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GRACEFUL LABELING OF CLOSED CATERPILLARS

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Abstract. For the sake of simplicity, we emphasize on simple, connected and finite graphs. After a long period of scramble over analysis and investigations, the notion of graceful labeling came into existence and therefore the credit goes to Rosa (1967) and then by Golomb (1972) for the first definition of graceful labeling. If there exists a bijective mapping $f : V(G) \to \mathbb{N} \cup \{0\}$ such that each edge $e \in E(G)$ has the induced label $\omega(f, V(G)) = \{|f(u) - f(v)| : u, v \in V(G)\}$ and $min\,\omega(f, V(G)) \leq \omega(e) \leq max\,\omega(f, V(G))$ such that the resulting edge labels are distinct, then f is said to be graceful labeling for the graph $G = (V, E)$. In this paper, we develop a new operation " superimposition" for joining two smaller graphs by which we can obtain a new larger graph. By this operation, we generate a new graph from a caterpillar $P_n \odot K_1$ and call it closed caterpillar. We prove that all closed caterpillars admit graceful labeling. We also conjecture that all two sided closed caterpillars have graceful labeling.

1. Introduction

Any graph $G = (V, E)$ use in this paper is simple, connected and finite. We have had a lot to say so far about graceful graphs but what about the graphs obtained from smaller graceful graphs? After a long period of scramble over analysis and investigations, the notion of graceful labeling (β-valuation) came into existence and therefore the credit goes to Rosa [14] and then by Golomb [8] for the first definition of graceful labeling. If there exists a bijective mapping $f : V(G) \to \mathbb{N} \cup \{0\}$ such that each edge $e \in E(G)$ has the induced label $\omega(f, V(G)) = \{|f(u) - f(v)| : u, v \in V(G)\}$ and $min\,\omega(f, V(G)) \leq \omega(e) \leq max\,\omega(f, V(G))$ and the resulting edge labels are distinct, then f is said to be graceful labeling for the graph $G = (V, E)$. A graceful labeling with the property that there exists an integer k so that for every edge $uv \in E$, either $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$ is called α - labeling (or α - valuation) [14]. An α - valuation and β - valuation of n-gon exists if and only if $n \equiv 0\ (mod\ 4)$ or $n \equiv 0\ or\ 3\ (mod\ 4)$ respectively made by Rosa [14]. A similar result made by Habbare [10] for cycles C_n that C_n is graceful if and only if $n \equiv 0\ or\ 3\ (mod\ 4)$.

A new operation between two graphs G_1 and G_2 is presented by Frucht and Harary [6]. They called this operation corona between G_1 and G_2 as follow: the corona $G_1 \circ G_2$ of the two graphs G_1 and G_2 (where G_1 has p_i points and q_i lines)

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is defined as the graph G obtained by taking one copy of G_1 and p_i copies of G_2, and then joining by a line the i^{th} point of G_1 to every point in the i^{th} copy of G_2. In this paper, we use the symbol $\odot$ in place of $\circ$ to denote the corona. As time is passing, new type of graph labeling techniques are emerging. Mohamed R.Zeen El Deen [5] obtained the even edge graceful labeling of the following graphs: y-tree, double star $B_{n,m}$, $K_{1,2n} : K_{1,2m}$, $P_{2n-1} \odot \bar{K_{2m}}$, double fan graph, prism graph, flag graph, flower graph and double cycle. Kumar et. al. [11] gave the idea of graph labeling in dental field. For a capacious survey on graph labeling, see the literature of Gallian [7] and for other resources one can see also [1], [2] [3], [4], [6], [9], [11], [12], [13].

2. Graceful Cycle Related Graphs obtained from Paths

In this section, we develop a new operation *superimposition for joining two graphs G_1 and G_2 in which G_1 has m pendant vertices and G_2 is a path of length m. We use the symbol 'sup' for the operation superimposition. It is defined as:*

$$V(G_1 \ sup \ G_2) = V(G_1)$$
$$E(G_1 \ sup \ G_2) = E(G_1) \cup E(G_2)$$

In other words, the vertices of G_2 are fused with the vertices of G_1 and the size $E(G_1 \ sup \ G_2)$ of $G_1 \ sup \ G_2$ remains same as the sum of the size of G_1 and G_2.

We also construct a graph from a path P_n in which each component is a cycle of length four. On adding exactly one edge to each vertex of P_n we obtain a caterpillar $P_n \odot K_1$. The vertex set V of this caterpillar $P_n \odot K_1$ is divided into two subvertex sets W and X, where $V = X \cup W$. We say that the set X is the set of ground vertices $x_1, x_2, ..., x_n$ and W is the set of celing vertices $w_1, w_2, ..., w_n$ (see Figure 1). If we join w_i to w_{i+1} by exactly one edge or a path P_n is superimposed on the pendant vertices w_i's of $P_n \odot K_1$, then we obtain a graph G and we call it closed caterpillar ($P_n \odot K_1 \ sup \ P_n$). In the next theorem, we prove that all closed caterpillar admists graceful labeling. Before proceeding, we have the following lemma:

Lemma 2.1. *All caterpillars $P_n \odot K_1$ have graceful labeling.*

Proof. Let g be the labeling of the $P_n \odot K_1$ with $2n$ vertices as follow:
$g(x_1) = 0$, $g(x_2) = 2n - 2$, $g(x_3) = 2$, $g(x_4) = 2n - 4$, $\ldots$, $g(x_n) = 2n - n$
and
$g(w_1) = 2n - 1$, $g(w_2) = 1$, $g(w_3) = 2n - 3$, $g(w_4) = 3$, $\ldots$, $g(w_n) = 2n - n - 1$.

Obviously, each edge of $P_n \odot K_1$ admits the absolute difference of the label of its end vertices and it is distinct. Therefore, the caterpillar $P_n \odot K_1$ is graceful. □

The ordinary and graceful labeling of caterpillar are shown in Figure 1 and Figure 2.

For the sake of illustration of graph labeling, we will lead to certain more profound theorem and prove that all closed caterpillars ($P_n \odot K_1 \ sup \ P_n$) admit graceful labeling.

Theorem 2.2. *All closed caterpillars ($P_n \odot K_1 \ sup \ P_n$) admit graceful labeling.*

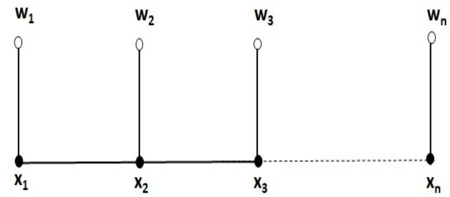

FIGURE 1. Caterpillar $P_n \odot K_1$ with ordinary labeling

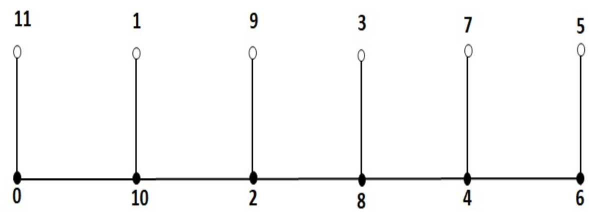

FIGURE 2. Caterpillar $P_6 \odot K_1$ with graceful labeling

Proof. From the lemma-1 defined above, we have prove that all caterpillars $P_n \odot K_1$ are graceful. Now, do the slightly change in the above lemma, Join w_i to w_{i+1} where $1 \leq i \leq (n-1)$ by an edge in the caterpillar $P_n \odot K_1$ and we obtain a closed caterpillar ($P_n \odot K_1$ *sup* P_n) (see Figure 3). Let l be the distance of the vertex x_i (or the vertex w_i) from the rightmost vertex x_n (or the vertex w_n). Let $h(t)$ be the labeling of ($P_n \odot K_1$ *sup* P_n) obtained by the graceful labeling $g(x)$ of $P_n \odot K_1$ defined in lemma 2.1 and $h(t)$ is defined as:

$$h(t) = \begin{cases} g(x_i) & \text{if } i \text{ is odd} \\ g(x_i) + l & \text{if } i \text{ is even} \\ g(w_i) & \text{if } i \text{ is even} \\ g(w_i) + l & \text{if } i \text{ is odd} \end{cases}$$

Clearly, we obtained the ground and celing vertices labels of the closed caterpillar as:

$$\begin{array}{cccc} g(x_1) & g(x_2) & \ldots & g(x_n) \\ \downarrow & \downarrow & \ldots & \downarrow \\ h(g(x_1)) & h(g(x_2)) & \ldots & h(g(x_n)) \\ \downarrow & \downarrow & \ldots & \downarrow \\ 0 & h(g(x_2)) + l & \ldots & h(g(x_n)) \text{ or } h(g(x_n)) + l \end{array}$$

and

$$\begin{array}{cccc}
g(w_1) & g(w_2) & \dots & g(w_n) \\
\downarrow & \downarrow & \dots & \downarrow \\
h(g(w_1)) & h(g(w_2)) & \dots & h(g(w_n)) \\
\downarrow & \downarrow & \dots & \downarrow \\
0 & h(g(w_2)) + l & \dots & h(g(w_n)) \text{ or } h(g(w_n)) + l
\end{array}$$

Therefore, we obtained the labels for celing edges as the sequence from left to right as: $3 \to 5 \to 9 \to 11 \to 15 \to 17 \to \dots$ and the labels for the ground edges as the sequence from left to right $2 \to 6 \to 8 \to 12 \to 14 \to 18 \to \dots$ and the labels for the vertical edges as the sequence from left to right $1 \to 4 \to 7 \to 10 \to 13 \to 16 \to \dots$. Therefore, each of the closed caterpillar gets the distinct labels $1 \to 2 \to 3 \to \dots \to (3n-2)$ and each vertex gets the distinct labels $0 \to 1 \to 2 \to 3 \to \dots \to (3n-2)$. So, every closed caterpillar $(P_n \odot K_1 \ sup \ P_n)$ admits graceful labeling.

□

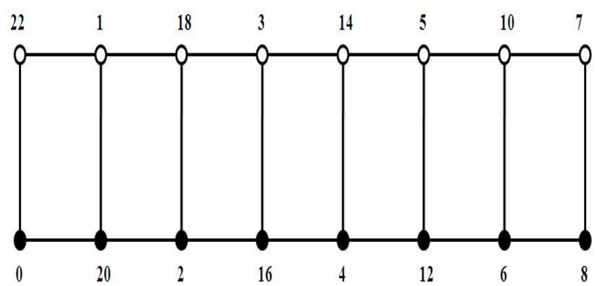

FIGURE 3. Graceful labeling of $(P_8 \odot K_1 \ sup \ P_8)$

3. *Graceful Graphs Obtained from Closed Caterpillars*

In this section, we generate new graceful graphs obtained from graceful closed caterpillars $(P_n \odot K_1 \ sup \ P_n)$ *which is defined in the previous section. This new class of graceful graph is obtained by adding diagonals to* $(P_n \odot K_1 \ sup \ P_n)$*. When an edge* $e = w_i x_{i+1}$ *or* $e = x_i w_{i+1}$ *join the vertices* w_i *to* x_{i+1} *or* x_i *to* w_{i+1} *respectively, then this edge e is said to be the diagonal. The next theorem asserts the graph* $(P_n \odot K_1 \ sup \ P_n)$ *with atmost two diagonals admits graceful labeling.*

Theorem 3.1. *The closed caterpillar* $(P_n \odot K_1 \ sup \ P_n)$ *with at most two diagonals admits graceful labeling.*

Proof. On adding one diagonal $w_1 x_2$ in the closed caterpillar (see Figure 1), the size is increased by one and the order remains same. The labels of odd and even w_i's are increased as the term of arithemetic progression 1, 3, 5, ... Similarly, the labels of even x_i's become $2x_i$ and odd x_i's become 2, 4, 6, ... respectively according to:

$$h'(t) = \begin{cases} 2h(g(x_i)) & \text{if } i \text{ is odd} \\ i + h(g(x_i) + l) & \text{if } i \text{ is even} \\ i + h(g(w_i)) & \text{if } i \text{ is even} \\ i + h(g(w_i) + l) & \text{if } i \text{ is odd} \end{cases}$$

After joining second diagonal w_2x_3, the odd w_i's and even x_i's are incresed by one and the remaining labelins of the vertices do not alter according to:

$$h''(t) = \begin{cases} h'(2h(g(x_i))) & \text{if } i \text{ is odd} \\ h'(i + h(g(x_i) + l)) + 1 & \text{if } i \text{ is even} \\ h'(i + h(g(w_i))) + i & \text{if } i \text{ is even} \\ h'(i + h(g(w_i) + l)) + 1 & \text{if } i \text{ is odd} \end{cases}$$

Therefore, $(P_n \odot K_1 \ sup \ P_n)$ with atmost two diagonals admits graceful labeling. □

For clearification, see the Figure 4.

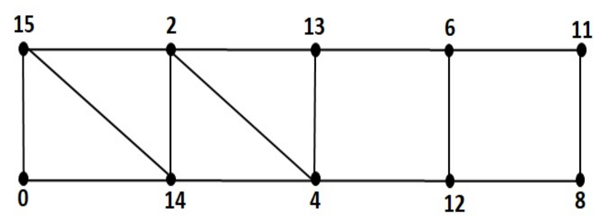

FIGURE 4. Graceful labeling of $(P_5 \odot K_1 \ sup \ P_5)$ with two diagonals

When we join x_1 to x_n and w_1 to w_n by an edge sepatetely in the closed caterpillar, then we obtain a new graph and we call it two sided closed caterpillar. We conjecture that all two sided closed caterpillars $(P_n \odot K_1 \ sup \ P_n)$ admit graceful labeling.

Conjecture 3.2. All two sided closed caterpillars $(P_n \odot K_1 \ sup \ P_n)$ are graceful when $n \equiv 0 \ or \ 3 \ (mod \ 4)$.

Conclusion and future work. *We develop a new operation superimposition for joining two graphs and from this operation, we prove that all closed caterpillars have graceful labeling. We also prove that the closed caterpillar with at most two diagonals admits graceful labeling and conjecture that all two sided closed caterpillars are graceful. In future, researcher can prove for the graceful labeling of closed caterpillar with many diagonal and the given conjecture.*

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