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A TENSOR PRODUCT OF FUZZY CYCLE C_n WITH K_2

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Abstract

In this paper some properties of fuzzy monotonic decomposition of graphs are studied. We established the tensor product of fuzzy cycle C_n with K_2 is a FCMD of fuzzy graphs under certain condition.
Key Words and Phrases: Fuzzy continuous monotonic decomposition, Tensor Product of two fuzzy graphs, Fuzzy Cycle C_n with K_2.

1. Introduction

The notion of fuzzy sets was introduced by L.A.Zadeh in 1965.It involves the concept of a membership function defined on a universal set. The value of the membership function lies in [0,1]. This concept has found applications in computer science, artificial intelligence, decision analysis, information science, pattern recognition, operations research and robotics. The fuzzy relations between fuzzy sets were also considered by Rosenfeld who developed the structure of fuzzy graphs. Later on, Bhattacharya gave some remarks on fuzzy graphs. The operations of union, join, Cartesian Product and composition on two fuzzy graphs were defined by Moderson J.N. and Peng[3]. C.S. In this paper some properties of fuzzy monotonic decomposition of graphs are studied. We have obtained the tensor product of fuzzy cycle C_n with K_2 is a FCMD of fuzzy graphs under certain condition.

Definition 1.1. A fuzzy set A in X is a set of ordered pairs $A = \{(x, \mu_A(x))/x \in X\}$ where $\mu_A : X \to [0, 1]$ is called membership function or grade of membership of x in A.

Definition 1.2. A fuzzy graph $G = (\sigma, \mu)$ on V is a pair of membership functions or fuzzy sets $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0.1]$ such $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$.

Definition 1.3. A fuzzy graph $G = (\sigma, \mu)$ is said to be a complete fuzzy graph if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $u, v \in \sigma^*$.

Definition 1.4. Let $G = (\sigma, \mu)$ be a fuzzy graph. A fuzzy path between two nodes v_0 ,v_n in G is a sequence of distinct nodes $v_0, v_1, \ldots, v_n$ such that $\mu(v_{i-1}, v_i) > 0$, $1 \leq i \leq n$. If $v_0 = v_n$ then the fuzzy path is called a fuzzy cycle.

Definition 1.5. The strength of the fuzzy path between two nodes u,v in a fuzzy graph G is $\mu^\infty(u, v) = sup\{\mu^K(u, v)K = 1, 2, 3, \ldots\}$ where

$$\mu^K(u, v)sup\{\mu(u, u_1) \wedge \mu(u_1, u_2) \wedge \cdots \wedge \mu(u_{K-1}, v)\}.$$

A fuzzy path with strength $\mu^\infty(u, v)$ is called strongest fuzzy the nodes u and v.

Definition 1.6. An arc (u,v) is said to be a strong arc or strong edge, if $\mu(u,v) \leq \mu^{\infty}(u,v)$ and the node is said to be a strong neighbor to v or v is said to be a strong neighbor u, otherwise it is called weak arc or weak edge. A node 'u' is said to be isolated in G if $\mu(u,v) = 0$ for all $u, v \in V$ and $v \neq u$.

2. FUZZY CONTINUOUS MONOTONIC DECOMPOSITION

Definition 2.1. Let G be a simple connected fuzzy graph. If $H_1, H_2, \ldots, H_k$ $\forall k \in N$ are edge disjoint fuzzy subgraphs of G such that $E(G) = E(H_1) \cup E(H_2) \cup, \ldots, \cup E(H_k)$ then $H_1, H_2, \ldots, H_k$ is said to be a decomposition of G.

Definition 2.2. $G_i/i = 1, 2, \ldots n$ be a collection of edge -disjoint fuzzy subgraphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup, \ldots, \cup E(G_n)$, if each G_i connected and $|E(G_i)| = i$ for each $i = 1, 2, \ldots, n$, then it is called a fuzzy continuous monotonic decomposition FCMD of fuzzy graph G.

Definition 2.3. Ascending fuzzy subgraph decomposition is a decomposition of fuzzy graph G into fuzzy subgraph H_i (Not necessarily connected) $|E(H_i)| = i$ and is isomorphic to a proper subgraph of H_{i+1}.

Definition 2.4. Tensor product of two simple fuzzy graphs G_1 and G_2 is the fuzzy graph G with vertex set $V(G) = V_1 \times V_2$. Let $_i$ be a fuzzy subset of V_i and let μ_i be a fuzzy subset of $X_i, i = 1, 2$, with the fuzzy subset $\sigma_1 \otimes \sigma_2$ of V and $\mu_1 \otimes \mu_2$ of X as

$$\sigma_1 \otimes \sigma_2(u_1, u_2) = min\{\sigma_1(u), \sigma_2(u_2)\} \ for \ all \ (u_1, u_2) \in V$$

$$(\mu_1 \otimes \mu_2)\{(u_1, u_2)(v_1, v_2)\} = min\{\mu_1(u_1v_1), \mu_2(u_2v_2)\}$$
$$(u_1, u_2) \in X_1 \ and \ (v_1, v_2) \in X_2$$

Lemma 2.5. *Let G be a connected fuzzy graph. If $H_1, H_2, \ldots, H_m$ are fuzzy continuous monotonic decomposition of G. Let $m \equiv 0(mod4)$. Then edge set of $H_1, H_2, \ldots, H_m$ (ie) $\{1, 2, \ldots, m\}$ can be partitioned into two sets S_1 and S_2 such that total number of edges in S_1 and S_2 are $\sum_{x \in S_1} x = \sum_{y \in S_2} y = n$. Here $\dfrac{m(m+1)}{2} = 2n$ with corresponding membership values.*

Proof. Let G be a connected fuzzy graph, since $H_1, H_2, \ldots, H_m$ are fuzzy continuous monotonic decomposition of G. Therefore each $H_1, H_2, \ldots, H_m$ are disjoint fuzzy subgraph of G. Also each H_i connected and $|E(H_i)| = i$ for each $i = 1, 2, \ldots m$.

Let $m = 4k, k1, k \in z$. Proof of the theorem by mathematical induction on k. When $k = 1, m = 4$ and $n = 5$. Let $S_1 = 1, 4$ and $S_2 = 2, 3$. Now $\sum_{x \in S_1} \equiv x = 1 + 4 = 5 = n$ and $\sum_{y \in S_2} \equiv y = 2 + 3 = 5 = n$. So that the result is true if $k = 1$.

Assume that the result is true for $k - 1$. Hence the set $\{1, 2, 3, \ldots, 4(K-1)\}$ can be partitioned into two fuzzy sets S_1 and S_2 such that $\sum_{x \in S_1} = x = \sum_{y \in S_2} =$

$y = n = (k-1)(4k-3)$.

To prove the result is true for k. The set $\{1, 2, 3, \ldots, 4k\}$ can be partitioned into two sets $S_1^{'}$ and $S_2^{'}$ where $S_1^{'} = S_1 \cup \{4k-3, 4k\}$ and $S_2^{'} = S_2 \cup \{4k-2, 4k-1\}$.

Now

$$\begin{aligned}\sum_{x \in S_1^{'}} x &= \sum_{x \in S_1} x + 4k - 3 + 4k \\ &= (k-1)(4k-3) + 8k - 3 \\ &= 4k^2 + k = k(4k+1) \\ &= n.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{y \in S_2^{'}} y &= \sum_{y \in S_2} y + 4k - 2 + 4k - 1 \\ &= (k-1)(4k-3) + 8k - 3 \\ &= 4k^2 + k \\ &= k(4k+1) \\ &= n.\end{aligned}$$

Hence by induction the lemma is true of all k. □

Lemma 2.6. *Let G be a connected fuzzy graph. If $H_1, H_2, \ldots, H_m$ are fuzzy continuous monotonic decomposition of G. Let $m+1 \equiv 0(mod4)$. Then edge set of $H_1, H_2, \ldots, H_m$ (ie) $\{1, 2, \ldots, m\}$ can be partitioned into two sets S_1 and S_2 such that total number of edges in S_1 and S_2 are$\sum_{x \in S_1} = x = \sum_{y \in S_2} = y = n$. Here $\dfrac{m(m+1)}{2} = 2n$ with corresponding membership values.*

Proof. Let G be a connected fuzzy graph, since $H_1, H_2, \ldots, H_m$ are FCMD of G. Therefore each $H_1, H_2, \ldots, H_m$ are disjoint fuzzy subgraph of G.

Let $m + 1 = 4k, k \geq 1, k \in z$ so that $m = 4k - 1$ proof is by induction on k. When $k = 1, m = 3$ and $n = \dfrac{m(m+1)}{4} = \dfrac{3 \times 4}{4} = 3$.Let $S_1 = 1, 2$ and $S_2 = 3$. Now $\sum_{x \in S_1} = x = 1 + 2 = 2 = n$ and $\sum_{y \in S_2} = y = 3 = 3 = n$. Hence the result is true if k=1.

Assume that the result is true for k-1. Hence the set $\{1, 2, 3, \times, 4(K-1) - 1\}$ can be partitioned into two fuzzy sets S_1 and S_2 such that $\sum_{x \in S_1} = x = \sum_{y \in S_2} = y = n = (k-1)(4k-5)$. To prove the result is true for k. The set $\{1, 2, 3, \ldots, 4k\}$ can be partitioned into two sets $S_1^{'}$ and $S_2^{'}$ where $S_1^{'} = S_1 \cup \{4k-4, 4k-1\}$ and

$S_2' = S_2 \cup \{4k-3, 4k-2\}$.

Now

$$\begin{aligned}\sum_{x \in S_1'} x &= \sum_{x \in S_1} x + 4k - 4 + 4k - 1 \\ &= (k-1)(4k-5) + 4k - 4 + 4k - 1 \\ &= 4k^2 - 9k + 5 + 8k - 5 \\ &= 4k^2 - k = k(4k-1) = n\end{aligned}$$

Also,

$$\begin{aligned}\sum_{y \in S_2'} y &= \sum_{y \in S_2} y + 4k - 3 + 4k - 2 \\ &= (k-1)(4k-5) + 8k - 5 \\ &= 4k^2 - k \\ &= k(4k-1) \\ &= n.\end{aligned}$$

Hence by induction the lemma is true of all k.

□

Theorem 2.7. *For any integer n the tensor product of fuzzy cycle C_n with complete fuzzy graph of two vertices K_2 ie $C_n \otimes K_2$ has a FCMD graphs $\{H_1, H_2, \ldots, H_m\}$ if and only if there exists an integer m satisfying the following properties.*

i $m = 4k$ *or* $4k - 1 (k \geq 1, k \in Z)$

ii $\frac{m(m+1)}{2} = 2n$

Proof. Let G be a tensor product of fuzzy cycle C_n with complete fuzzy graph of two vertices K_2. (i.e) $G = C_n \otimes K_2$. By definition $|E(G)| = 2n$. Assume $G = C_n \otimes K_2$ has a FCMD. By the result $|E(G)| = m + I_{C_2}$. Hence $2n = \frac{m(m+1)}{2}$ since $C_n \otimes K_2$ has a CMD of fuzzy graphs.

$$\begin{gathered}2n = 1 + 2 + 3 + \ + m \\ \implies 2n = \frac{m(m+1)}{2} \\ Hence\ m(m+1) = 4n \\ ie\ m(m+1) = 0(mod4) \\ \implies m(m+1) = 4k \\ \implies m = 4k \ or\ m+1 = 4k \\ \implies m = 4k \ or\ m = 4k-1 \ where\ k \geq 1, k \in Z.\end{gathered}$$

Conversely, assume $m(m+1) = 0(mod4)$. Let $G = C_n \otimes K_2$. Let fuzzy cycle $(u_1, u_2, \ldots, u_n, u_1)$ and $\mu(u_1, u_2) \geq 0$ and $K_2 = \{v_1, v_2\}$ where $1 \leq i \leq n, i \leq j \leq$

2. Now $V(G) = \{w_{ij}; 1 \leq i \leq n, i \leq j \leq 2\}$ and $|E(G)| = 2n$.

Case (i): Suppose n is even

Define

$$T_1 = \{(w_i1, w_{(i+1)2}) : 1 \leq i \leq n, i - odd\} \cup \{(w_{i2}, w_{(i+1)1}) : 1 \leq i \leq n-1, \\ i - even\} \cup \{(w_{i2}, w_{i1}); i = n\}$$

and

$$T_2 = \{(w_i2, w_{(i+1)1}) : 1 \leq i \leq n, i - odd\} \cup \{(w_{i1}, w_{(i+1)2}) : 1 \leq i \leq n-1, \\ i - even\} \cup \{(w_{i2}, w_{i1}); i = n\}$$

Here $|T_1| = n$ and $|T_2| = n$. Also $|T_1| + |T_2| = 1 + 2 + 3, \ldots, m = m + 1_{C_2}$. By Lemma 2.1 & 2.2.

o $\{1, 2, 3, \ldots, m\} = S_1 \cup S_2$ where $\sum_{x \in S_1} x = n$ and $\sum_{y \in S_2} y = n$. Decompose $T_1 \& T_2$ into fuzzy trees $\{H_i\}$ as follows $T_1 = \bigcup_{i \in S_1} H_i$, $T_2 = \bigcup_{i \in S_2} H_i$ and $|E(H_i)| = i, 1 \leq i \leq m$ clearly $\{H_1, H_2, \ldots, H_m\}$ forms a continuous monotonic decomposition $C_n \otimes K_2$.

□

Example 2.8. In this example let us decompose the tensor product of fuzzy cycle with complete fuzzy graph of two vertices (ie) $C_5 \otimes K_2$. Let $V(C_5) = \{u_1, u_2, u_3, u_4, u_5\}$ and $V(K_2) = \{v_1, v_2\}$.

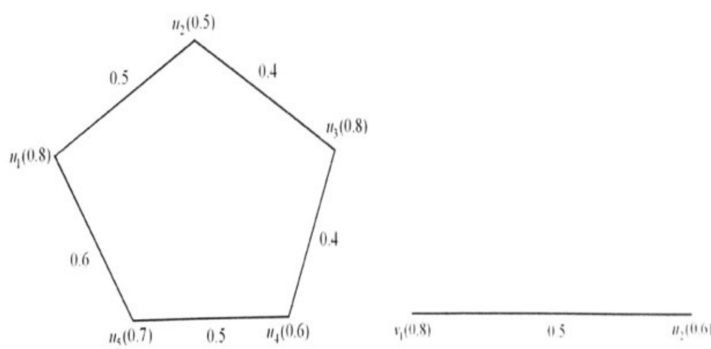

FIGURE 1. **Fuzzy cycle C_5, and Complete fuzzy graph K_2**

Tensor product of fuzzy cycle C_5 with complete fuzzy graph is given by
Here $|E(G)| = 10$ and $m = 4$. Let $e_{ij} = ((u_i, v_1), (u_j, u_2))$ where $1 \leq i, j \leq 5$

$$T_1 = \{e_{12}, e_{32}, e_{34}, e_{54}, e_{51}\}, T_2 = \{e_{21}, e_{23}, e_{43}, e_{45}, e_{48}\}$$

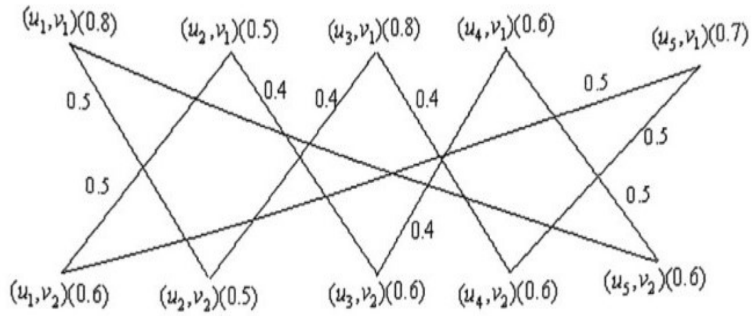

FIGURE 2. $C_5 \otimes K_2$

$|T_1| = |T_2| = 5$. Hence $|T_1| + |T_2| = 10 = 1 + 2 + 3 + 4 = 5C_2$
$S_1 = \{1, 4\}$and $S_2 = \{2, 3\}$. T_1 is decomposed as $T_1 = H_1 \cup H_4$ where $H_1 = \{e_{12}\}$, $H_4 = \{e_{32}, e_{34}, e_{54}, e_{51}\}$, T_2 is decomposed as $T_2 = H_2H_3$ where $H_2 = \{e_{12}, e_{23}\}$and $H_4 = \{e_{43}, e_{45}, e_{15}\}$,
$\{H_1, H_2, H_3, H_4\}$ formed a FCMD of $C_5 \otimes K_2$.

Theorem 2.9. *A complete fuzzy graph K_m accepts fuzzy continuous monotonic decomposition of $H_1, H_2, \ldots, H_(m-1)$ $\forall m \in N$.*

CONCLUSION

In this paper, we have illustrated the operation of tensor product of fuzzy cycle C_n with K_2 is a FCMD of fuzzy graphs. Some properties of fuzzy continuous monotonic decomposition of graphs are investigated.

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