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AN APPLICATION OF A GENERALIZED DISTRIBUTION SERIES ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

S. D. BHOURGUNDE, S. D. THIKANE, M. G. SHRIGAN AND M. T. GOPHANE

Abstract. The purpose of the present paper is to establish connections between various subclasses of analytic univalent functions by applying certain convolution operator involving generalized distribution series. Also, we investigate several mapping properties involving these subclasses.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

normalized by the conditions $f(0) = f'(0) - 1 = 0$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Further, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and univalent in $\mathbb{C}$. Moreover, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad z \in \mathbb{U}. \tag{1.2}$$

A function $f(z) \in \mathcal{A}$ is said to be in the class $S^*(\beta)$ of starlike function of order β $(0 \leq \beta < 1)$ if it satisfies the condition

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (z \in \mathbb{U}). \tag{1.3}$$

We also write $S(0) := S^*$ (for details, see [7], [8], [11], [16], [19]). Furthermore, a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{K}(\beta)$ of convex functions of order $\beta(0 \leq \beta < 1)$ if it satisfies the condition

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \quad (z \in \mathbb{U}). \tag{1.4}$$

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Clearly, we have $\mathcal{K}(0) := \mathcal{K}$ (for details, see [7], [10], [11], [16], [17], [18], [19], [20]). It is well known fact that $f \in \mathcal{K}(\beta) \Leftrightarrow zf' \in S^*(\beta)$. Bharati et al.[2] investigated and studied the certain subclasses $\mathcal{UCV}(\alpha, \beta)$ and $S_p(\alpha, \beta)$ as follows:

$$\mathcal{UCV}(\alpha, \beta) =: \left\{ f : f \in \mathcal{A} \,\text{and}\, Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \alpha \left|\frac{zf''(z)}{f'(z)}\right| + \beta \right\}, \tag{1.5}$$

$$S_p(\alpha, \beta) =: \left\{ f : f \in \mathcal{A} \,\text{and}\, Re\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \right\}, \tag{1.6}$$

for $z \in \mathbb{U}, \alpha \geq 0, 0 \leq \beta < 1$.

Indeed, it follows from (1.5) and (1.6) that $f \in \mathcal{UCV}(\alpha, \beta) \Leftrightarrow zf' \in S_p(\alpha, \beta)$. Also, we write $\mathcal{TUCV}(\alpha, \beta) = \mathcal{UCV}(\alpha, \beta) \cap \mathcal{T}$ and $\mathcal{T}S_p(\alpha, \beta) = S_p(\alpha, \beta) \cap \mathcal{T}$.

Several mapping properties of the classes β-starlike and β-uniformly convex functions of order α in the open unit disk $\mathbb{U}$ were studied recently by Murugusundaramoorthy [6]. Altınkaya and Yalçın [1] gave obligatory conditions for the Poisson distribution series belonging to the class $\mathcal{T}(\gamma, \delta)$. Using numerous techniques, the authors determined necessary and sufficient conditions on the parameters involved in various distribution series belong to the class of univalent functions or in its subclasses of functions that are convex, starlike, close-to-convex, uniformly convex and so forth. In recent years, the univalent function theorists have shown good affinity towards study of various distribution series by relating them with the area of geometric function theory (see [5], [6], [13], [14]). We need the following lemmas to prove our main results.

Lemma 1.1. (see [2, p. 21, Theorem 2.3]) *A function $f \in \mathcal{A}$ of the form* (1.1) *is in the class $\mathcal{UCV}(\alpha, \beta)$, if it satisfies the following condition:*

$$\sum_{n=2}^{\infty} n[n(1+\alpha) - (\alpha + \beta)]\, |a_n| \leq 1 - \beta. \tag{1.7}$$

Lemma 1.2. (see [2, p. 23, Theorem 2.6]) *A function $f \in \mathcal{A}$ of the form* (1.1) *is in the class $S_p(\alpha, \beta)$, if it satisfies the following condition:*

$$\sum_{n=2}^{\infty} [n(1+\alpha) - (\alpha + \beta)]\, |a_n| \leq 1 - \beta. \tag{1.8}$$

In 2018, Porwal [12] introduce the probability distribution whose probability mass function is

$$\zeta(n) = \frac{a_n}{T}, \qquad n = 0, 1, 2, 3, ..., \tag{1.9}$$

where $T = \sum\limits_{n=0}^{\infty} a_n$ is convergent series for $a_n \geq 0, \forall n \in N$. Also, we introduce the series

$$\psi(p) = \sum_{n=0}^{\infty} a_n p^n. \tag{1.10}$$

It is easy to see that the series given by (1.10) is convergent for $|p| \leq 1$. Further, Porwal [12] introduce a power series whose coefficients are probabilities of the generalized distribution for functions as follows:

$$K_\psi(z) = z + \sum_{n=2}^{\infty} \frac{a_{n-1}}{T} z^n. \tag{1.11}$$

Now, we define

$$PK_\psi(z) = 2z - K_\psi(z) = z - \sum_{n=2}^{\infty} \frac{a_{n-1}}{T} z^n. \tag{1.12}$$

The Hadamard product (or convolution) of two power series $f(z) = \sum\limits_{n=2}^{\infty} a_n\, z^n$ and $g(z) = \sum\limits_{n=2}^{\infty} b_n\, z^n$ is defined as

$$(f * g)(z) = (g * f)(z) = \sum_{n=2}^{\infty} a_n b_n z^n.$$

Next, we introduce the convolution operator $PK_\psi(f, z)$ for function f of the form (1.1) as follows

$$PK_\psi(f, z) = PK_\psi(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{a_{n-1}}{T} |a_n| z^n. \tag{1.13}$$

Motivated by results on connections between various subclasses of analytic univalent functions by using special functions (for details, see [3], [9], [15], [21], [22]), we establish connections between distribution function and univalent functions. We also studied relation between the classes $\mathcal{UCV}(\alpha, \beta)$ and $S_p(\alpha, \beta)$ by applying the convolution operator given by (1.13).

2. Main Results

Theorem 2.1. *A function $PK_\psi(z)$ is in $\mathcal{TS}_p(\alpha, \beta)$ if and only if*

$$\frac{1}{T}\Big[(1+\alpha)\psi'(1) + (1-\beta)[\psi(1) - \psi(0)]\Big] \le 1 - \beta. \tag{2.1}$$

Proof. Since

$$PK_\psi(z) = z - \sum_{n=2}^{\infty} \frac{a_{n-1}}{T} z^n,$$

in view of Lemma 1.2, it is sufficient to show that

$$\sum_{n=2}^{\infty} [n(1+\alpha) - (\alpha+\beta)] \left|\frac{a_{n-1}}{T}\right| \le 1 - \beta. \tag{2.2}$$

Now, let

$$\begin{aligned} \varrho_1(\alpha, \beta, T) &= \sum_{n=2}^{\infty} [n(1+\alpha) - (\alpha+\beta)] \left|\frac{a_{n-1}}{T}\right| \\ &= \frac{1}{T} \sum_{n=2}^{\infty} [n + n\alpha - \alpha - \beta] |a_{n-1}| \end{aligned}$$

Writing $n = (n-1)+1$, we get

$$\begin{aligned}\varrho_1(\alpha,\beta,T) &= \frac{1}{T}\sum_{n=2}^{\infty}[(1+\alpha)(n-1)+1-\beta]\,|a_{n-1}| \\ &= \frac{1}{T}\left[(1+\alpha)\sum_{n=2}^{\infty}(n-1)a_{n-1}+(1-\beta)\sum_{n=2}^{\infty}a_{n-1}\right] \\ &= \frac{1}{T}\left[(1+\alpha)\sum_{n=1}^{\infty}na_n+(1-\beta)\sum_{n=1}^{\infty}a_n\right] \\ &= \frac{1}{T}\left[(1+\alpha)\psi^{'}(1)+(1-\beta)\Big[\psi(1)-\psi(0)\Big]\right]\end{aligned}$$

which evidently completes the proof of Theorem 2.1. □

Theorem 2.2. *A function $PK_\psi(z)$ is in $\mathcal{UCV}(\alpha,\beta)$ if and only if*

$$\frac{1}{T}\Big[(1+\alpha)\psi''(1)+(3+2\alpha-\beta)\psi'(1)+(1-\beta)[\psi(1)-\psi(0)]\Big] \leq 1-\beta. \quad (2.3)$$

Proof. Since

$$PK_\psi(z) = z - \sum_{n=2}^{\infty}\frac{a_{n-1}}{T}z^n,$$

in view of Lemma 1.1, it is sufficient to show that

$$\sum_{n=2}^{\infty} n\Big[n(1+\alpha)-(\alpha+\beta)\Big]\left|\frac{a_{n-1}}{T}\right| \leq 1-\beta. \quad (2.4)$$

Now, let

$$\begin{aligned}\varrho_2(\alpha,\beta,T) &= \sum_{n=2}^{\infty} n\Big[n(1+\alpha)-(\alpha+\beta)\Big]\left|\frac{a_{n-1}}{T}\right| \\ &= \frac{1}{T}\sum_{n=2}^{\infty}\Big[n^2(1+\alpha)-n(\alpha+\beta)\Big]|a_{n-1}|\end{aligned}$$

Writing $n^2 = (n-2)(n-1) + 3(n-1) + 1$ and $n = (n-1) + 1$, we get

$$\begin{aligned}
\varrho_2(\alpha, \beta, T) &= \frac{1}{T}\sum_{n=2}^{\infty}\Big[(1+\alpha)[(n-2)(n-1)+3(n-1)+1] + n(\alpha+\beta)\beta\Big]\, a_{n-1} \\
&= \frac{1}{T}\Bigg[(1+\alpha)\sum_{n=2}^{\infty}(n-2)(n-1)a_{n-1} \\
&\qquad +(3+2\alpha-\beta)\sum_{n=2}^{\infty}(n-1)a_{n-1} + (1-\beta)\sum_{n=2}^{\infty}a_{n-1}\Bigg] \\
&= \frac{1}{T}\Bigg[(1+\alpha)\sum_{n=1}^{\infty}n(n-1)a_n \\
&\qquad +(3+2\alpha-\beta)\sum_{n=1}^{\infty}na_n + (1-\beta)\sum_{n=1}^{\infty}a_n\Bigg] \\
&= \frac{1}{T}\Big[(1+\alpha)\psi''(1) + (3+2\alpha-\beta)\psi'(1) + (1-\beta)[\psi(1)-\psi(0)]\Big].
\end{aligned}$$

which evidently completes the proof of Theorem 2.2. □

Corollary 2.3. *A function $PK_\psi(z)$ is in $\mathcal{S}^*(\beta)$ if and only if*

$$\frac{1}{T}\Big[\psi'(1) + (1-\beta)[\psi(1)-\psi(0)]\Big] \leq 1-\beta. \tag{2.5}$$

Corollary 2.4. *A function $PK_\psi(z)$ is in $\mathcal{K}(\beta)$ if and only if*

$$\frac{1}{T}\Big[\psi''(1) + (3-\beta)\psi'(1) + (1-\beta)[\psi(1)-\psi(0)]\Big] \leq 1-\beta. \tag{2.6}$$

3. Inclusion Properties

we say that a function $f \in \mathcal{A}$ is in the class $\mathcal{R}^\tau(\gamma, \delta)$ if it satisfies the following inequality:

$$\left|\frac{f'(z)-1}{(\gamma-\delta)\tau - \delta[f'(z)-1]}\right| < 1, \quad (z \in \mathbb{U}) \tag{3.1}$$

where $\tau \in \mathbb{C}\backslash\{0\}, -1 \leq \gamma < \delta \leq 1$. This class was introduced by Dixit and Pal [4]. Making use of the following lemma, study the action of the generalized distribution series on the class $PK_\psi(f, z)$.

Lemma 3.1. [4] *If $f \in \mathcal{R}^\tau(\gamma, \delta)$ is of the form* (1.1) *then*

$$|a_n| = (\gamma-\delta)\frac{|\tau|}{n}, \;\; n \in N\backslash\{1\}. \tag{3.2}$$

Theorem 3.2. *Let $f \in \mathcal{R}^\tau(\gamma, \delta)$ ($\tau \in \mathbb{C}\backslash\{0\}, -1 \leq \gamma < \delta \leq 1$) is of the form* (1.1) *if the inequality*

$$\frac{(\gamma-\delta)|\tau|}{T}\Big[(1+\alpha)\psi'(1) + (1-\beta)[\psi(1)-\psi(0)]\Big] \leq 1-\beta. \tag{3.3}$$

is satisfied, then $PK_\psi(f, z) \in \mathcal{UCV}(\alpha, \beta)$.

Proof. In view of Lemma 1.1, it is sufficient to show that

$$\sum_{n=2}^{\infty} n\Big[n(1+\alpha)-(\alpha+\beta)\Big]\left|\frac{a_{n-1}}{T}\right||a_n| \le 1-\beta. \tag{3.4}$$

Since $f \in \mathcal{R}^{\tau}(\gamma,\delta)$, then by Lemma 3.1 we have

$$|a_n| = \frac{(\gamma-\delta)|\tau|}{n}.$$

Let

$$\begin{aligned} \varrho_3(\alpha,\beta,T) &= \frac{1}{T}\sum_{n=2}^{\infty} n\Big[n(1+\alpha)-(\alpha+\beta)\Big]|a_{n-1}||a_n| \\ &\le \frac{(\gamma-\delta)|\tau|}{T}\sum_{n=2}^{\infty}\Big[n(1+\alpha)-(\alpha+\beta)\Big]a_{n-1} \end{aligned}$$

Writing $n=(n-1)+1$, we get

$$\begin{aligned} \varrho_3(\alpha,\beta,T) &= \frac{(\gamma-\delta)|\tau|}{T}\sum_{n=2}^{\infty}\Big[(1+\alpha)(1-n)+(1-\beta)\Big]a_{n-1} \\ &= \frac{(\gamma-\delta)|\tau|}{T}\left[(1+\alpha)\sum_{n=2}^{\infty}(n-1)a_{n-1}+(1-\beta)\sum_{n=2}^{\infty}a_{n-1}\right] \\ &= \frac{(\gamma-\delta)|\tau|}{T}\left[(1+\alpha)\sum_{n=1}^{\infty}na_n+(1-\beta)\sum_{n=1}^{\infty}a_n\right] \\ &= \frac{(\gamma-\delta)|\tau|}{T}\Big[(1+\alpha)\psi'(1)+(1-\beta)[\psi(1)-\psi(0)]\Big]. \end{aligned}$$

which evidently completes the proof of Theorem 3.2. □

Corollary 3.3. *Let $f \in \mathcal{R}^{\tau}(\gamma,\delta)$ ($\tau \in \mathbb{C}\setminus\{0\}, -1 \le \gamma < \delta \le 1$) is of the form* (1.1) *if the inequality*

$$\frac{(\gamma-\delta)|\tau|}{T}\Big[\psi'(1)+(1-\beta)[\psi(1)-\psi(0)]\Big] \le 1-\beta. \tag{3.5}$$

is satisfied, then $PK_{\psi}(f,z) \in \mathcal{K}(\beta)$.

4. An Integral Operator

In this section, we introduce an integral operator $PI_{\psi}(z)$ as follows:

$$PI_{\varphi}(f,z) = \int_0^z \frac{PK_{\varphi(t)}}{t}dt \tag{4.1}$$

and we obtain a necessary and sufficient condition for $PI_{\varphi}(z)$ belonging to the class $\mathcal{U}CV(\alpha,\beta)$.

Theorem 4.1. *If $PK_{\varphi}(z)$ is defined by* (1.11)*, then $PI_{\varphi}(z)$ defined by* (4.1) *in the class $\mathcal{U}CV(\alpha,\beta)$, if and only if* (4.1) *satisfies*

$$\frac{1}{T}\Big[(1+\alpha)\psi'(1)+(1-\beta)[\psi(1)-\psi(0)]\Big] \le (1-\beta). \tag{4.2}$$

Proof. Since
$$PI_{\varphi}(f,z) = z - \sum_{n=2}^{\infty} \frac{a_{n-1}}{nT} z^n.$$
In view of Lemma 1.1, it is sufficient to show that
$$\sum_{n=2}^{\infty} n[n(1+\alpha) - (\alpha+\beta)] \frac{a_{n-1}}{nT} \leq 1-\beta. \tag{4.3}$$
Now, let
$$\begin{aligned}\varrho_4(\alpha,\beta,T) &= \sum_{n=2}^{\infty} n[n(1+\alpha) - (\alpha+\beta)] \frac{a_{n-1}}{nT} \\ &= \frac{1}{T}\sum_{n=2}^{\infty}[n + n\alpha - \alpha - \beta]a_{n-1}\end{aligned}$$
Writing $n = (n-1)+1$, we get
$$\begin{aligned}\varrho_4(\alpha,\beta,T) &= \frac{1}{T}\sum_{n=2}^{\infty}[(1+\alpha)(n-1) + 1 - \beta]\, a_{n-1} \\ &= \frac{1}{T}\left[(1+\alpha)\sum_{n=2}^{\infty}(n-1)a_{n-1} + (1-\beta)\sum_{n=2}^{\infty} a_{n-1}\right] \\ &= \frac{1}{T}\left[(1+\alpha)\sum_{n=1}^{\infty} na_n + (1-\beta)\sum_{n=1}^{\infty} a_n\right] \\ &= \frac{1}{T}\left[(1+\alpha)\psi^{'}(1) + (1-\beta)\Big[\psi(1) - \psi(0)\Big]\right]\end{aligned}$$
which evidently completes the proof of Theorem 4.1. □

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S. D. Bhourgunde: Department of Mathematics, Bhivrabai Sawant Institute of Technology and Research, Pune-412007 (MS), India.
Email address: `sdbhourgunde@gmail.com`

S. D. Thikane: Department of Mathematics, Jaysingpur college, Jaysingpur-416101 (MS), India.
Email address: `sdtikhane@gmail.com`

M. G. Shrigan: Department of Mathematics, Bhivrabai Sawant Institute of Technology and Research, Pune-412007 (MS), India.
Email address: `mgshrigan@gmail.com`

M. T. Gophane: Department of Mathematics, Shivaji University , Kolahapur-416004 (MS), India
Email address: `gmachchhindra@gmail.com`