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# APPROXIMATE SOLUTION OF FRACTIONAL INTEGRO - DIFFERENTIAL EQUATIONS USING LEAST SQUARES FRAMEWORK BASED ON CONSTRUCTED ORTHOGONAL POLYNOMIAL

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Abstract. In this study, we utilize, a numerical approach to solve a fractional order linear integro - differential equation in the Caputo sense. We constructed orthogonal polynomials as basis functions by proposing the standard least squares method (SLSM) and employ the general weight function of the type $w(x) = (c + dx^i)^n$ to generate orthogonal polynomials with $c = 1, d = -1, n = 1$, and $i = 4$. This type of problem is reduced to a system of linear algebraic equations, which is then solved through SciLab 6.1.1 programming language. To demonstrate the method, some numerical examples have been solved.

## 1. Introduction

In science, engineering, and other fields, fractional calculus involves the integration, derivatives, and applications of non-integer order. Fractional calculus application areas have expanded to include real-world problems such as earthquake modelling, fluid dynamic traffic modelling with fractional derivates, fluid mechanics, assessment of viscoelastic material properties etc. In the literature many authors have contributed interesting books on fractional derivatives and fractional integrations [6], [10]]. Fractional Integro-Differential Equations( FIDEs) do not yield any analytical or precise solutions so, solving them can be difficult task. As a result, approximate solutions based on numerical methods are extremely beneficial. Several numerical methods are proposed in the literature to solve the FIDEs such as, Least Squares Method(LSM) [1], [2], [3], [4] , [12], [13], [14]], [15], modified Adomian decomposition method [5], Adomian decomposition method [7], homotopy perturbation method [13], modified homotopy perturbation method (MHPM)[16], Bernstein collocation method [17], perturbed least square method(PLSM)[18].
Ajisope et.al.[1], constructed orthogonal polynomials as fundamental functions using SLSM for solving the Volterra FIDEs. In [2], the authors presented a numerical solution to linear FIDE using the Least Square approach with shifted Laguerre collocation method. In this paper numerical solution was found at m = 7, which

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creates complexity in computaton. The authors in [3], developed numerical solutions for linear FIDEs utilising shifted Chebyshev polyniomials of the third order and the Least Square Method. Using shifted Chebyshev polyniomials of the first kind and the Least Square Method, the authors in [4] found numerical solutions for linear FIDEs. The modified Adomian decomposition method was used by Hamoud et.al.[5] to estimate the solution of Caputo fractional Volterra-Fredholm integro-differential equations. Mittal [7] used the Adomain decompisition technique to get good results for FIDE and compared the results to the collocation method. The authors in [8] utilized Bernstein polynomials to find the numerical solution of FIDEs and improved the results in [9] by applying Hermite polynomials even at the lower values of $n$. Osama et.al.[11] used Bernstein piecewise polynomial to approximate the solution of the FIDEs. The fractional derivatives are mentioned in the Caputo sense. The authors of [12] proposed constructed orthogonal polynomials to solve the FIDEs using two numerical methods: SLSM and MHPM. In [13], the Bernsetin polynomials are used as the basis function to calculate approximate solutions of FIDEs using two numerical methods: LSM and HPM. Oyedepo et.al.[14], solved FIDEs using SLSM and PLSM with Bernstein polynomials as basis function. The Bernstein Least Squares methodology, which uses the Bernsetin polynomial for solving the FIDEs, is proposed in [15]. MHPM and Bernstein polynomials are the techniques utilised in [16] to solve FIDEs. For solving FIDEs, Oyedepo et.al. [17] proposed the Bernstein collocation approach. To solve Volterra FIDEs [18] proposed PLSM using constructed orthogonal polynomials(OP). The major goal of our research is to use the Least Square approach to obtain the numerical solution to the FIDEs utilising the designed orthogonal polynomials as the basis function. We employ the weight function $w(x) = 1 - x^4$ to create the orthogonal polynomials, which improves the absolute error of SLSM when compared to [12] and [13]. The basic form of the problem considered in this study is as follows:

$$D_*^\alpha v(x) = F(x) + \int_0^1 k(x,t)v(t)dt, \quad 0 \leq x, t \leq 1 \tag{1.1}$$

with the following initial conditions:

$$v^{(j)}(0) = c_j, j = 0, 1, ......n-1, n-1 < \alpha \leq n, n \in N \tag{1.2}$$

where $D_*^\alpha v(x)$ is the Caputo fractional derivative of $v(x)$, $k(x,t)$, $F(x)$ are known functions, $x$ and $t$ are real variables that vary between $[0, 1]$, and $v(x)$ is the unknown function that needs to be identified.

## 2. Preliminaries

In this section, we present some fundamental fractional calculus definitions and properties that will aid us in formulating a method for obtaining a numerical solution to a given problem.

**Definition 2.1:**[6]The Caputo fractional derivative operator $D^\alpha$ of order $\alpha$ is defined as:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}} dt, \quad \alpha > 0$$

where $m-1<\alpha\leq m, m\in N, x>0$
We have the following properties:

(1) $J^{\alpha}J^{\nu}f=J^{\alpha+\nu}, \alpha,\nu>0, f\in C_{\mu}, \mu>0$
(2) $J^{\alpha}x^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}x^{\beta+\alpha}, \quad \alpha>0, \quad \beta>-1, \quad x>0$
(3) $J^{\alpha}D^{\alpha}f(x)=f(x)-\sum_{k=0}^{m-1}f^{k}(0^{+})\frac{x^{k}}{k}, x>0, m-1<\alpha\leq m$
(4) $D^{\alpha}J^{\alpha}f(x)=f(x), \quad x>0, \quad m-1<\alpha\leq m$
(5) $D^{\alpha}$ $C$=0, $C$ is a constant
(6) $D^{\alpha}x^{\beta}=\begin{cases}0, & \beta\in N_0, \quad \beta<\lceil\alpha\rceil\\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, & \beta\in N_0, and \quad \beta\geq\lceil\alpha\rceil\end{cases}$

where $\lceil\alpha\rceil$ denoted the smallest integer greater than or equal to $\alpha$ and $N_0=\{0,1,2,....\}$

**Definition 2.2: Orthogonality:**
$v_p(x)$ and $v_q(x)$ are said to be orthogonal they are defined on the interval $c\leq x\leq d$ if:

$$<v_p(x),v_q(x)>=\int_c^d v_p(x)v_q(x)dx=0 \tag{2.1}$$

In the other hand, there is a weight function $w(x)>0$ exists then:

$$<v_p(x),v_q(x)>=\int_c^d w(x)v_p(x)v_q(x)dx=0 \tag{2.2}$$

Then we mentioned that $v_p(x)$ and $v_q(x)$ are orthogonal to each other in terms of the weight function $w(x)$.
In general, we write:

$$\int_c^d w(x)v_p(x)v_q(x)dx=\begin{cases}0, & p\neq q\\ \int_c^d w(x)v_p^2(x)dx, & p=q\end{cases} \tag{2.3}$$

## 3. Construction of Orthogonal Polynomials

We built our orthogonal polynomials in this section by employing the general weight function of the type: $w(x)=(c+dx^i)^n$. This equates to quartic function that fulfils the orthogonality conditions with in interval [c,d] for c=1,d=-1,n=1 and i=4 respectively. The orthogonal polynomial $v_j(x)$ defined with in range [c,d] with the leading term $x^j$ is given as a result of the Gram-Schmidt orthogonalization procedure.

$$v_j(x)=x^j-\sum_{i=0}^{j-1}b_{j,i}v_i(x), i=0,1,2,...,j-1 \quad and \quad j\geq 1, \tag{3.1}$$

where $v_j(x)$ is a $j^{th}$ degree rising polynomial and the values of the approximation function in 'x' are represented by $v_i(x)$. The linear polynomial $v_j(x)$ having following term $x$ is expressed as, starting with $v_0(x)=1$,

$$v_1(x)=x-b_{1,0}v_0(x), \tag{3.2}$$

where $b_{1,0}$ is an unknown constant to determined. Since $v_0(x)$ and $v_1(x)$ are orthogonal. We have,

$$\int_c^d w(x)v_1(x)v_0(x)dx = 0 = \int_c^d xw(x)v_0(x)dx - b_{1,0}\int_c^d w(x)v_0^2(x)dx \quad (3.3)$$

From the above, we have

$$b_{1,0} = \frac{\int_c^d w(x)xv_0(x)dx}{\int_c^d w(x)v_0^2(x)dx} \quad (3.4)$$

Substitute (3.4) into (3.2)

$$v_1(x) = x - \frac{\int_c^d w(x)xv_0(x)dx}{\int_c^d w(x)v_0^2(x)dx}v_0(x), where \quad v_0(x) = 1 \quad (3.5)$$

As a result of this, the procedure can be generalised and expressed as,

$$v_j(x) = x^j + b_{j,0}v_0(x) + b_{j,1}v_1(x) + b_{j,2}v_2(x) + ... + b_{j,j-1}v_{j-1}(x) \quad (3.6)$$

Where $b_{j,0}$ is chosen so that $v_j(x)$ is orthogonal to $v_0(x), v_1(x), v_2(x), ...v_{j-1}(x)$. These conditions defer,

$$b_{j,i} = \frac{\int_c^d w(x)x^jv_0(x)dx}{\int_c^d w(x)v_0^2(x)dx} \quad (3.7)$$

We use the weight function in the form,

$$w(x) = 1 - x^4, v_0(x) = 1$$

We have $n = 1, j = 1$ and $v_0(x) = 1$, we can write (3.1) as,
$v_1(x) = x - b_{1,0}v_0(x)$ where $b_{1,0} = 0$. Thus,

$$v_1(x) = x, v_2(x) = x^2 - \frac{5}{21}, v_3(x) = x^3 \quad (3.8)$$

Define shifted orthogonal polynomia $v_i^*(x)$ interms of the orthogonal polynomial $v_i(x)$ valid in $[0,1]$ by the following relation:

$$v_i^*(x) = v_i(2x-1) \quad (3.9)$$

Therefore, we get

$$v_0^*(x) = 1, v_1^*(x) = 2x-1, v_2^*(x) = 4x^2-4x+\frac{16}{21}, v_3^*(x) = 8x^3-12x^2+6x-1 \quad (3.10)$$

The approach used in this study assumes an approximate solution using an orthogonal polynomial as the basis function, as follows:

$$v(x) \cong v_n(x) = \sum_{i=0}^{n} b_i v_i^*(x) \quad (3.11)$$

where $b_i$, i=0,1,2,...are constants, and $v_i^*(x)$ signifies the orthogonal polynomial of degree N.

## 4. Demonstration of the Proposed method

The least squares method with construction of orthogonal polynomials is applied to find the approximate solution of FIDE of the type(1.1) and (1.2). Operating $I^\alpha$ on both sides of (1.1), we obtain

$$I^\alpha D^\alpha v(x) = I^\alpha F(x) + I^\alpha \left[\int_0^1 k(x,t)v(t)dt\right] \tag{4.1}$$

$$v(x) = \sum_{k=0}^{m-1} v^k(0^+)\frac{x^k}{k!} + I^\alpha F(x) + I^\alpha \left[\int_0^1 k(x,t)v(t)dt\right] \tag{4.2}$$

To determine the approximate solution of (1.1), we use the orthogonal polynomial basis on $[c,d]$ as

$$v(x) \cong v_n(x) = \sum_{i=0}^{n} b_i v_i^*(x) \tag{4.3}$$

where $b_i (i = 0, 1, ......n)$ are unknown constants to be determined. Substituting (4.3) into (4.2)

$$\sum_{i=0}^{n} b_i v_i^*(x) = \sum_{k=0}^{m-1} v^k(0^+)\frac{x^k}{k!} + I^\alpha F(x) + I^\alpha \left[\int_0^1 k(x,t)\sum_{i=0}^{n} b_i v_i^*(t)dt\right]$$

As a result, the residual equation is as follows:

$$R(b_0, b_1, ..........b_n) = \sum_{i=0}^{n} b_i v_i^*(x) - \left\{\sum_{k=0}^{m-1} v^k(0^+)\frac{x^k}{k!} + I^\alpha \left[F(x) + \int_0^1 k(x,t)\sum_{i=0}^{n} b_i v_i^*(t)dt\right]\right\} \tag{4.4}$$

Let

$$S(b_0, b_1, ...b_n) = \int_0^1 \left[R(b_0, b_1, ...b_n)\right]^2 w(x)dx \tag{4.5}$$

where $w(x)$ is the positive weight function defined on $[c,d]$. For the sake of simplicity, we'll choose $w(x) = 1$. Thus,

$$S(b_0, b_1, ........b_n) =$$
$$\int_0^1 \left[\sum_{i=0}^{n} b_i v_i^*(x) - \{\sum_{k=0}^{m-1} v^k(0^+)\frac{x^k}{k!} + I^\alpha \left[F(x) + \int_0^1 k(x,t)\sum_{i=0}^{n} b_i v_i^*(t)dt\right]\}\right]^2 dx \tag{4.6}$$

We obtain the value of $b_i$ by finding the minimum value of S as :

$$\frac{\partial S}{\partial b_i} = 0, \quad i = 0, 1, .......n \tag{4.7}$$

Applying (4.7) on (4.6) we obtain,

$$\int_0^1 \left[\sum_{i=0}^{n} b_i v_i^*(x) - \{\sum_{k=0}^{m-1} v^k(0^+)\frac{x^k}{k!} + I^\alpha \left[F(x) + \int_0^1 k(x,t)\sum_{i=0}^{n} b_i v_i^*(t)dt\right]\}\right] dx *$$
$$\int_0^1 \{v_i^*(x) - I^\alpha(\int_0^1 k(x,t)v_i^*(t)dt)\}dx \quad (4.8)$$

By simplifying the above equation we can obtain a $(n+1)$ algebraic system of equations in $(n+1)$ unknown constants $b_i's$.
This system can be formed by using matrices form as follows:

$$A = \begin{pmatrix} \int_0^1 R(x,b_0)h_0dx & \int_0^1 R(x,b_1)h_0dx & \cdots\cdots & \int_0^1 R(x,b_n)h_0dx \\ \int_0^1 R(x,b_0)h_1dx & \int_0^1 R(x,b_1)h_1dx & \cdots\cdots & \int_0^1 R(x,b_n)h_1dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^1 R(x,b_0)h_ndx & \int_0^1 R(x,b_1)h_ndx & \cdots\cdots & \int_0^1 R(x,b_n)h_ndx \end{pmatrix}$$

$$B = \begin{pmatrix} \int_0^1 \left[I^\alpha F(x) + \sum_{k=0}^{m-1} v^k(0^+)\frac{x^k}{k!}\right] h_0dx \\ \int_0^1 \left[I^\alpha F(x) + \sum_{k=0}^{m-1} v^k(0^+)\frac{x^k}{k!}\right] h_1dx \\ \vdots \\ \int_0^1 \left[I^\alpha F(x) + \sum_{k=0}^{m-1} v^k(0^+)\frac{x^k}{k!}\right] h_ndx \end{pmatrix}$$

where

$$h_i = v_i^*(x) - I^\alpha \left[\int_0^1 k(x,t)v_i^*(t)dt\right], i = 0, 1, ...n$$

$$R(x,b_i) = \sum_{i=0}^{n} b_i v_i^*(x) - I^\alpha \left[\int_0^1 k(x,t)\left(\sum_{i=0}^{n} b_i v_i^*(t)\right)dt\right], i = 0, 1, ...n$$

By solving above system, we obtain the unknown co-efficients and the approximate solution of (1.1).
In this work we defined absolute error has

$$Absolute \quad error = |v(x) - v_m(x)|, \quad 0 \le x \le 1 \quad (4.9)$$

where $v(x)$ is the exact solution and $v_m(x)$ is the approximate solution.

## 5. Numerical Examples

To demonstrate the Proposed method, we choose two illustrations. All of the results were generated using SciLab 6.1.1.

**Example 5.1.** Consider the fractional integro-differential equation[12]

$$D^{\frac{1}{2}}v(x) = \frac{\frac{8}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtv(t)dt, \quad 0 \le x \le 1, \quad v(0) = 0 \quad (5.1)$$

with exact solution $v(x) = x^2 - x$.
By taking the fractional integration on both sides of the equation (5.1), we get

$$v(x) = \sum_{k=0}^{m-1} v^k(0^+)\frac{x^k}{k!} + I^\alpha\left\{\frac{\frac{8}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtv(t)dt\right\} \tag{5.2}$$

To determine the approximate solution of (5.1) we set

$$v(x) = \sum_{i=0}^{3} b_i v_i^*(x) \tag{5.3}$$

After substituting in equation (5.2)

$$\sum_{i=0}^{3} b_i v_i^*(x) = I^\alpha\left\{\frac{\frac{8}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xt\left[\sum_{i=0}^{3} a_i v_i^*(t)\right]dt\right\}$$

After simplifying the above equation we get,

$$b_0\left[1 - \frac{x^{1.5}}{2\Gamma(2.5)}\right] + b_1\left[2x - 1 - \frac{x^{1.5}}{6\Gamma(2.5)}\right] + b_2\left[4x^2 - 4x + \frac{16}{21} - \frac{x^{1.5}}{21\Gamma(2.5)}\right] +$$
$$b_3\left[8x^3 - 12x^2 + 6x - 1 - \frac{x^{1.5}}{10\Gamma(2.5)}\right] - \frac{8\Gamma(2.5)x^2}{6\sqrt{\pi}} + \frac{2\Gamma(1.5)x}{\sqrt{\pi}} - \frac{x^{1.5}}{12\Gamma(2.5)} = 0$$

Also substituting $x = 0.1, 0.2, 0.3$ and $0.4$ in above equation, we get a linear system of equations:

$$(0.7344667)b_0 + (-0.1028397)b_1 + (0.065173)b_2 + (-0.0614433)b_3 + 0.1236052 = 0$$
$$(-0.1028397)b_0 + (0.2942772)b_1 + (-0.0113863)b_2 + (0.1766532)b_3 - 0.0167419 = 0$$
$$(0.065173)b_0 + (-0.0113863)b_1 + (0.094641)b_2 + (-0.006807)b_3 - 0.0112463 = 0$$
$$(-0.0614433)b_0 + (0.1766532)b_1 + (-0.006807)b_2 + (0.1289011)b_3 - 0.0100017 = 0$$

Solving the above equations we get:
$b_0 = -0.1904762, b_1 = 0.0000002, b_2 = 0.2499995$, and $b_3 = -0.0000005$
The values are then substituted into equation(5.3)we get the approximate solution of(5.1).
Approximate solution is,

$$v(x) = -0.1904762 + 0.0000002(2x - 1) + 0.2499995(4x^2 - 4x + 16/21)$$
$$- 0.0000005(8x^3 - 12x^2 + 6x - 1)$$

Following Table 1 represent comparison between the approximate solution when $\alpha = 1/2$ with the exact solution $v(x) = x^2 - x$.

| x | exact solution | Approximate solution of OP ($n = 3$) | Approximate solution [12] | Absolute error of OP | Absolute error [12] |
|---|---|---|---|---|---|
| 0.1 | -0.09 | -0.0900001 | -0.08998135725 | 0.0000001 | 0.000018642 |
| 0.2 | -0.16 | -0.1600001 | -0.15996634990 | 0.0000001 | 0.00003365 |
| 0.3 | -0.21 | -0.21 | -0.20995511270 | 0 | 0.000044887 |
| 0.4 | -0.24 | -0.2399999 | -0.2399476690 | 0.0000001 | 0.000052331 |
| 0.5 | -0.25 | -0.2499999 | -0.24994404190 | 0.0000001 | 0.000055958 |
| 0.6 | -0.24 | -0.2399999 | -0.23994425520 | 0.0000001 | 0.000055744 |
| 0.7 | -0.21 | -0.2099999 | -0.20994833220 | 0.0000001 | 0.000051667 |
| 0.8 | -0.16 | -0.1600001 | -0.15995629600 | 0.0000001 | 0.000043704 |
| 0.9 | -0.09 | -0.0900003 | -0.08996816998 | 0.0000003 | 0.00003183 |

TABLE 1. Example 1 numerical results

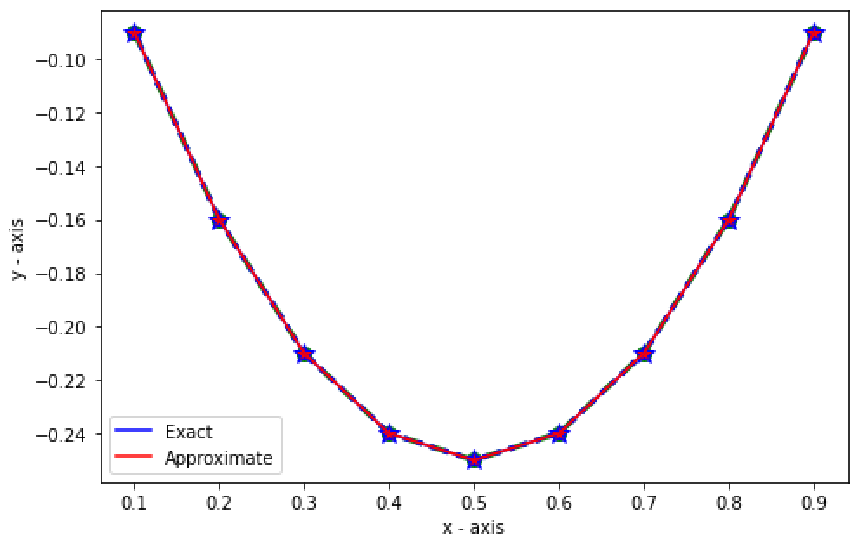


FIGURE 1. Comparison between approximate and exact solution of Example 1

**Example 5.2.** Consider the fractional integro-differential equation[12]

$$D^{\frac{5}{6}}v(x) = -\frac{3}{91}\frac{x^{1/6}\Gamma(5/6)(-91+216x^2)}{\pi} + (5-2e)x + \int_0^1 xe^t v(t)dt$$

subject to $v(0) = 0$, with exact solution $v(x) = x - x^3$.
Approximate solution is,

$$v(x) = 0.2857142+0.1249994(2x-1)-0.3749996(4x^2-4x+16/21)-0.1249991(8x^3-12x^2+6x-1)$$

Following Table 2 represent comparison between the approximate solution with the exact solution $v = x - x^3$.

| x | exact solution | Approximate solution of OP ($n = 3$) | Approximate solution [12] | Absolute error of OP | Absolute error [12] |
|---|---|---|---|---|---|
| 0.1 | 0.099 | 0.0990001 | 0.09895741497000 | 0.0000001 | 0.0000425 |
| 0.2 | 0.192 | 0.1920001 | 0.19191319860000 | 0.0000001 | 0.000086801 |
| 0.3 | 0.273 | 0.2730001 | 0.2728702740000 | 0.0000001 | 0.00012972 |
| 0.4 | 0.336 | 0.3359999 | 0.33583118160000 | 0.0000001 | 0.000168818 |
| 0.5 | 0.375 | 0.3749998 | 0.37479846150000 | 0.0000002 | 0.000201538 |
| 0.6 | 0.384 | 0.0.3839997 | 0.38377465420000 | 0.0000003 | 0.000225345 |
| 0.7 | 0.357 | 0.3569997 | 0.35676229990000 | 0.0000003 | 0.0002377 |
| 0.8 | 0.288 | 0.2879998 | 0.28776393890000 | 0.0000002 | 0.000236061 |
| 0.9 | 0.171 | 0.1710001 | 0.17078211170000 | 0.0000001 | 0.000217888 |

TABLE 2. Example 2 numerical results

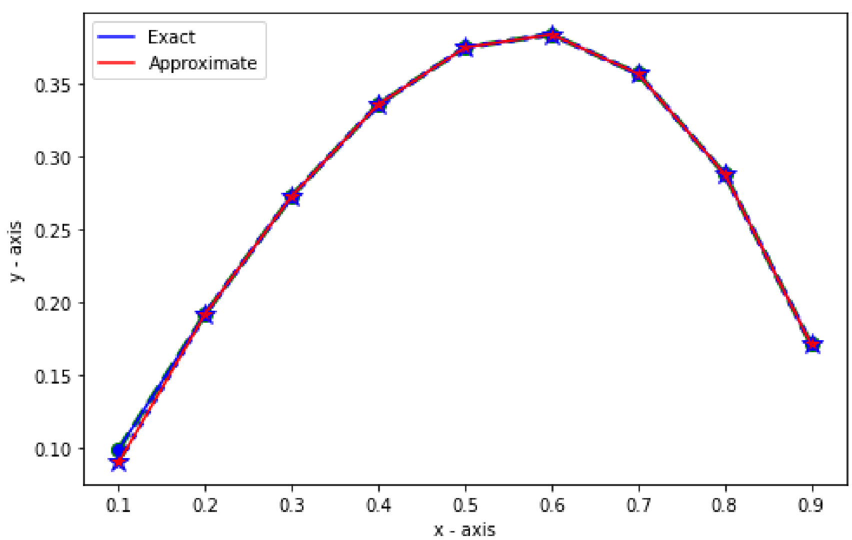


FIGURE 2. Comparison between approximate and exact solution of Example 2

## 6. Conclusion

In this paper, we use the constructed orthogonal polynomials to deduce the numerical solution to FIDE's using the least square approach. The usage of SLSM is useful since, it requires less processing work and is an effective method for solving such equations. The numerical findings achieved are superior to those found in [12].

The numerical solutions and the exact solutions are much better in accordance. To demonstrate our method is better in solving FIDE's, comparisons between approximate and exact solutions are shown in Table 1 and Table 2 with graphical formats. We used SciLab 6.1.1 for programming to demonstrate numerical results.

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