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f-vector Computation of Disconnected Union of Wheel Graphs tW_n as Spanning Forest Complex

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Abstract

Graph theory has a great significance in every field specially in the area of science and engineering. Our aim is to apply the concept of graphs on the very basic object of Algebra called a set. Invention of spanning simplicial complex [1] was the revolutionary idea in the field of Commutative Algebra. In this manuscript, our main contribution is to evaluate dimensions of spanning forest complexes of disconnected union of wheel graphs tW_n. Also f-vectors of these spanning forest complexes of different disconnected wheel graphs are computed to verify the total number of subsets of the certain set.

1 Introduction

Commutative algebra is the study of commutative rings and ideals. Commutative Algebra is a tool for solving problems that are involved in algebraic geometry and algebraic number theory. In algebraic geometry, the problems are formulated in polynomials and solved with the theorems and techniques of commutative algebra. There are many branches of algebra. Algebraic combinatorics is one of the branches of algebra that deals with the objects that base on geometry. Graph theory is a part of combinatorics and there is always a connection of combinatorics and commutative algebra for a long time.
Simplicial complex is a very basic object in combinatorial commutative algebra because of the interaction of algebra and graph theory. Simplicial complex was firstly introduced by R.P. Stanley in 1970. Every branch of mathematics bases on set. Similarly, algebra is also based on set. Set is a collection of well defined and distinct objects. Objects that require no more information and that are different in properties are referred in collection of a set. simplicial complex is a set that satisfies some certain properties. R. P Stanley was the first to introduce the algebraic study of simplicial complexes in the 1970s. In [17] Stanley's work laid the foundation for Combinatorial Commutative Algebra. Following that, in [18] Villarreal and his collaborators provide a new concept in commutative algebra by associating the edge ideal with a simple graph. Following that, many researchers worked to improve the relationship between the algebraic characteristics of edge ideals associated with simple graphs and the graph's combinatorial structure. Due to its links with algebraic geometry and topology, the study of simplicial complexes is one of the most critical topics in combinatorial commutative algebra. In [1,2,3] Anwar was the key inspiration, as he proposed the concept of spanning simplicial complexes in relation to spanning trees and explored its algebraic implications. In [2] Anwar also explain

the concept of spanning simplicial complexes of r-cyclic networks. In [9] Jin Guo later developed the concept of a spanning tree complex and came up with the term spanning forest complex, which refers to a complex associated with a disconnected graph. The algebraic properties of the spanning forest complex remain unexplored. Our goal is to look into some of the algebraic and combinatorial properties of spanning forest complexes that are related to disjoint wheel graphs.

In [1] we start our work uni cyclic graph. In [1] Firstly, we understand uni-cyclic graph then find its spanning trees. After finding the spanning trees of uni cyclic graph we create a set of spanning trees and called that set, set of spanning trees. This set is denoted as $s(G)$. Future, we understand simplicial complex. After understanding simplicial complex then understand the idea of face and facets and learn who to find the dimension of face, facets and simplicial complex see [1,2]. After that we have work on Spanning Simplicial complex and find the dimensional components of Spanning simplicial complex by using different f-vectors see [1,2,3]. Secondly, we have work on r-cyclic graphs see [2]. Find its spanning forests set and spanning forest complex and also computed its dimension by using f-vectors. Further, we move on wheel graph see [5]. In wheel graph we have to learn who to find its spanning trees, understand its some important results like who to find the number of cycles in wheel graph see [5]. At the end our main work on disconnect wheel graphs. We calculate its spanning forests and spanning forests complex also computed its dimension by using f-vector. Then Villarreal and his fellows gave the new concept of edge ideals of simple graphs. This idea was revolutionary idea on which research is going on and some new terms were found which extends this idea. For a connected graph $G = (V, E)$, spanning simplicial complex $\Delta_s(G)$ involves the edges of spanning trees of G and spanning trees contains are trees of G that involves every vertex of G.

By extending this idea, Anwar [1] worked on the edges of spanning trees of simple graphs and introduced the new concept of spanning simplicial complex of simple connected graphs. He also discussed the algebraic properties of these spanning simplicial complexes and found f-vector, h vector and Stanley Reiesner Rings of simple connected graphs. He worked on spanning simplicial complexes of uni-cyclic graphs and r-cyclic graphs and algebraic properties of these graphs. After that J. Guo [8] introduced the new term spanning forest complex. This idea was relevant to disconnected graphs and it was not known. That idea was also the extended idea of simplicial complex and spanning simplicial complex. For a disconnected graph G, spanning forest complex involves the edges of forest complex of G instead of spanning trees. Spanning forest complex H is a subgraph of G whenever intersection of H with every connected component K results in the spanning tree of K i.e. $H \cap K$ is spanning tree of K. Spanning forest complexes of multi-cyclic graphs connected by edges are still unknown. We will study about spanning forest complexes of multi-cyclic graphs connected by edges. We will specially focus on spanning forest complex of ladder graphs and f-vector of these ladder graphs. Our aim is to extend the work and explore algebraic and combinatorial properties of new work.

In section 2, we explain some basic definitions in detail relevant to our work that will help us to understand next chapters and our main work. These terminologies will play a great role in understanding of recent work and main topics. We discuss here some types of graphs, subgraphs, spanning trees and some other important definition.

In section 3, we define and explain simplicial complex Δ and spanning simplicial complex Δ. We will see here how the simple connected graph G is a simplicial complex. We will study about spanning simplicial complexes of non-cyclic graphs, uni-cyclic graphs, multi-cyclic graphs of different graphs and discuss algebraic and combinatorial properties of these spanning simplicial complexes that will help us to go through the next chapter of thesis.

section 4 consists of main work and results that were not known before this work. It includes the spanning forest complexes $\Delta_s(G)$ and f-vector of disconnected union of multi-cyclic graphs connected by edges. Our focus will be on spanning forest complexes and f-vector of wheel graphs with different number of steps k.

1.0.1

2 Preliminaries

Definition 2.1. ***Set***

Set is a collection of objects that are well defined having different properties.

Definition 2.2. ***Graph***

A graph G is 2-tuple set consisting of vertices and edges. Vertex set of a graph G is denoted by $V(G)$ and edge set is denoted by $E(G)$. See fig 1. In mathematical notations we write graph as follows

$$G = (V(G), E(G))$$

Definition 2.3. ***Order of Graph***

No. of vertices in a graph G are called the order of G. $V(G)$ is called size of a graph G. Order of a graph G as shown in fig 1 is 8.

Definition 2.4. ***Size of Graph***

No. of edges in a graph G are called the size of G. $E(G)$ is called size of a graph G. Size of a graph G as shown in fig 1 is 16.

Definition 2.5. ***Loop of a Graph***

If there is a cycle on single vertex then it is called loop. In fig 2, there are four loops on a single vertex v.

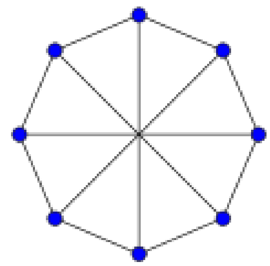

Figure 1: Graph G

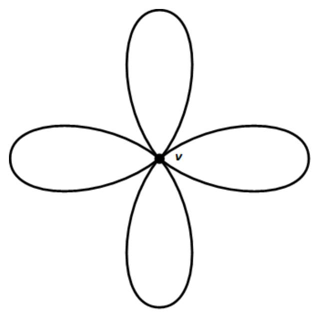

Figure 2: Loop

Definition 2.6. ***Multiple Edges***

If there are two different cycles on two same two vertices then these are called multiple edges. See fig 3

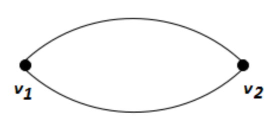

Figure 3: Multiple Edges

Definition 2.7. ***Acyclic graph***

A graph G acyclic if it has no cycle. A graph free of cycles is called acyclic graph. See fig 4.

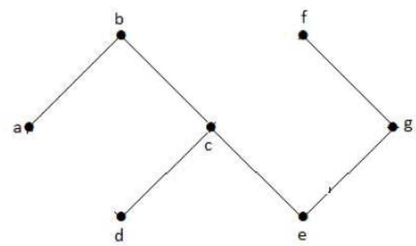

Figure 4: Acyclic Graph

Definition 2.8. ***Connected Graph***

If all the vertices of a graph G are connected by a path, then it is called connected graph. See fig 5

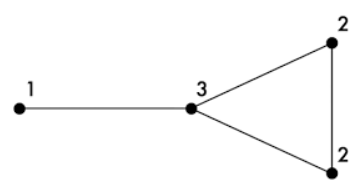

Figure 5: Connected Graph

Definition 2.9. ***Simple Graph***

Simple graph is a graph G with no loops or multiple edges. See fig 6.

Definition 2.10. ***Tree Graph***

Tree is a graph G such that it is connected and acyclic. A graph in which there is a path between every pair of vertices and there is no cycle in that graph is called tree graph. See fig 7

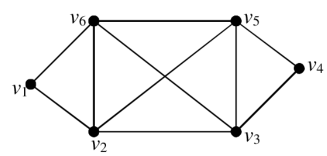

Figure 6: Simple Graph

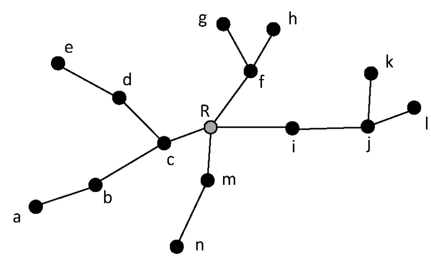

Figure 7: Tree Graph

Definition 2.11. ***Subgraph***

A subgraph H of a graph G is a graph such that its vertices are the vertices of G and its edges are the edges of G.

$$H = (V(H), E(H)), where$$

$V(H) \subset (V(G))$ *and* $E(H) \subset (E(G))$.

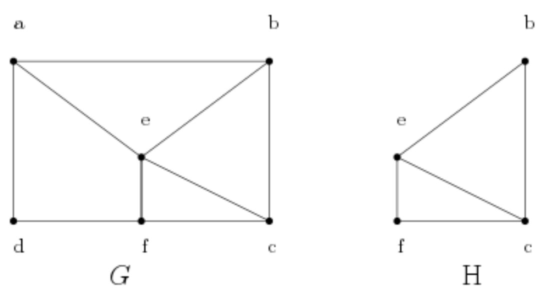

Figure 8: Subgraph H of a graph G

Definition 2.12. ***Spannning Tree***

A spanning tree of a graph G is a subgraph T of G containing all the vertices of G such that it is a tree.i.e. H is a spanning tree if $|V(H)| = |V(G)|$ and it is connected acyclic graph.

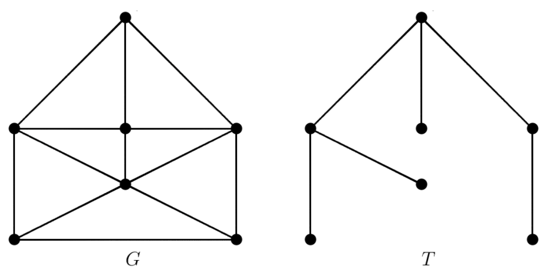

Figure 9: Spanning Tree T of a graph G

3 Simplicial Complex

Definition 3.1. *Simplicial complex is defined on a set say $n = 1, 2, ..., n$. It is denoted as Δ. A simplicial complex Δ is a class of sets defined on a set n such that the following conditions hold.*

- *All the singleton set are in the set Δ. i.e. $\forall\ a \in n$, $\{a\} \in \Delta$*
- *if $X \subset Y$ then $X \in \Delta$, where $Y \in \Delta$*

Definition 3.2. ***(Face/Facets)***

Elements of a simplicial complex Δ are the faces. Faces having maximal dimension are called its facets. Dimension of a face F denoted by $dim(F)$ is defined as

$$dim(F) = |F| - 1$$

where $|F|$ = Length of face.

Definition 3.3. ***(Dimension)***

Dimension of a simplicial complex Δ *denoted by* $dim(\Delta)$ *is the dimension of its maximal facet.*

$$dim(\Delta) = \max(dim(F))$$

.

Definition 3.4. ***(f-vector)***

For a 'd' dimensional simplicial complex Δ*, f-vector is d+1 tuple and it is defined as*

$$f = (f_0, f_1, ...f_d)$$

Example 3.1. *A graph shown in fig 10 is a simplicial complex because it satisfies all the conditions of a simplicial complex.*
Singleton sets $\{v_0\}$,$\{v_1\}$,$\{v_2\}$,$\{v_3\}$,$\{v_4\}$,$\{v_5\} \in \Delta$*. These all are* 0 *dimensional faces. Faces having dimension* 1 *are* $\{v_0, v_1\}$,$\{v_0, v_2\}$,$\{v_0, v_5\}$,$\{v_1, v_2\}$,$\{v_2, v_3\}$$\{v_2, v_4\}$, $\{v_2, v_5\}$,$\{v_3, v_4\}$*. So their subsets are also* $\in \Delta$*. Faces having dimension* 2 *are* $\{v_0, v_1, v_2\}$,$\{v_0, v_2, v_5\}$ *and* $\{v_2, v_3, v_5\}$ *which are also the facets.*
As the dimension of a facet is 2 *becuase* $dim(F) = |F| - 1$*. So* $dim(\Delta) = 2$ *because* $dim(\Delta) = \max dim(F)$*.*
So

$$f = (f_0, f_1, f_2)$$

f_0*,* f_1 *and* f_2 *denotes the number of* 0*,* 1 *and* 2 *dimensional faces respectively, where* $f_0 = 6$*,* $f_1 = 8$*,* $f_2 = 3$*.*

3.1 Spanning Simplicial Complex

Definition 3.5. *Let* G *be a simple connected graph. Spanning simplicial complex of a graph* G *denoted by* $\Delta_s(G)$ *will be a class of sets generated by edges (edge set) of all of the spanning trees of* G*.*

$$\Delta_s(G) = \langle T_i | T_i \in k(G) \rangle$$

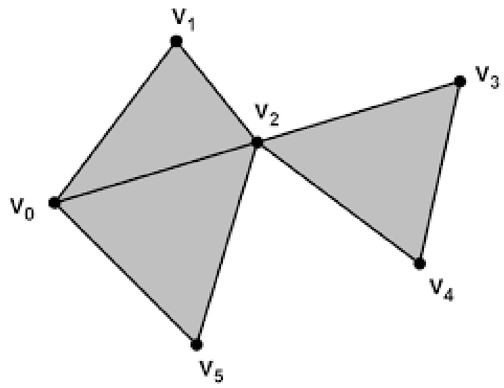

Figure 10: Simplicial Complex

where $k(G)$ is edge set of all spanning trees of G.

Definition 3.6. ***(Uni Cyclic Graph)***

Uni cyclic graph denoted by $U_{n,m}$ is a connected graph having one cycle of length of length m on n number of vertices for $m \leq n$. .

Example 3.2. *Let $s(G) = \{E_1, E_2, E_3, ...E_s\}$ be the edge set of all spanning trees of a cyclic graph C_6 shown in fig 11. It is also denoted by $U_{6,6}$. Spanning trees of C_6 can be obtained by cutting one edge one by one. Its spanning trees are 6. So*

$$\Delta_s(C_6) = <E_i|E_i \in s(C_6)>$$

and

$s(C_6) = \{e_2, e_3, e_4, e_5, e_6\}, \{e_1, e_3, e_4, e_5, e_6\}, \{e_1, e_2, e_4, e_5, e_6\}, \{e_1, e_2, e_3, e_5, e_6\}, \{e_1, e_2, e_3, e_4, e_6\}, \{e_1, e_2, e_3, e_4, e_5\}$.

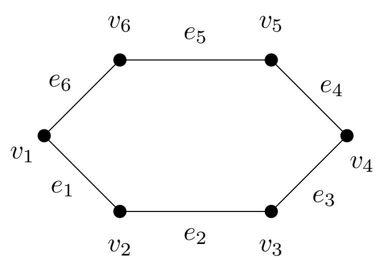

Figure 11: $U_{6,6}$

3.1.1 Characterization of $\Delta_s(C_6)$

Each spanning tree has 5 *number of edges in it. So Every facet of* $\Delta_s(C_6)$ *has a dimension 4 and* $dim(C_6) = 4$.

$$f = (f_0, f_1, f_2, f_3, f_4)$$

Faces of dimension 0 *are* $\{e_1\}$,$\{e_2\}$,$\{e_3\}$,$\{e_4\}$,$\{e_5\}$,$\{e_6\}$. *So* $f_0 = 6$. *Faces of dimension* 1 *are* $\{e_1, e_2\}$,$\{e_1, e_3\}$,$\{e_1, e_4\}$,$\{e_1, e_5\}$,$\{e_1, e_6\}$,$\{e_2, e_3\}$,$\{e_2, e_4\}$,$\{e_2, e_5\}$,$\{e_2, e_6\}$,$\{e_3, e_4\}$, $\{e_3, e_5\}$,$\{e_3, e_6\}$,$\{e_4, e_5\}$,$\{e_4, e_6\}$,$\{e_5, e_6\}$. *So* $f_1 = 15$. *Faces of dimension* 2 *are* $\{e_1, e_2, e_3\}$,$\{e_1, e_2, e_4\}$,$\{e_1, e_2, e_5\}$,$\{e_1, e_2, e_6\}$,$\{e_1, e_3, e_4\}$,$\{e_1, e_3, e_5\}$,$\{e_1, e_3, e_6\}$,$\{e_1, e_4, e_5\}$, $\{e_1, e_4, e_6\}$,$\{e_1, e_5, e_6\}$,$\{e_2, e_3, e_4\}$,$\{e_2, e_3, e_5\}$,$\{e_2, e_3, e_6\}$,$\{e_2, e_4, e_5\}$,$\{e_2, e_4, e_6\}$,$\{e_2, e_5, e_6\}$, $\{e_3, e_4, e_5\}$,$\{e_3, e_4, e_6\}$,$\{e_3, e_5, e_6\}$,$\{e_4, e_5, e_6\}$. *So* $f = 20$. *Faces of dimension* 3 *are* $\{e_1, e_2, e_3, e_4\}$,$\{e_1, e_2, e_3, e_5\}$,$\{e_1, e_2, e_3, e_6\}$,$\{e_1, e_2, e_4, e_5\}$,$\{e_1, e_2, e_4, e_6\}$,$\{e_1, e_2, e_5, e_6\}$,$\{e_1, e_3, e_4, e_5\}$, $\{e_1, e_3, e_4, e_6\}$,$\{e_1, e_3, e_5, e_6\}$,$\{e_1, e_4, e_5, e_6\}$,$\{e_2, e_3, e_4, e_5\}$,$\{e_2, e_3, e_4, e_6\}$,$\{e_2, e_3, e_5, e_6\}$,$\{e_2, e_4, e_5, e_5\}$, $\{e_3, e_4, e_5, e_6\}$. *So* $f_3 = 15$. *Faces of dimension* 4 *are the facets of* $\Delta_s(C_6)$. *So* $f_4 = 6$. *So f-vector of* $\Delta_s(C_6)$ *is*

$$f = (6, 15, 20, 15, 6)$$

Theorem 3.1. *Let* $\Delta_s(U_{n,m})$ *be a spanning simplicial complex of a uni cyclic* *graph. Then* $dim(U_{n,m}) = n-2$ *and* $f = (f_0, f_1, ..., f_{n-2})$ *and*

$$f_i = \begin{cases} \binom{n}{i+1}, \text{ when } i \leq m-2 \\ \binom{n}{i+1} - \binom{n-m}{i+1-m}, \text{ when } m-2 < i \leq n-2 \end{cases}$$

Definition 3.7. ***(Multi cyclic Graph)***

Multi cyclic graph denoted by $G_{n,r}$ *is a connected graph having* r *cycles and* n *number of edges such no two cycles have a common edge.*

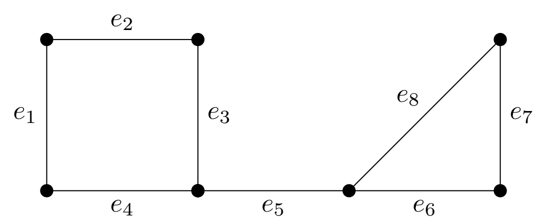

Figure 12: $G_{8,2}$

Example 3.3. *Let us find spanning simplicial complex of* $G_{8,2}$ *having* 2 *cycles and* 8 *edges as shown in fig 12. Spanning trees of this graph can be obtained by cutting one edge from its cycle. So every facet of* $\Delta_s(G_{8,2})$ *will contain* 6 *edges. So* $\Delta_s(G_{8,2}) = \langle\{e_2, e_3, e_4, e_5, e_6, e_7\}, \{e_2, e_3, e_4, e_5, e_6, e_8\}, \{e_2, e_3, e_4, e_5, e_7, e_8\}$ $, \{e_1, e_3, e_4, e_5, e_6, e_7\}, \{e_1, e_3, e_4, e_5, e_6, e_8\}, \{e_1, e_3, e_4, e_5, e_7, e_8\}, \{e_1, e_2, e_3, e_5, e_6, e_7\},$ $\{e_1, e_2, e_3, e_5, e_6, e_8\}, \{e_1, e_2, e_3, e_5, e_7, e_8\}, \{e_1, e_2, e_4, e_5, e_6, e_7\}, \{e_1, e_2, e_4, e_5, e_6, e_8\},$ $\{e_1, e_2, e_4, e_5, e_7, e_8\}\rangle$.

Theorem 3.2. *Let* $\Delta_s(G_{n,r})$ *be a connected graph having* r *cycles of lengths say* $m_1 \leq m_2 \leq m_3, ..., m_r$ *having* n *number of edges. Then* $dim(\Delta_s(G_{n,r})) = n-r-1$. *And* $f = (f_0, f_1, ..., f_{n-r-1})$ *with*

$$f_i = \binom{n}{i+1} + \sum_{v=1}^{r}(-1)^v \left[\sum_{(j_1,j_2,...,j_r)=1, j_1 \neq j_2 ... \neq j_r}^{r} \binom{n - \sum_{s=1}^{v} m_{j_s}}{i+1-\sum_{s=1}^{v} m_{j_s}} \right]$$

4 Spanning Forest Complex

Definition 4.1. *Let G be a disconnected graph and A be the connected component of G. A subgraph B of G is called a spanning forest of G if intersection of B with every connected component A is a spanning tree of A. i.e. Spanning forest complex of a graph G denoted by $\Delta_s(G)$ will be a class of sets generated by edges (edge set) of all of the spanning forests of G.*

$$\Delta_s(G) = \langle T_i | T_i \in k(G) \rangle$$

where $k(G)$ is edge set of all spanning forests of G.

4.1 Spanning Forest Complexes of tC_n

Definition 4.2. *tC_n is a disconnected graph having t copies of cyclic graphs C_n i.e. $tC_n = \cup_{i=1}^{n} C_i$. C_n is a cyclic graph of length n on n number of vertices as shown in figure 13.*

Theorem 4.1. *Let tC_n be a disconnected graph having disjoint union of t copies of c_n. Then number of edges of tC_n will be tn and number of spanning forests will be n^t. Also $dim(\Delta_s(tC_n)) = t(n-1) - 1$.*

Example 4.1. *A graph shown in figure 13 is $3C_4$ having 3 copies of cyclic graph C_4. So total number of edges are $(3)(4) = 12$ and total number of spanning forests are $4^3 = 64$.*

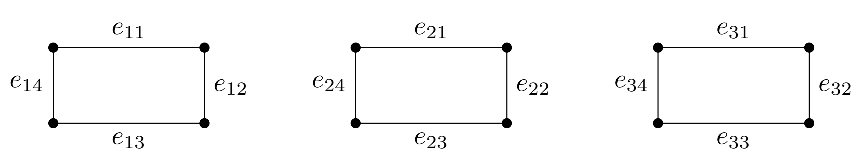

Figure 13: $3C_4$

Every facet of $\Delta_s(3C_4)$ will contain 3 elements from each copy because every spanning forest has 9 number of edges in it. So $dim(\Delta_s(3C_4)) = 3(4-1)-1 = 8$ and $f = (f_0, f_1, ...f_8)$.

Theorem 4.2. *Let $\Delta_s(2C_n)$ be a disconnected graph having 2 copies of cyclic graph C_n of length n. Then $dim\Delta_s(2C_n) = 2n-3$ and*

$$f_i = \binom{2n}{i+1} - 2\binom{n}{i+1-n}$$

Theorem 4.3. *Let $\Delta_s(3C_n)$ be a disconnected graph having 3 copies of cyclic graph C_n of length n. Then $dim\Delta_s(3C_n) = 3n-4$ and*

$$f_i = \binom{3n}{i+1} - 3\binom{2n}{i+1-n} + 3\binom{n}{i+1-2n}$$

Theorem 4.4. *Let $\Delta_s(4C_n)$ be a disconnected graph having 4 copies of cyclic graph C_n of length n. Then $dim\Delta_s(4C_n) = 4n-5$ and*

$$f_i = \binom{4n}{i+1} - 4\binom{3n}{i+1-n} + 6\binom{2n}{i+1-2n} - 4\binom{n}{i+1-3n}$$

Theorem 4.5. *Let $\Delta_s(tC_n)$ be a disconnected graph having t copies of cyclic graph C_n of length n. Then $dim\Delta_s(tC_n) = t(n-1)-1$ and $f = (f_0, f_1, ..., f_{t(n-1)-1})$ where*

$$f_i = \binom{tn}{i+1} + \sum_{v=1}^{t-1}(-1)^v\binom{t}{v}\binom{(t-v)n}{i+1-vn}$$

Theorem 4.6. *In [2] Let $\Delta_s(2C_n)$ represents the Spanning Smplicial Complex(n represents the same length)of 2 cycles graph then the $dim(\Delta_s(2C_n))$=$2n-3$ with* f-vector f_i= $\binom{2n}{i+1}$- $2\binom{n}{i+1-n}$ *when* $0 \leq i \leq 2n-3$

Theorem 4.7. *In [2] Let $\Delta_s(tC_n)$ represents the spanning forest complex of t-cycle graph tC_n,then the $dim(\Delta_s(tC_n))$ =$t(n-1)-1$ with*

$$\text{f}(\Delta_s(tC_n)) = (\text{f}_0, \text{f}_1, ..., \text{f}_{t(n-1)-1})$$

$$\text{f}_i = \binom{tn}{i+1} + \sum_{k=1}^{t-1}(-1)^k\binom{t}{k}\binom{(t-k)n}{i+1-kn}$$

4.2 Spanning Forest Complex of $tC_{n,k}$

4.2.1 Spanning forest complex of $2C_{4,2}$

This graph is the result of the union of two copies of a two-step ladder graph. 2 steps of C_4 are connected by edges and further it is union of two copies of $C_{4,2}$ as shown in Fig.1.2. As this disconnected graph contains two copies of $C_{4,2}$ see [6].

4.2.2 Characterization of $\Delta_s(2C_{4,2})$

The union of two copies of a two-step ladder graph yields this graph. Spanning forest of $2C_{4,2}$ is the subgraph of $2C_{4,2}$

4.2.3 Dimension of $\Delta_s(2C_{4,2})$

As the spanning tree is connected acyclic graph. So we obtain spanning forests of $2C_{4,2}$ by cutting down one edge from each cycle to make it cycle free and connected. Each copy contains 7 edges and two cycles. So, we cut 2 edges from 7 edges and do it for 2 copies. So, each spanning forest of $2C_{4,2}$ will contain $2(7) - 2(2) = 10$ elements in it.

4.2.4 *f*-vector of $\Delta_s(2C_{4,2})$

since

$$\dim(2C_{4,2}) = (14 - 4) - 1 = 9$$

$$f = (f_0, f_1, \ldots, f_9)$$

$$\Delta_s(2C_{4,2}) = \langle E(T_i) | T_i \in s(G) \rangle$$

Theorem 4.8. *Let $\Delta_s(2C_{4,2})$ be a spanning forest complex and e represents the total number of edges. The dim $\Delta_s(2C_{4,2})$ is 9 and* $\mathrm{f} = (\mathrm{f}_0, \mathrm{f}_1, \ldots, \mathrm{f}_9)$ *where*

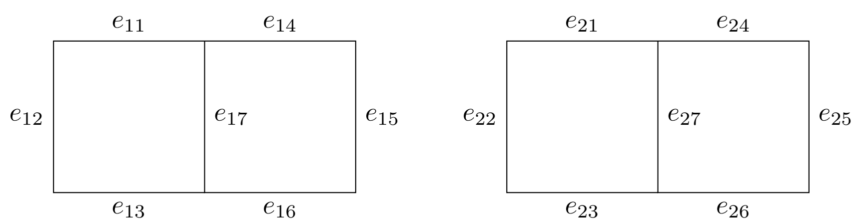

Figure 14: $2C_{4,2}$.

$e = 2(2(4) - 1) = 14$

$$f_i = \begin{cases} \binom{e}{i+1} & when\ i = 0, 1, 2 \\ \binom{e}{i+1} - \binom{4}{1} & when\ i = 3 \\ \binom{e}{i+1} - \binom{4}{1}[\binom{3}{1} + \binom{7}{1}] & when\ i = 4 \\ 2\binom{7}{5}\binom{7}{1} + 2\binom{7}{4}\binom{7}{2} + \binom{7}{3}\binom{7}{3} - \binom{4}{1}[\binom{3}{1}\binom{7}{1} + \binom{7}{2}] & when\ i = 5 \\ 2\binom{7}{5}\binom{7}{2} + 2\binom{7}{4}\binom{7}{3} - \binom{4}{1}[\binom{3}{1}\binom{7}{2} + \binom{7}{3}] & when\ i = 6 \\ 2\binom{7}{5}\binom{7}{3} + \binom{7}{4}\binom{7}{4} - \binom{4}{1}[\binom{3}{1}\binom{7}{3} + \binom{7}{4}] + [\binom{4}{2} - 2] & when\ i = 7 \\ 2\binom{7}{5}\binom{7}{4} - \binom{4}{1}[\binom{3}{1}\binom{7}{4} + \binom{7}{5}] + [\binom{4}{2} - 2][2\binom{3}{1}] & when\ i = 8 \\ \binom{7}{5}\binom{7}{5} - \binom{4}{1}[\binom{3}{1}\binom{7}{5}] + [\binom{4}{2} - 2][\binom{3}{1}\binom{3}{1}] & when\ i = 9 \end{cases}$$

Proof. As we know that $2C_{4,2}$ is a graph having 2 copies of $C_{4,2}$ is 2 step ladder graph of a cycle C_4 and these step are connected by edges. As spanning forest is actually a subgraph associated to union of spanning trees of these 2 copies. As each copy having 2 steps of C_4 and one edge is common. We need to cut 2 edges from each copy to make its single spanning tree. As we have two copies so we have to cut 4 edges from total edges to make its spanning forest. Total edges of $2C_{4,2}$ are 14. Every spanning forest or facet will contain 10 edges. So by definition of dimension of spanning forest

$$\dim(2C_{4,2}) = \max(\dim(F))$$

Where F is a facet. Since each facet has dimension 9 because dim(F)=$|F|$-1

$$\dim(2C_{4,2})=9$$

and by definition of $f-vector$

$$f=(f_0,f_1,...,f_9)$$

Now we move on f-vector calculation of f_0,f_1,f_2 these can be computed very easily because these $f-vector$ no cycles appears in these sets because the length of smallest cycle is 4.

For $i = 3$, first of all we find all sets having 4 elements in it and they can be computed by using combination$\binom{14}{4}$ These sets also include the cycles of length 4. Since these cycles of length 4 are 4. So by using Inclusion and Exclusion Principle, we cut these cycles from the total number of sets and we get the result. For $i = 4$, We find all the sets having 5 number of elements in it we count them using mathematical expression as$\binom{14}{5}$. They also include the sets having cycles of length 4. Since these cycles are 4. By using Inclusion Exclusion Principle we cut these cycles of length 4 from the sets having length 5. So, 1 blank spot is there for variation of set. This blank spot can be the element from 1st copy of $2C_4$ except from element making the cycle and these elements will be 3 in number. So we can write it in mathematical as$\binom{3}{1}$. Here is also the possibility of filling this blank spot by the 7 elements of the 2nd copy. So we can write it as $\binom{7}{1}$. By combining above possibility we get the result.

For $i = 5$, We find all the sets having 6 number of elements in it with the condition that maximum of 5 edges we can take from one copy irrespective of the choice of a copy. So will fill the blank sport of a set having length 6 and write it in mathematical from as $2\binom{7}{5}\binom{7}{1}$+$2\binom{7}{4}\binom{7}{2}$. Here also include the sets having cycles of length 4. Since these cycles are 4. But there is a possibility of variation of sets including these 4 cycles of length 4. By using Inclusion Exclusion Principle we cut these cycles of length 4 from the sets having length

6. So 2 blank spots are there for variation os sets. These blank spots are filled by the 1 element from 1st copy of $2C_{4,2}$ apart from element making the cycle and these are 3 in number and 2nd spot can be filled by any element of 2nd copy of$2C_{4,2}$ and these are 7 in numbers. So, we write this in mathematical from as $\binom{3}{1}\binom{7}{1}$. Also here is the possibility of filling these two spots by the 2nd copy of $2C_{4,2}$. So it will be written as$\binom{7}{2}$. By above possibilities we get the result.

For $i = 6$, we find all the sets having 7 number of elements in it with the condition that maximum of 5 edges we can take from one copy irrespective of the choice of a copy. So we fill the blank spots of a set having length 7 and write it in mathematical from as $2\binom{7}{5}\binom{7}{2}+2\binom{7}{4}\binom{7}{3}$. They also include the sets having cycles of length 4 we cut these cycles by using Inclusion Exclusion Principle from the sets having length 7. So here 3 blank spots are for the variation of sets. These blank spots can be filled by the 1 element from 1st copy of $2C_{4,2}$ apart from element making the cycle and these are 3 in numbers and other two spots can be filled by any two elements of 2nd copy of $2C_{4,2}$ and these are 7 in numbers. So, we write this in mathematically from as $\binom{3}{1}\binom{7}{2}$. Also here is the possibility of filling these spots by the 2nd copy of $2C_{4,2}$,so it will be written as $\binom{7}{3}$.

For $i = 7$, We find all the sets having 8 number of elements in it with the condition that maximum of 5 edges we can take from one copy irrespective of the choice of a copy. So,we fill the blank spots of a set having length 8 and write it in mathematically from as $2\binom{7}{5}\binom{7}{3}+\binom{7}{4}\binom{7}{4}$. They also include the sets having cycles of length 4.Since these cycles are 4 in number. By using Inclusion Exclusion Principle, we cut these cycles of length 4 from the sets having length 8. So blank spots are there for variation of sets. These blank spots can be filled by the 1 element from 1st copy of $2C_{4,2}$ apart from element making the cycle and these are 3 in numbers and other four spots can be filled by any four elements of 2nd copy of$2C_{4,2}$ and these are 7 in numbers. So we write this in mathematical form as $\binom{3}{1}\binom{7}{3}$. Also here is the possibility of filling these four spots by the 2nd copy of $2C_{4,2}$, So it will be Written as$\binom{7}{4}$. Now here is also the possibility of two cycles of length 8 and these cycles are 4 in numbers which can be include in the sets having length 8 so, we add these sets and get the result.

For $i = 8$, We find all sets having 9 number of elements in it with the condition that maximum of 5 edges we can take from one copy irrespective of the choice of a copy. So we filled the blank spots of a set having length 9 and write in mathematically from as $2\binom{7}{5}\binom{7}{4}$. They also include the sets having cycles of length 4. Since these cycles are 4 in number. By using Inclusion Exclusion Principle we cut these cycles of length 4 from the sets having length 9. So 5 blank sport are there for variation of sets. These blank spots can be filled by the 1 element from 1st copy of $2C_{4,2}$ apart from element making the cycle and these are 3 in number and other five spots can be filled by any five elements of 2nd copy of $2C_{4,2}$ and these are 7 in numbers. So, we write this in mathematics form as$\binom{3}{1}\binom{7}{4}$. Also here is the possibility of filling these five spots by the 2nd copy of $2C_{4,2}$. So it will be written as $\binom{7}{5}$. Now here is also the possibility of two cycles of length 8 and these cycles are 4 in numbers which can be included in the sets having length 8 and still one spot is there to be filled up. So we add these sets and spot can be filled up either by 1st copy or can be filled up by 2nd copy of $2C_{4,2}$ and we write it as $2\binom{3}{1}$.

For $i = 9$, We find all the sets having 10 number of elements and we write it in mathematically from as $\binom{7}{5}\binom{7}{5}$. They also include the sets having cycles of length 4. By using Inclusion Exclusion Principle, we cut these cycles of length 4 from the sets having length 10. So, 6 blank spots are there. These blank spots can be filled by the 1 element from 1st copy of $2C_{4,2}$ and other five spots can be filled by any five elements of 2nd copy of $2C_{4,2}$. So,we write this mathematically form as$\binom{3}{1}\binom{7}{5}$. Now here also the possibility of two cycles of length 8 and these cycles are 4 in numbers which can be included in the sets having length 8 and still one spot is there to be filled up. So, we add these sets. It will be filled by 1st copy 2nd copy of $2C_{4,2}$ and write it as $\binom{3}{1}\binom{3}{1}$. □

Definition 4.3. ***Wheel Graph***

Let G a uni-cyclic graph, if we insert a vertex inside the graph G and joint this vertex to every other vertices of G, then the obtaining new Graph is called Wheel graph see [5]. It is denoted by $W_n (n \geq 4)$*.* W_5 *is shown in figure 15 A wheel graph also involves cycles* $C_1, C_2, ..., C_t$*. The letter t denotes the number of consecutive cycles in the wheel graph, where* $t = n - 1$*. The edge set of* W_n *is as follows:* $E = \{e_{11}, e_{21}, e_{21}, e_{22}, ..., e_{m1}, e_{m2}\}$.

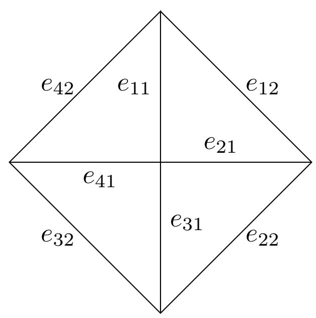

Figure 15: W_5.

4.3 Characterizations of Spanning Tree of Wheel Graph

In [5] The spanning tree of wheel graph W_n is obtained by pull out t edges from t cycles. Cutting down is the name of the procedure for obtaining spanning trees. we get spanning trees of wheel graph W_n by using the following methods see [6].

- Remove any unusual edges from each cycle.

- Remove all of the similar edges from subsequent cycles, but only one from the larger cycle.

- Remove some of the most common and rare edges from various cycles.

Theorem 4.9. *Let G be a graph, if G is wheel graph W_n then it has a total $2^m - 1$ cycles.*

Proof. Let G be a graph and G be a wheel graph W_n. Here n represents the number of vertices appearing in the wheel graph. m represents the number of consecutive cycles in a wheel graph W_n, that is always equal to $m = n - 1$. In a wheel graph C_{i1} and C_{i2} are the shared edges between the cycles. The cycles of wheel graph represented as C_1,C_2,...,C_m. While C_1,C_2,...,C_m represents the number of successive cycles in wheel graph W_n. We obtained $C_{1,2}, C_{2,3}, ..., C_{m-1,m}, C_{m,1}$ when one of the common edge is removed. We obtained $C_{1,2,3}, C_{2,3,4}, C_{m-1,m,1}, C_{m,1,2}$. When two of the common edges are removed and continuous in this way we get $C_{1,2,3,...,m}, C_{2,3,4,...,m,1}, C_{m,1,2,...,m-1}$. A quick count indicates that this number is $2^m - 1$ as in [5] □

4.4 Spanning Forest complex of tW_n

4.4.1 Spanning Forest Complex of $2W_5$

This Graph is union of two copies of two wheel graphs. In wheel graph more than one cycles. These cycles are convective cycles connected by with common edges. These edges represent as C_{i1} and C_{i2}. The spanning threes of wheel graph can be obtained as :Remove any unusual edges from each cycle.Remove all of the similar edges from subsequent cycles, but only one from the larger cycle. Remove some of the most common and rare edges from various cycles.

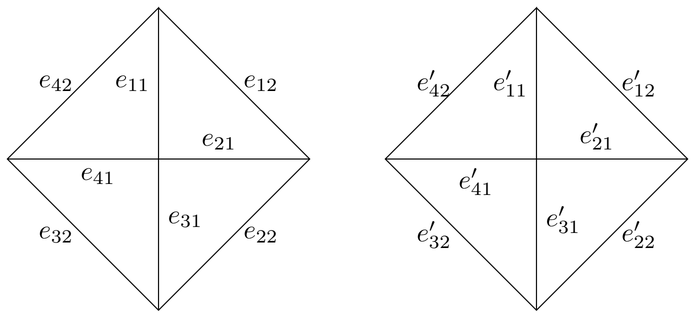

Figure 16: $2C_{4,2}$.

4.4.2 Dimension of $2W_5$

As the spanning tree is connected acyclic graph. So we obtain spanning forests of $2W_5$ by cutting down four edges from each cycles to make it cycle free and connected. Each copy contained 8 edges and more than one cycles. So, we cut 4 edges from 8 edges and do it for 2 copies. So, each spanning forest of $2W_5$ will contained $2(8) - 2(4) = 8$ elements in it.

4.4.3 *f* vector $2W_5$

Since

$$dim(2W_5) = (16 - 8) - 1 = 7$$

$$f = (f_0, f_1, \ldots, f_7)$$

Theorem 4.10. *Let $\Delta_s(2W_5)$ be a Spanning Forest complex where e be the total number of edges. Then dim($\Delta_s(2W_5)$)is 7 and* f=(f_0,f_1,....f_7) *where* $e = 2(10-2) = 16$ *and*

$$f_i = \begin{cases} \binom{e}{i+1} - \binom{p}{1}[\binom{e/2-3}{i+1-3} + \binom{e/2}{i+1-3}] - \binom{q}{1}[\binom{e/2-4}{i+1-4}] \quad \text{when } 0 \leq i \leq 3 \\ 2\binom{e/2}{4}\binom{e/2}{i-3} + \binom{e/2}{3}\binom{e/2}{i-2} - \binom{p}{1}[\binom{e/2-3}{1}\binom{e/2}{i-3} + \binom{e/2}{i-2}] - \binom{q}{1}[\binom{e/2}{i-3}] \quad \text{when } i = 4 \\ 2\binom{e/2}{4}\binom{e/2}{2} + \binom{e/2}{3}\binom{e/2}{3} - \binom{p}{1}[\binom{e/2-3}{1}\binom{e/2}{2} + \binom{e/2}{3}] - \binom{q}{1}[\binom{e/2}{2}] \\ +[\binom{e/2}{2} - 12] \quad \text{when } i = 5 \\ 2\binom{e/2}{4}\binom{e/2}{3} - \binom{p}{1}[\binom{e/2-3}{1}\binom{e/2}{3} + \binom{e/2}{4})] - \binom{q}{1}[\binom{e/2}{3}] + [\binom{e/2}{2} - 12]\binom{e/2-3}{1} \\ +2[\binom{e/2+1}{2} - 16] \quad \text{when } i = 6 \\ \binom{e/2}{4}\binom{e/2}{4} - \binom{p}{1}[\binom{e/2-3}{1}\binom{e/2}{4}] - [\binom{q}{1}]\binom{e/2}{4} + [\binom{e/2}{2} - 12][\binom{e/2-3}{1}\binom{e/2-3}{1}] \\ +2[\binom{e/2+1}{2} - 16]\binom{e/2-3}{1} + [\binom{10}{2} - 20] \quad \text{when } i = 7 \end{cases}$$

Proof. The graph $2W_n$ having 2 copies of wheel graph. Each copies having 8 edges. so, that's why total number of edges are 16. For getting spanning tree,we remove 4 edges from each copies. As we have two copies so we cut total 8 edges to make it spanning forest. As we cut total 8 edges so left behind 8 edges. Every spanning Forests contains 8 edges. So, according to the definition of dimension of spanning forest

$$\dim(\Delta_s(2W_n)=\max(\dim(F))$$

where F represents facet and dimension of each is 7 determine as dim(F)= $|F|$-1. so,

$$\dim(\Delta_s\ (2W_n))=7$$

and according to the definition of 8 f-vector

$$f=(f_0,f_1,...,f_7)$$

Now we calculate f-vectors .f_0 andf_1 are very easily calculated because these pairs of sets cannot contains any cyclic as the smallest length of cyclic is 3.

For i=2 first we calculate all the sets having 3 elements by using combination i.e. $\binom{16}{3}$. These sets also include the cycles of length 3. Since these cycles are 8. So , by using Inclusion Exclusion Principle we remove these cycles from the total number of sets and we get result.

For i=3 Here also we find the set having 4 elements by using combination$\binom{16}{4}$. In these sets also include the cycles of length of 3 and 4. cycles length of 3 are 8 and length of 4 are 10. By using Inclusion Exclusion Principle we cut these cycles from the set having elements 4. Here for cycle length 1 blank spot for variation of sets. This blank spot can be the element of first copy of wheel graph or the element of second copy of wheel graph, If the element of first copy then we can write it in mathematical expression as$\binom{5}{1}$ and if the element of 2nd copy then it mathematically written as$\binom{8}{1}$. After combining above mentions we get the result.

For i=4, We identify all the sets with 5 elements,with the restriction that we can only pick 4 edges from one, whatever of the copy we choose. So we fill the blank spots of a set having length 5 and write it in mathematically from as $2\binom{8}{4}\binom{8}{1}+2\binom{8}{3}\binom{8}{2}$. They also offer a set with 3 and 4 cycle lengths. These are 8 and 10,respectively. However, there is the option of altering the sets to include these 8 and 10 cycles. By using Inclusion Exclusion Principle, we cut these cycles of length 3 and 4. Here 2 blank sport for variation of sets. It may possible 1 blank sport in the 1st copy of wheel graph $2W_5$ and 1 sport can be filled by any element of 2nd copy and these are 8 in numbers so we write in mathematical as $\binom{5}{1}\binom{8}{1}$. Here also possible 2 spots comes to 2nd copy of wheel graph. In mathematically we write it as$\binom{8}{2}$. Since, cycles of length 4 also appears here. So for cycles of length 4, 1 blank spot for variation of sets. This blank spot can be the element of 1st copy or the element of 2nd copy so by mathematically we can write it as$\binom{4}{1}$+$\binom{8}{1}$. So if we combine all the possibilities above mention we get the result.

For i=5, we find all the sets having 6 elements with the condition that maximum 4 edges pick up from any copy. we can write mathematically form as $2\binom{8}{4}\binom{8}{2}$+$\binom{8}{3}\binom{8}{3}$. Here also includes the sets having cyclic length 3 and 4. These are 8 and 10. We cut these cycles by using Inclusion Exclusion Principle. For cycles of length 3, there are 3 blank sports for variations of sets, but for cycles of length 4, blank sports for variations for sets. For cycles of length 3, these blank spots can be filled as, filled 1 blank sport in first copy and 2 sport from 2nd copy. so can write mathematically as$\binom{5}{1}\binom{8}{2}$. Here also possible filled these three blank spots from 2nd copy so it will be written as $\binom{8}{3}$

For cycles of length 4, there are 2 blank spots for variation of sets. we can filled these blank spots as filled blank sport in 1st copy and 1 blank spots in 2nd copy, also possible we filled these spots by the 2nd copy of wheel graph. So by mathematically we write it as $\binom{4}{1}\binom{8}{1}$+$\binom{8}{2}$. Now here is also the possibility of two cycles of length 6 and these are 8 in numbers which can be include in the set having length 6. So we add these sets and get result.

For $i = 6$, we find all the sets having 7 number of elements in it with the condition that maximum of 4 edges we can take from one copy irrespective of the choice of a copy. So we fill the blank spots of a set having length 7 and write it in mathematically form as $2\binom{8}{4}\binom{8}{3}$. Here also the cycles of length 3 and 4, we cut these cycles by using Inclusion Exclusion Principle. For cycles of length 3 there are 4 blank sports that can be filled as filled 1 spots from first copy and 3 from 2nd copy or filled all 4 spots from 2nd copy of wheel graph. So, we can write it mathematically as $\binom{5}{1}\binom{8}{3}$+$\binom{8}{4}$. For cycles of length 4, there are 3 blank spots for the variations of sets. These 3 blank spots filled as take 1 spot in 1st copy and 2 in 2nd copy or take all three spots in 2nd copy. So, by mathematically we can write it as $\binom{4}{1}\binom{8}{2}$+$\binom{8}{3}$. Now here is also the possibility of two cycles of length 6 and 7 these are 8 and 10. For 6 length still one blank spot is there to be filled up. so we add these sets and the spot can be filled up either by 1st copy or can be filled up by 2nd copy of $2W_5$ and we can write it as $[\binom{8}{2} - 12]\binom{5}{1}$ For$i = 7$, we find all the sets having 8 elements in it with condition that maximum of 4 edges we can take from one copy irrespective of the choice of a copy. we find all the sets having 8 of elements by using mathematical expression $\binom{8}{4}\binom{8}{4}$. They also include the sets having cycles of length 3 and 4. By Inclusion Exclusion Principal, we cut these cycles of length 3 and 4 from the sets having length 8. So here 5 blank spots for length of cycle 3 and 4 blank spots for cycle of length 4. For cycle of length 3 these spots are filled $\binom{5}{1}\binom{8}{4}$ and for length 4 it can be filled as $\binom{8}{4}$. Now here it is possible that cycle of 4 length include that are 10 and also include cycle of length 3 also include that 8. So we add these possibilities and get the final result. □

5 Spanning Forest Complex of $3W_5$

This graph is union of three copies of wheel graph W_5, connected by edges and further it is union of three copies. as shown in Fig.1.5. As this disconnected graph contains three copies of W_5, And single copies of W_5 has 45 spanning trees.

5.1 Characterization of $\Delta_s(3W_5)$

This graph is union of 3 copies of wheel graph as shown in Fig.1.5. Spanning forest of $3W_5$ is the subgraph of $3W_5$.Spanning forest of $3W_5$ is actually the union of three copies of spanning trees of W_5. As single copy contains 45 spanning trees, so total number of spanning forests of $3W_5$ are $45^3 = 91125$. As the spanning tree is connected acyclic graph. So we obtain spanning forests of $3W_5$ by cutting down some of the common and non-common edges to make it cycle *free and connected. Each copy contains 8 edges. Each copy contains 8 edges, we get spanning forest by cut down 4 edges from 8 edges and do it for 3 copies. So, each spanning forest of* $3W_5$ *will contain* $3(8) - 3(4) = 12$ *element in it. So*

$$dim(\Delta_s(3W_5))=(24-12)-1=11$$

5.1.1 *f*-vector $(3W_5)$

since $dim(\Delta_s(3W_5))=11$

$$\mathrm{f}=(\mathrm{f}_0,\mathrm{f}_1,\ldots,\mathrm{f}_{11})$$

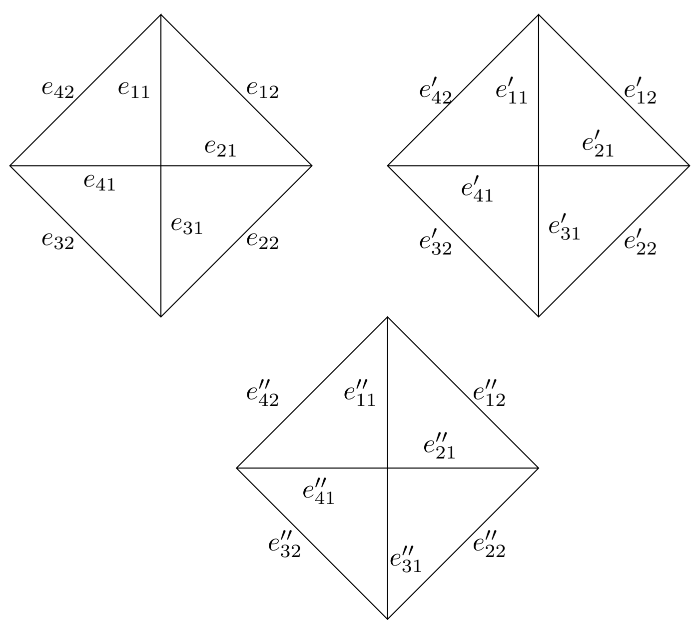

Figure 17: $3W_5$.

Theorem 5.1. *Let* $\Delta_s(3W_5)$ *be a spanning forest complex and e be the total number of edges. Then* $dim(\Delta_s(3W_5))$ *is 11 and* $\mathrm{f}=(\mathrm{f}_0,\mathrm{f}_1,\ldots,\mathrm{f}_{11})$ *where e=24* *and p represents the cycles of length 3 and q represents the cycle of length 4.*

$$f_i = \begin{cases} \binom{e}{i+1} \quad \textit{when } i = 0,1 \\ \binom{e}{i+1} - \binom{p}{1} \quad \textit{when } i = 2 \\ \binom{e}{i+1} - \binom{p}{1}[\binom{e/3-3}{1} + 2\binom{e/3}{1}] - \binom{q}{1} \quad \textit{when } i = 3 \\ 6\binom{e/3}{4}\binom{e/3}{1} + 3\binom{e/3}{3}\binom{e/3}{2} + 3\binom{e/3}{3}\binom{e/3}{1}\binom{e/3}{1} + 6\binom{e/3}{2}\binom{e/3}{2}\binom{e/3}{1} - \binom{p}{1}[2\binom{e/3-3}{1}\binom{e/3}{1}) \\ +2\binom{e/3}{2} + \binom{e/3}{1}\binom{e/3}{1}] - \binom{q}{1}[\binom{e/3-4}{1} + 2\binom{e/3}{1}] \quad \textit{when } i = 4 \\ 3\binom{e/3}{4}\binom{e/3}{2} + \binom{e/3}{3}\binom{e/3}{3} + 3\binom{e/3}{4}\binom{e/3}{1}\binom{e/3}{1} + 6\binom{e/3}{3}\binom{e/3}{2}\binom{e/3}{1} + \binom{e/3}{2}\binom{e/3}{2}\binom{e/3}{2} \\ -\binom{p}{1}[2\binom{e/3-3}{1}\binom{e/3}{2} + 2\binom{e/3}{2}\binom{e/3}{1} + \binom{e/3}{3} + \binom{e/3-3}{1}\binom{e/3}{1}\binom{e/3}{1}] - \binom{q}{1}[2\binom{e/3}{2} + \binom{e/3}{1}\binom{e/3}{1}] \\ +[\binom{e/2}{2} - 18] \quad \textit{when } i = 5 \\ 6\binom{e/3}{4}\binom{e/3}{3} + 3\binom{e/3}{4}\binom{e/3}{2}\binom{e/3}{1} + 3\binom{e/3}{3}\binom{e/3}{2}\binom{e/3}{2} - \binom{p}{1}[2\binom{e/3-3}{1}\binom{e/3}{3} + 2\binom{e/3}{4} + \binom{e/3}{2}\binom{e/3}{2} \\ +2\binom{e/3}{3}\binom{e/3}{1} + 2\binom{e/3-3}{1}\binom{e/3}{2}\binom{e/3}{1}] - \binom{q}{1}[2\binom{e/3}{2}\binom{e/3}{1} + 2\binom{e/3}{3}] + [\binom{e/2}{2} - 18][\binom{e/3-3}{1} \\ +2\binom{e/3}{1} + 2[\binom{13}{2} - 22] \quad \textit{when } i = 6 \\ 3\binom{e/3}{4}\binom{e/3}{4} + 6\binom{e/3}{4}\binom{e/3}{2}\binom{e/3}{3}) + 6\binom{e/3}{4}\binom{e/3}{3}\binom{e/3}{1} + 3\binom{e/3}{3}\binom{e/3}{3}\binom{e/3}{2} + 3\binom{e/3}{4}\binom{e/3}{2}\binom{e/3}{2}) \\ -\binom{p}{1}[2\binom{e/3-3}{1}\binom{e/3}{4} + 2\binom{e/3}{4}\binom{e/3}{1} + 2\binom{e/3}{3}\binom{e/3}{2}] - \binom{q}{1}[\binom{e/3}{2}\binom{e/3}{2} + 2\binom{e/3}{4} + 2\binom{e/3}{3}\binom{8}{1}] \\ +[\binom{e/2}{2} - 18][2\binom{e/3-3}{1}\binom{e/3}{1} + 2\binom{e/3}{2} + \binom{e/3}{1}\binom{e/3}{1}] + [\binom{15}{2}) - 30] + 2[\binom{13}{2} - 22][\binom{e/3-3}{1} \\ +2\binom{e/3}{1}] \quad \textit{when } i = 7 \\ 6\binom{e/3}{4}\binom{e/3}{4}\binom{e/3}{1} + 6\binom{e/3}{4}\binom{e/3}{3}\binom{e/3}{2} + \binom{e/3}{3}\binom{e/3}{3}\binom{e/3}{3} - \binom{p}{1}[2\binom{e/3-3}{1}\binom{e/3}{4}\binom{e/3}{1} + \binom{e/3}{3}\binom{e/3}{3} \\ +2\binom{e/3}{4}\binom{e/3}{2}] - \binom{q}{1}[2\binom{e/3-3}{1}\binom{e/3}{4}) + 2\binom{e/3}{4}\binom{e/3}{1} + 2\binom{e/3}{3}\binom{e/3}{2}] \\ +[\binom{e/2}{2} - 18][2\binom{e/3-3}{1}\binom{e/3}{2} + 2\binom{e/3}{2}\binom{e/3}{1} + 2\binom{e/3}{3} + \binom{e/3-3}{1}\binom{e/3}{1}\binom{e/3}{1}] + [\binom{15}{2} - 30] \\ [2\binom{e/3}{1}] + 2[\binom{13}{2} - 22][2\binom{e/3-3}{1}\binom{e/3}{1} + 2\binom{e/3}{2} + \binom{e/3}{1}\binom{e/3}{1}] - [\binom{e/2}{3}) - 156] \quad \textit{when } i = 8 \end{cases}$$

$$
f_i = \begin{cases}
3\binom{e/3}{4}\binom{e/3}{4}\binom{e/3}{2} + 3\binom{e/3}{4}\binom{e/3}{3}\binom{e/3}{3} - \binom{p}{1}[2\binom{5}{1}\binom{e/3}{4}\binom{e/3}{2} + \binom{5}{1}\binom{e/3}{3}\binom{e/3}{3} + 2\binom{e/3}{4}\binom{e/3}{3}] \\
-\binom{q}{1}[2\binom{5}{1}\binom{e/3}{4}\binom{e/3}{1} + 2\binom{e/3}{4}\binom{e/3}{2} + 2\binom{e/3}{3}\binom{e/3}{3}][\binom{12}{2} - 18][2\binom{5}{1}\binom{e/3}{3} + \binom{e/3}{2}\binom{e/3}{2} \\
+2\binom{e/3}{4} + 2\binom{e/3}{3}\binom{e/3}{1} + 2\binom{5}{1}\binom{e/3}{2}\binom{e/3}{1}] + [\binom{15}{2} - 30][2\binom{5}{1}\binom{e/3}{1} + 2\binom{e/3}{2} + \binom{e/3}{1}\binom{e/3}{1}]2[\binom{13}{2} - 22] \\
[2\binom{5}{1}\binom{e/3}{2} + 2\binom{e/3}{2}\binom{e/3}{1} + 2\binom{e/3}{3} + \binom{5}{1}\binom{e/3}{1}\binom{e/3}{1}] - [\binom{12}{3} - 156]\binom{5}{1} - [\binom{15}{3} - 375] \quad when\ i = 9 \\
3\binom{e/3}{4}\binom{e/3}{4}\binom{e/3}{3} - \binom{12}{1}[2\binom{5}{1}\binom{e/3}{3}\binom{e/3}{4} + 2\binom{e/3}{4}\binom{e/3}{4}] - \binom{15}{1}[2\binom{e/3}{3}\binom{e/3}{4}] + [\binom{12}{2} - 18] \\
[\binom{5}{1}\binom{5}{1}\binom{e/3}{3} + 2\binom{5}{1}\binom{e/3}{4}][\binom{15}{2} - 30]\binom{e/3}{3} + 2[\binom{13}{3} - 22][\binom{5}{1}\binom{e/3}{3} + \binom{e/3}{4}] - [\binom{12}{3} - 156] \\
[3\binom{5}{1}\binom{5}{1}] - [\binom{13}{3} - 206]\binom{5}{1} - [\binom{14}{3} - 264]\ when\ i = 10 \\
\binom{e/3}{4}\binom{e/3}{4}\binom{e/3}{4} - \binom{p}{1}[\binom{5}{1}\binom{e/3}{4}\binom{e/3}{4}] - \binom{q}{1}[\binom{e/3}{4}\binom{e/3}{4}] + [\binom{12}{2} - 18]\binom{5}{1}\binom{5}{1}\binom{e/3}{4} + [\binom{15}{2} - 30]\binom{8}{4} \\
+2[\binom{13}{2} - 22]\binom{5}{1}\binom{e/3}{4} - [\binom{12}{3} - 156]\binom{5}{1}\binom{5}{1}\binom{5}{1} - [\binom{15}{3} - 330] - [\binom{13}{3} - 206]\binom{5}{1}\binom{5}{1} \\
-[\binom{14}{3} - 264]\binom{5}{1} \quad when\ i = 11
\end{cases}
$$

Proof. Spanning forest of $2W_5$ is actually the union of three copies of spanning trees of $3W_5$. As single copy contains 45 spanning trees so, total number of spanning forests of $3W_5$ are $45^3 = 91125$. we obtain spanning forests of $3W_5$ by cutting down some of the common and non-common edges from each cycles to make it cycle free and connected. Each copy contains 8 edges. So, we cut 4 edges from 8 edges and do it for 3 copies. So, each spanning forests of $3W_5$ will contains 3(8)-3(4)=12 elements in it. So $dim(\Delta_s(3W_5))$=(24-12)-1=11 and

$$f=(f_0,f_1,...,f_{11})$$

To prove f-vector,f_0,f_1 can easily be computed because spanning forests or facets of $dim(\Delta_s(3W_5))$ generate all pairs of sets having 1, 2 elements or edges. These pairs of sets cannot contain any cycle because length of smallest cycle is 3. So we can write it as $\binom{24}{1}$,$\binom{24}{2}$ respectively. For $i = 2$, first of all we find all sets having 3 elements in it and they can be counted by using combination formula i.e. $\binom{24}{3}$. These sets will also include the cycles of length 3. Since these cycles of length 3 are 12 see Fig.1.5. So, by using Inclusion Exclusion principal, we cut these cycles from the total number of sets and we have $\binom{12}{1}$.

For $i = 3$, We find all the sets having 4 number of elements in it we count them using mathematical expression as $\binom{24}{4}$. They also includes the sets having cycles of length 3 and 4. Since these are 12 and 15 respectively. But there is a possibility of variation of sets including these are 12 cycles of length 3 and cycles of length 4 are 15. By using Inclusion Exclusion Principal,we cut these cycles of length 3 and 4 from the sets having from the sets having length 4. So 1 blank spot is there for variation of sets. This blank spot can be the element from 1st copy of $3W_5$ apart from element making the cycle and these elements will be 5 in number. So we can write it in mathematically as$\binom{5}{1}$. Here is also the possibility of filling this blank spot by 8 elements of either $2nd$ or $3rd$ copy. So we can write it as $2\binom{8}{1}$. Here also contains the cycle of length 4 so we cut these cycles by using Inclusion Exclusion Principal and these are 15 so, we write it mathematically as $\binom{15}{1}$.

For $i = 4$, We find all the sets having 5 number of elements in it with the condition that maximum of 5 edges can be taken from one copy irrespective of the choice of a copy. So we fill the blank spots os a set having length 5 and write it in mathematical form as $6\binom{8}{4}\binom{8}{1} + 3\binom{8}{3}\binom{8}{2} + 3\binom{8}{3}\binom{8}{1}\binom{8}{1} + 6\binom{8}{2}\binom{8}{2}\binom{8}{1}$. Since we have three copies so we get the result because of the variation of copies. They also include the sets having cycles of length 3 and 4. Since these are 12 and 15 respectively. By using inclusion Exclusion principal, we cut these cycles of length 4 and 3 from the set having length 5. So for cycles of length 3 here 2 blank spots are there for variation of sets. These blank spots can be filled by the 1 element from $1st$ copy of $3W_5$ apart from element making the cycle and $2nd$ spot can be filled by either of any other 2 copies having 8 elements in each. Also there is the possibility of filling these two spots by $2nd$ and $3rd$ copy by taking 1, 1 element from each i.e. $\binom{12}{1}[2\binom{5}{1}\binom{8}{1} + 2\binom{8}{2} + \binom{8}{1}\binom{8}{1}]$. For cycles of length 4, 1 blank spot is there for variation of sets. This blank spot can be the element from $1st$ copy of $3W_3$ apart from element making the cycle and these elements will be 3 in number. So we can write it in mathematical expression as $\binom{4}{1}$. Here is also the possibility of filling this blank spot by the 8 elements of either $2nd$ or $3rd$ copy.So we can write it as $2\binom{8}{1}$.

For $i = 5$, We find all the sets having 6 number of elements in it with the condition that maximum of 4 edges we can take from one copy irrespective of the choice of a copy. So we fill the blank spots of a set having 6 length and vary it in 3 copies and we write it in mathematical form as $6\binom{8}{4}\binom{8}{2}+\binom{8}{3}\binom{8}{3}+3\binom{8}{4}\binom{8}{1}\binom{8}{1}+6\binom{8}{3}\binom{8}{2}\binom{8}{1}+\binom{8}{2}\binom{8}{2}\binom{8}{2}$. They also include the sets having cycles of length 3 and 4. Since these are 12 and 15 in number. But there is a possibility of variation of sets including these cycles. By using Inclusion Exclusion Principle,we cut these cycles of length 3 from the set having length 6. So 3 blank spots are there for variation of sets. These blank spots can be filled by 1 element from $1st$ copy and 2 element of either $2nd$ or $3rd$ copy i.e. $2\binom{5}{1}\binom{8}{2}$. Also there is a possibility of taking 1 and 2 elements from either of $2nd$ and $3rd$ copy i.e.$\binom{8}{2}\binom{8}{1}$. There could be 3 elements from either of $2nd$ or $3rd$ copy i.e. $2\binom{8}{3}$. There could be possibility 1, 1 and 1 element possibility from three copies i.e. $2\binom{5}{1}\binom{8}{1}\binom{8}{1}$. For cycle of length 4, 2 blank spots are there for variations of sets. These blank spots can be filled by the 1 element from $1st$ copy of $3W_5$ apart from element making the cycle and second spot can be filled by either of any other 2 copies having 8 elements in each. Also there is the possibility of filling these two spots either by $2nd$ or $3rd$ copy. There is also the possibility of filling these two spots by $2nd$ and $3rd$ copy by taking 1, 1 element from each.i.e. $\binom{15}{1}[2\binom{4}{1}\binom{8}{1}+2\binom{8}{2}+\binom{8}{1}\binom{8}{1}]$

For $i = 6$, We find all the sets having 7 number of elements in it with the condition that maximum of 4 edges we can take from one copy irrespective of the choice of a copy. So we fill the bank spots of a set having length 8 and vary it in 3 copies and we write it in mathematical form as $6\binom{8}{4}\binom{8}{3}+3\binom{8}{4}\binom{8}{2}\binom{8}{1}+3\binom{8}{3}\binom{8}{2}\binom{8}{2}$. They also include the sets having cycles of length 3 and 4. Since cycles of length

3 are 12 in number. But there is a possibility of variation of sets including these 12 cycles of length 3. By using Inclusion Exclusion Principle, we cut these cycles of length 3 from the sets having length 8. So 4 blank spots are there for variation of sets. These blank spots can be filled by 1 element from 1st copy and 3 elements of either 2nd or 3rd copy i.e $2\binom{5}{1}\binom{8}{3}$. Also there is a possibility of taking 2 and 2 elements from either of 2nd and 3rd copy i.e $\binom{8}{2}\binom{8}{2}$. There could be 3 elements from of 2nd copy and 1 element from 3rd copy i.e. $2\binom{8}{3}\binom{8}{1}$. There could be 1, 2, 1 element possibility from three copies i.e. $2\binom{5}{1}\binom{8}{2}\binom{8}{1}$. Since there are also cycles of length 4 and these are 15 in number. But there is a possibility of variation of sets including these 15 cycles of length 4. By using Inclusion Exclusion Principle, we cut these cycles of length 4 from the sets having length 8. So 3 blank spots are there for variation of sets. These blank spots can be filled by 1 element from 1st copy and 2 elements of either 2nd or 3rd copy i.e. $2\binom{8}{2}\binom{8}{1}$. Also there is a possibility of taking 3 elements either 2nd or 3rd copy i.e. $2\binom{8}{3}$. Here also include the double cycles of length 6 and 7 so by using Inclusion Exclusion principle we add these cycles i.e. $[\binom{12}{2}$-18$][\binom{5}{1}+2\binom{8}{1}]$ and also included $2[\binom{13}{2}$-22$]$

For $i = 7$, We find all the sets having 8 number of elements in it with the condition that maximum of 4 edges we can take from one copy irrespective of the choice of a copy. So we fill the blank spots of a set having length 8 and write it in mathematical form as $3\binom{8}{4}\binom{8}{4}+6\binom{8}{4}\binom{8}{2}\binom{8}{3}+6\binom{8}{4}\binom{8}{3}\binom{8}{1}+3\binom{8}{3}\binom{8}{3}\binom{8}{2}+3\binom{8}{4}\binom{8}{2}\binom{8}{2}$. Here also include the sets having cycles of length 3 and 4. Since these cycles are 12 and 15 in number. For cycles and length 3 we use Inclusion Exclusion Principle, we cut these cycles of length 3 from the sets having length 8. So 5 blank spots are there for variation of sets. These blank spots can be filled by the 1 element from 1st element from 1st copy of $3W_5$ apart from element making the cycle and these are 5 in numbers and other four spots can be filled by any three elements of either 2nd or 3rd copy of $3W_5$ i.e.$2\binom{5}{1}\binom{8}{4}$. Also here is the possibility of filling these five spots by the 1 element from 2nd copy and 4 elements 3rd copy of $3W_5$ i.e. $2\binom{8}{4}\binom{8}{1}$. Also here is the possibility of filling these five spots by the 3 and 2 elements either these are in 2nd copy or 3rd copy of $3W_5$ i.e.$2\binom{8}{3}\binom{8}{2}$. For the cycle of length 4 We cut these cycles by using Inclusion Exclusion principle, from the sets having length 8. So 4 blank spots are there for variation of sets. These blank spots can be filled by the 2, 2 elements either from the 2nd or 3rd copy of $3W_5$ i.e.$\binom{8}{2}\binom{8}{2}$. Also possibly filled these 4 blank spots 3 and 1 element by using 2nd or 3rd copy of $3W_5$ i.e.$2\binom{8}{3}\binom{8}{1}$. There is also the possibility of 2 cycles of length 6. Also there is the possibility of variation of sets in two spots. There is the possibility of 1, 1 element from 1st and 2nd copy. There is the possibility of 2 element from 2nd or 3rd copy. Also possible 1, 1 from 2nd and 3rd copy i.e.$2\binom{5}{1}\binom{8}{1} + 2\binom{8}{2} + \binom{8}{1}\binom{8}{1}$. There is also the possibility of 2 cycles of length 8. There is also the possibility of 2 cycles of length 3 and 4. We add these cycles by using Inclusion Exclusion principle here also 1 blank spots so we filled this spot as $\binom{5}{1} + 2\binom{8}{1}$

For $i = 8$, We find all the sets having 9 number of elements in it and we write it in mathematical form as $6\binom{8}{4}\binom{8}{4}\binom{8}{1} + 6\binom{8}{4}\binom{8}{3}\binom{8}{2} + \binom{8}{3}\binom{8}{3}\binom{8}{3}$. They also include the sets having cycles of length 3 and 4. Since these cycles are 12 and 15 in numbers. By using Inclusion Exclusion Principle, We cut these cycles of length 3 and 4 from the sets having length 9. So 6 blank spots for the cycle of length 3 and 5 spots for cycle of length 4 are there for variation of sets. For the cycle of length 3 these blank spots can be filled by the 1 element from 1st copy of $3W_5$ apart from element making the cycle and these are 5 in numbers possible 4 element in 2nd or 1 element in 3rd copy of $3W_5$ i.e. $2\binom{5}{1}\binom{8}{4}\binom{8}{1}$. It also possible that these blank spots can be filled by using 3, 3 element from the 2nd and 3rd copy of $3W_5$ i.e.$\binom{8}{3}\binom{8}{3}$. Also these blank spots can be filled by using 4 element and 2 elements either in 2nd copy or 3rd copy i.e.$2\binom{8}{4}\binom{8}{2}$. For the cycles of length 4,5 blank spots are there for variation of sets. These blank spots can be filled by the 1 element from 2nd copy and 4 elements from 3rd copy of $3W_5$ i.e. $2\binom{8}{4}\binom{8}{1}$. Also it is possible that these 5 blank spots can be filled by using 3 and 2 elements from either 2nd or 3rd copy of $3W_5$. Here also possible exist the double cycles of length 6 i.e. $[\binom{12}{2}-18]$. Here also 3 blank spots for the variation of sets. We filling these blank spots as 1 element and 2 elements, these elements are comes from 1st copy or 2nd or 3rd copy i.e. $2\binom{5}{1}\binom{8}{2}$ also is it possible that these 3 spots comes from 2 elements and 1 elements either 2nd or 3rd copy i.e. $2\binom{5}{1}\binom{8}{2}$. Also possible these 3 elements comes form 2nd or 3rd copy i.e. $2\binom{8}{3}$ also possible these 3 spots comes from each copies of $3W_5$ i.e. $\binom{5}{1}\binom{8}{1}\binom{8}{1}$. Here also possible that double cycle of length 8 exist that are [$\binom{15}{2}$-30], here also 1 blank spot exist for the variations of sets. It possible that this spot exist from either 1st copy or 2nd copy i.e. $2\binom{8}{1}$. Here also exist the double cycles of length 3 and 4 exist that2[$\binom{13}{2}$-22]. Here, there are 2 blank spots for the variations of sets that can be filled as [$\binom{5}{1}\binom{8}{1}$+$2\binom{8}{2}$+$\binom{8}{1}\binom{8}{1}$]. Here also exist the triple cycles. By using Inclusion Exclusion Principle we add these cycles i.e. [$\binom{12}{3}$-156]. Similarly way we get result for $i = 9, 10$.

Now For $i = 11$, We find all the sets having 12 number of elements by including all the possibilities and we write it in mathematical form as $\binom{8}{4}\binom{8}{4}\binom{8}{4}$. Here also include the sets having cycles of length 3 and 4. Here for cycles of length 3 by using Inclusion Exclusion Principle, we cut these cycles of length 3 from the sets having length 12. So 9 blank spots are there. These blank spots can be filled by the 1 element from 1st copy of $3W_5$ and other eight spots can be filled by 4, 4 elements from 2nd copy and 3rd copy i.e. $\binom{5}{1}\binom{8}{4}\binom{8}{4}$. Here for cycles of length 4 by using Inclusion Exclusion Principle, we cut these cycles of length 4 from the sets having length 12. So 8 blank spots are there. These blank spots can be filled by the eight spots can be filled by 4, 4 elements from 2nd copy and 3rd copy i.e. $\binom{8}{4}\binom{8}{4}$. There is also the possibility of 2 cycles of length 6. These are $[\binom{12}{2}-18]$. Also there is the possibility of variation of sets in 6 spots.

There is possibility of 1, 1, 4 element from 1st, 2nd and 3rd copy respectively i.e. $\binom{5}{1}\binom{5}{1}\binom{8}{4}$. There is also the possibility of 2 cycles of length 8. These are $[\binom{15}{2} - 30]$. Also there is the possibility of variation of sets in 4 spots. There is possibility of 4 element from 3rd copy i.e. $\binom{8}{4}$. There is also the possibility of 2 cycles of length 7. These are $2[\binom{13}{2} - 22]$. Also there is the possibility of variation of sets in 5 spots. There is possibility of 1, 4 element from 1st and 3rd copy i.e. $\binom{5}{1}\binom{8}{4}$. Now here can be the sets including 3 cycles of length 9 and variation of sets. Three blank spots are there and can be filled out 1, 1, 1 from all three copies i.e.$\binom{5}{1}\binom{5}{1}\binom{5}{1}$. Now here can be the sets including 3 cycles of length 12 and variation of sets. Now here can be the sets including 3 cycles of length 10 and variation of sets. 2 blank spots are there and can be filled out 1 from all three copies i.e. $\binom{5}{1}$. Now here can be the sets including 3 cycles of length 11 and variation of sets. Three blank spots are there and can be filled out 1, 1 from all three copies i.e. $\binom{5}{1}\binom{5}{1}$.

□

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