f_p$, $f_{p+2}$ and the space $\mathcal{V}'_{s_{p+1}}$, where $p = 1, 2$.

In order to enumerate all possible cases, we note the following obvious statements.

**Lemma 3.2.**

(1) $\rho(s_1) = \rho(s_4)$ *if and only if either one of the linear maps* $f_{p+2}$ *and* $f_p$ *is the linear map* $m_{t_1}$*, while another is the linear map* $\Delta_{t_2}$*, where* $p = 1, 2$ *and* $t_1,\ t_2 \in \{1, 2, 3\}$*, or both linear maps are* $\eta$*.*

(2) $|\rho(s_1) - \rho(s_4)| = 1$ *if and only if exactly one of the linear maps, either* $f_{p+2}$ *or* $f_p$*, where* $p = 1, 2$*, is the linear map* $\eta$*, while another is the linear map* $m_t$*, if* $\rho(s_1) > \rho(s_4)$*, or the linear map* $\Delta_t$*, if* $\rho(s_1) < \rho(s_4)$*, where* $t \in \{1, 2, 3\}$*.*

(3) $|\rho(s_1) - \rho(s_4)| = 2$ *if and only if both linear maps* $f_{p+2}$ *and* $f_p$ *are either* $m_t$*, if* $\rho(s_1) > \rho(s_4)$*, or* $\Delta_t$*, if* $\rho(s_1) < \rho(s_4)$*, where* $t \in \{1, 2, 3\}$ *can takes different values for* $f_{p+2}$ *and* $f_p$ *and* $p = 1, 2$*.*

*Proof.* For all $t = 1, 2, 3$, the linear map $m_t$ decreases the number of curves by one, and the linear map $\Delta_t$ increases the number of curves by one. In its turn, the linear map $\eta$ remains the number of curves the same. □

**Lemma 3.3.** *The linear map* $\eta$ *is defined only on a vector space of the form*

$$\mathcal{V}'_s = V^{\otimes\gamma'(s)} \otimes W.$$

*In other words,* $\delta'(s) = 1$ *in formula* (3.1)*.*

*Proof.* Indeed, consider a dashed arc $\ell$, the endpoints of which belong to a nontrivial curve $\zeta$ on the torus. If both endpoints of $\ell$ belong to the same side of $\zeta$, then, as a result of switching the type of the crossing, the nontrivial curve $\zeta$ splits into two curves having different types, i.e. trivial and nontrivial. The nontrivial curve maps to the nontrivial one, i.e. the map $\eta : W \mapsto W$ takes place, if endpoints of $\ell$ belong to different sides of the curve $\zeta$. In this case, on the torus $T$, the complement to the union $\zeta \cup \ell$ is a 2-disk $D^2$ and cannot contain other nontrivial curves. □

It is easy to see that for each fixed pair $\rho(s_1)$ and $\rho(s_4)$, there exist only those pairs of the spaces $\mathcal{V}'_{s_1}$ and $\mathcal{V}'_{s_4}$ and the superpositions $f_{p+2} \circ f_p : \mathcal{V}'_{s_1} \to \mathcal{V}'_{s_4}$ that are given below, where $p = 1, 2$. The corresponding superpositions are equal, since the corresponding products of the matrices (2.6) (or the obvious extensions of these matrices) of the linear maps are equal.

Suppose that $\rho(s_1) = \rho(s_4)$, then we obtain Cases 1 – 3 given below, where we take into account item (1) of Lemma 3.2.

**Case 1:** Suppose that $\rho(s_1) = \rho(s_4) = 1$.

(1) $f_{p+2} \circ f_p : V \to V$, where $f_{p+2} \circ f_p \in \{m_1 \circ \Delta_1, m_3 \circ \Delta_3\}$:
- $V \xrightarrow{\ \Delta_1\ } V \otimes V \xrightarrow{\ m_1\ } V$,
- $V \xrightarrow{\ \Delta_3\ } W \otimes W \xrightarrow{\ m_3\ } V$.

(2) $f_{p+2} \circ f_p : W \to W$, where $f_{p+2} \circ f_p \in \{m_2 \circ \Delta_2, \eta \circ \eta\}$:
- $W \xrightarrow{\ \Delta_2\ } V \otimes W \xrightarrow{\ m_2\ } W$,

- $W \xrightarrow{\eta} W \xrightarrow{\eta} W$.

**Case 2:** Suppose that $\rho(s_1) = \rho(s_4) = 2$.

(1) $f_{p+2} \circ f_p : V \otimes V \to V \otimes V$, where $f_{p+2} \circ f_p \in \{\Delta_1 \circ m_1, m_1 \circ \Delta_1\}$:
- $V \otimes V \xrightarrow{m_1} V \xrightarrow{\Delta_1} V \otimes V$,
- $V \otimes V \xrightarrow{\Delta_1} V \otimes V \otimes V \xrightarrow{m_1} V \otimes V$.

(2) $f_{p+2} \circ f_p : V \otimes V \to W \otimes W$, where $f_{p+2} \circ f_p \in \{\Delta_3 \circ m_1, m_2 \circ \Delta_3\}$:
- $V \otimes V \xrightarrow{m_1} V \xrightarrow{\Delta_3} W \otimes W$,
- $V \otimes V \xrightarrow{\Delta_3} V \otimes W \otimes W \xrightarrow{m_2} W \otimes W$.

(3) $f_{p+2} \circ f_p : V \otimes W \to V \otimes W$, where $f_{p+2} \circ f_p \in \{\Delta_2 \circ m_2, m_2 \circ \Delta_1, m_3 \circ \Delta_3, m_1 \circ \Delta_2\}$:
- $V \otimes W \xrightarrow{m_2} W \xrightarrow{\Delta_2} V \otimes W$,
- $V \otimes W \xrightarrow{\Delta_1} V \otimes V \otimes W \xrightarrow{m_2} V \otimes W$,
- $V \otimes W \xrightarrow{\Delta_3} W \otimes W \otimes W \xrightarrow{m_3} V \otimes W$,
- $V \otimes W \xrightarrow{\Delta_2} V \otimes V \otimes W \xrightarrow{m_1} V \otimes W$.

(4) $f_{p+2} \circ f_p : W \otimes W \to W \otimes W$, where $f_{p+2} \circ f_p \in \{\Delta_3 \circ m_3, m_2 \circ \Delta_2\}$:
- $W \otimes W \xrightarrow{m_3} V \xrightarrow{\Delta_3} W \otimes W$,
- $W \otimes W \xrightarrow{\Delta_2} V \otimes W \otimes W \xrightarrow{m_2} W \otimes W$.

(5) $f_{p+2} \circ f_p : W \otimes W \to V \otimes V$, where $f_{p+2} \circ f_p \in \{\Delta_1 \circ m_3, m_3 \circ \Delta_2\}$:
- $W \otimes W \xrightarrow{m_3} V \xrightarrow{\Delta_1} V \otimes V$,
- $W \otimes W \xrightarrow{\Delta_2} V \otimes W \otimes W \xrightarrow{m_3} V \otimes V$.

**Case 3:** Suppose that $\rho(s_1) = \rho(s_4) = 3$. The second case described in item (1) of Lemma 3.2 is impossible, since each curve should take part in at least one linear map. In the first case, we have a pair of curves connected by a dashed arc (switching brings to the linear map $m_{t_1}$, $t_1 \in \{1, 2, 3\}$), while the third curve is disjoint and is endowed with a dashed arc (switching brings to the linear map $\Delta_{t_2}$, $t_2 \in \{1, 2, 3\}$). Therefore, the face $F$ is commutative, since any curve in each of the sets $\mathcal{D}'_s$ is involved in only one map, $m_{t_1}$ or $\Delta_{t_2}$, i.e. the two possible walks along the edges differ only in the choice in which of the two crossings the smoothing type switches first. Hence, $f_1 = f_4$ and $f_2 = f_3$, while $f_1$ and $f_2$ act independently.

Suppose that $\rho(s_1) < \rho(s_4)$, then we obtain Cases 4 – 6 given below, where we take into account items (2) and (3) of Lemma 3.2.

**Case 4:** Suppose that $\rho(s_1) = 1$, $\rho(s_4) = 2$.

(1) $f_{p+2} \circ f_p : W \to V \otimes W$, where $f_{p+2} \circ f_p \in \{\eta \circ \Delta_2, \Delta_2 \circ \eta\}$:
- $W \xrightarrow{\Delta_2} V \otimes W \xrightarrow{\eta} V \otimes W$,
- $W \xrightarrow{\eta} W \xrightarrow{\Delta_2} V \otimes W$.

Note that, for example, the superposition $V \xrightarrow{\Delta_3} W \otimes W \xrightarrow{\eta} W \otimes W$ does not exist, see Lemma 3.3.

**Case 5:** Suppose that $\rho(s_1) = 1$, $\rho(s_4) = 3$.

(1) $f_{p+2} \circ f_p : V \to V \otimes W \otimes W$, where $f_{p+2} \circ f_p \in \{\Delta_3 \circ \Delta_1, \Delta_2 \circ \Delta_3\}$:
- $V \xrightarrow{\Delta_1} V \otimes V \xrightarrow{\Delta_3} V \otimes W \otimes W$,
- $V \xrightarrow{\Delta_3} W \otimes W \xrightarrow{\Delta_2} V \otimes W \otimes W$.

(2) $f_{p+2} \circ f_p : V \to V \otimes V \otimes V$, where $f_{p+2} \circ f_p = \Delta_1 \circ \Delta_1$:
- $V \xrightarrow{\Delta_1} V \otimes V \xrightarrow{\Delta_1} V \otimes V \otimes V$.

(3) $f_{p+2} \circ f_p : W \to W \otimes W \otimes W$, where $f_{p+2} \circ f_p = \Delta_3 \circ \Delta_2$:
- $W \xrightarrow{\Delta_2} V \otimes W \xrightarrow{\Delta_3} W \otimes W \otimes W$.

(4) $f_{p+2} \circ f_p : W \to V \otimes V \otimes W$, where $f_{p+2} \circ f_p \in \{\Delta_1 \circ \Delta_2, \Delta_2 \circ \Delta_2\}$:
- $W \xrightarrow{\Delta_2} V \otimes W \xrightarrow{\Delta_1} V \otimes V \otimes W$,
- $W \xrightarrow{\Delta_2} V \otimes W \xrightarrow{\Delta_2} V \otimes V \otimes W$.

**Case 6:** Suppose that $\rho(s_1) = 2$, $\rho(s_4) = 3$.

$f_{p+2} \circ f_p : V \otimes W \to V \otimes V \otimes W$, where $f_{p+2} \circ f_p \in \{\Delta_1 \circ \eta, \eta \circ \Delta_1\}$:
- $V \otimes W \xrightarrow{\eta} V \otimes W \xrightarrow{\Delta_1} V \otimes V \otimes W$,
- $V \otimes W \xrightarrow{\Delta_1} V \otimes V \otimes W \xrightarrow{\eta} V \otimes V \otimes W$.

Suppose that $\rho(s_1) > \rho(s_4)$, then we use the symmetry to obtain Cases 7 – 9 given below from Cases 4 – 6 given above by replacement $m_t$ with $\Delta_t$, $t \in \{1, 2, 3\}$, and vice versa.

**Case 7:** Suppose that $\rho(s_1) = 2$, $\rho(s_4) = 1$.

(1) $f_{p+2} \circ f_p : V \otimes W \to W$, where $f_{p+2} \circ f_p \in \{\eta \circ m_2, m_2 \circ \eta\}$:
- $V \otimes W \xrightarrow{m_2} W \xrightarrow{\eta} W$,
- $V \otimes W \xrightarrow{\eta} V \otimes W \xrightarrow{m_2} W$.

**Case 8:** Suppose that $\rho(s_1) = 3$, $\rho(s_4) = 1$.

(1) $f_{p+2} \circ f_p : V \otimes V \otimes V \to V$, where $f_{p+2} \circ f_p = m_1 \circ m_1$:
- $V \otimes V \otimes V \xrightarrow{m_1} V \otimes V \xrightarrow{m_1} V$.

(2) $f_{p+2} \circ f_p : W \otimes W \otimes W \to W$, where $f_{p+2} \circ f_p = m_2 \circ m_3$:
- $W \otimes W \otimes W \xrightarrow{m_3} V \otimes W \xrightarrow{m_2} W$.

(3) $f_{p+2} \circ f_p : V \otimes V \otimes W \to W$, where $f_{p+2} \circ f_p \in \{m_2 \circ m_1, m_2 \circ m_2\}$:
- $V \otimes V \otimes W \xrightarrow{m_1} V \otimes W \xrightarrow{m_2} W$,
- $V \otimes V \otimes W \xrightarrow{m_2} V \otimes W \xrightarrow{m_2} W$.

(4) $f_{p+2} \circ f_p : V \otimes W \otimes W \to V$, where $f_{p+2} \circ f_p \in \{m_3 \circ m_2, m_1 \circ m_3\}$:
- $V \otimes W \otimes W \xrightarrow{m_2} W \otimes W \xrightarrow{m_3} V$,
- $V \otimes W \otimes W \xrightarrow{m_3} V \otimes V \xrightarrow{m_1} V$.

**Case 9:** Suppose that $\rho(s_1) = 3$, $\rho(s_4) = 2$.

(1) $f_{p+2} \circ f_p : V \otimes V \otimes W \to V \otimes W$, where $f_{p+2} \circ f_p \in \{m_1 \circ \eta, \eta \circ m_1\}$:
- $V \otimes V \otimes W \xrightarrow{\eta} V \otimes V \otimes W \xrightarrow{m_1} V \otimes W$,
- $V \otimes V \otimes W \xrightarrow{m_1} V \otimes W \xrightarrow{\eta} V \otimes W$.

STEP 2. 3. Let $\rho = 4$. In this case, obviously, the face $F$ is commutative. Indeed, any curve in each of the sets $\mathcal{D}'_s$ is involved in only one map, therefore, the two possible walks along the edges differ only in the choice in which of the two crossings the smoothing type switches first. Hence, $f_1 = f_4$ and $f_2 = f_3$, while $f_1$ and $f_2$ act independently.

STEP 3. Let us construct the face $F$ to be anticommutative. Similarly to the proof proposed in [4] (see also [3]) for a chain complex constructed for a classical knot, we can set each commutative face $F$ to be anticommutative. To this end, we endow each linear map corresponding to an edge of the face $F$ with the sign determined by formula (2.7) introduced when constructing a chain complex..

STEP 4. Since each two-dimensional face $F$ of the unit $\mathbf{n}-$ dimensional cube is anticommutative, then $d^{n+1}d^n = 0$, see [3]. □

## 4. Invariance

**Lemma 4.1.** *The proposed construction of the Khovanov homology of knots in the thickened torus $T \times I$ is invariant.*

*Proof.* Direct verification of invariance with respect to all three Reidemeister moves $\Omega_1 - \Omega_3$ is carried out similarly to the corresponding proof given in [3]. An analogy takes place, since the constructed linear maps $m_t$ and $\Delta_t$, $t = 1, 2, 3$, have all the properties used for proof in [3]. Namely, the element $v^+$ is a unit for the maps $m_t$, $t = 1, 2, 3$ (this property is used to prove invariance under $\Omega_1$) and the maps $\Delta_t$, $t = 1, 2, 3$, are isomorphisms modulo $v^+ = 0$ (this property is used to prove invariance under $\Omega_2$ and $\Omega_3$). The linear map $\eta$ does not appear in the Reidemeister moves. □

## 5. Main Theorem

A factor group $\mathcal{H}^n(\mathcal{C}) = ker(d^n)/Im(d^{n-1})$ is called an $n-$dimensional cohomology group of a chain complex $\mathcal{C}$ (2.1). Nevertheless, following historically established terminology, when constructing Khovanov's construction, we talk about homology instead of cohomology.

A chain complex $\mathcal{C}$ is called finitely generated if the number of its nonzero chain groups (i.e., nonzero $\mathcal{C}^n$) is finite.

For a fixed value of the homological grading $i(\nu_s) = i$, consider a graded vector space

$$\mathcal{V}^i = \oplus_{j,k}\mathcal{V}^{i,j,k}.$$

Then by the graded dimension of the graded vector space $\mathcal{V}^i$ we mean a power series

$$\operatorname{qg}\dim \mathcal{V}^i = \sum_{j,k} q^j \cdot g^k \cdot \dim \mathcal{V}^{i,j,k}, \tag{5.1}$$

where $\dim \mathcal{V}^{i,j,k}$ is the usual dimension of the vector space $\mathcal{V}^{i,j,k}$, and $q$ and $g$ are variables.

By the graded Euler characteristic $\hat{\mu}(\mathcal{C})$ of a finitely generated graded chain complex $\mathcal{C}$ we mean the alternating sum of the graded dimensions (5.1) of homology groups of the chain complex $\mathcal{C}$ [3].

**Theorem 5.1.** *Up to the change of variables $a = (-q)^{-1/2}$ and $x = g + g^{-1}$, the generalized Kauffman bracket $\mathcal{X}(a, x)_K$ (1.2) of a knot $K \in T \times I$ is equivalent to the graded Euler characteristic $\hat{\mu}(\mathcal{C}(K))$:*

$$\mathcal{X}(a, x)_K = \hat{\mu}(\mathcal{C}(K)) = \sum_i (-1)^i \cdot qg\dim \mathcal{H}^i(K) = \sum_{i,j,k} (-1)^i \cdot q^j \cdot g^k \cdot \dim \mathcal{H}^{i,j,k}(K).$$

*Proof.* For a finitely generated graded chain complex $\mathcal{C}$ such that $d^{n+1}d^n = 0$ and all chain groups $\mathcal{C}^n$ have finite dimension, the graded Euler characteristic $\hat{\mu}(\mathcal{C})$ is equal to the alternating sum of the graded dimensions of the chain groups $\mathcal{C}^n$, see [4, 3].

It is easy to see that definition (5.1) of the graded dimension $\operatorname{qg}\dim \mathcal{V}^i$ of the graded space $\mathcal{V}^i = \oplus_{j,k}\mathcal{V}^{i,j,k}$ implies the equalities

$$\operatorname{qg}\dim(V^{\otimes\gamma(s)} \otimes W^{\otimes\delta(s)}) = (q + q^{-1})^{\gamma(s)}(g + g^{-1})^{\delta(s)}$$

and

$$\operatorname{qg}\dim(\mathcal{V}_1 \oplus \mathcal{V}_2) = \operatorname{qg}\dim \mathcal{V}_1 + \operatorname{qg}\dim \mathcal{V}_2.$$

Therefore, the terms in the sum of the Kauffman bracket over all states satisfying the condition $i = \frac{\omega(D)-\sigma(s)}{2} = n$ are associated with elements of the chain group $\mathcal{C}^n$. □

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ALYONA A. AKIMOVA: DEPARTMENT OF MATHEMATICAL AND COMPUTER MODELLING, SOUTH URAL STATE UNIVERSITY, CHELYABINSK, RUSSIA

*E-mail address*: `akimovaaa@susu.ru`