Global and Stochastic Analysis
Vol. 4 No. 1, January (2017), 171-181

MUK PUBLICATIONS
Open Access Publisher

ON MODULATION-TYPE SPACES $\mathcal{H}^p_\omega(\mathcal{G})$

DEEPAK KARTIKEY*, ASHISH KUMAR, AND J.K.MAITRA

ABSTRACT. The aim of this paper is to construct Modulation -type spaces $\mathcal{H}^p_\omega(\mathcal{G})$, $1 \leq p < \infty$. By applying Gabor Transform $V_g f$ function $f \in L^2(\mathcal{G})$ with respect to window function $g \in L^2(\mathcal{G})$, $\mathcal{G}$ being locally compact abelian group and ω a Beurling-Domar weight function.We define suitable norm on this space and prove that$\mathcal{H}^p_\omega(\mathcal{G})$ becomes a Banach space and is an essential Banach Convolution module over $L^1_\omega(\mathcal{G})$.Also we define the space $S^p_\omega(\mathcal{G}) = L^1_\omega(\mathcal{G}) \bigcap \mathcal{H}^p_\omega(\mathcal{G}), 1 \leq p < \infty$, and endow it with the sum norm and show that $S^p_\omega(\mathcal{G})$ becomes a Banach convolution observed that it is segal algebra.

1. Introduction

Modulation spaces were, originally investigated by H.G.Feichtinger [2]. For a detailed of the theory of modulation spaces we refer K.Gröchenig's text [5, ch. 11-13 (215-299)]. And also account of the development of modulation spaces,including the Feichtinger algebra in particular .

My work outcome from the Research papers [7,8],Gürkanli and Sandikci has studied some properties on Lorentz-type modulation spaces $M(p,q)(R^d)$ and Lorentz mixed norm on modulation spaces $M(P,Q)$ respectively. Used them we construct modulation-type spaces $H^p_\omega(\mathcal{G})$ on locally compact abelian group and define the space $S^p_\omega(\mathcal{G})$.

In the present paper is organized as follows. In section 2,we provide necessary notation and concepts,In section 3,define the space $H^p_\omega(\mathcal{G})$ with Gabor transform $V_g f$ on locally compact abelian group $\mathcal{G}$ and prove that it is Banach Space.and also show that translation invariant and is an essential Banach convolution module over $L^1_\omega(G)$.In last section we define a space $S^p_\omega(\mathcal{G})$ and prove that it is Banach convolution algebra.Finally we observed that it is seal algebra.

2. Preliminaries

we adopt notation and definitions from [6,9,10] Let $\mathcal{G}$ be a locally compact abelian group and $\hat{\mathcal{G}}$ its dual group consisting of all continuous characters on $\mathcal{G}$. Let dx and $d\xi$ be the normalized Haar measures on $\mathcal{G}$ and $\hat{\mathcal{G}}$ respectively.We assume that $\omega : \mathcal{G} \mapsto R^+$ is weight function on $\mathcal{G}$ satisfying the Beurling-Domar

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2010 *Mathematics Subject Classification.* 43A70,43A15,43A10,42C40.

Key words and phrases. Gabor Transform, Modulation space, seal algebra, BD-Condition, LCA groups.

* The author is thankful to UGC for the financial support under the RGN Fellowship Government of India.

(BD) Condition:

$$\sum_{n\geq 1}^{\infty} n^{-2} log\omega(x^n) < \infty, \forall x \in \mathcal{G}$$

A function ω on $\mathcal{G}$ is called an m-moderate weight function provided

$$\omega(x+y) \leq \omega(x)m(y) \forall\, x, y \in \mathcal{G}.$$

We denote the translation (Time-shifts) and modulation (Frequency-shifts)by T_x and M_ξ such that

$$T_x f(t) = f(t-x)\,\forall x, t \in \mathcal{G}$$

and

$$M_\xi f(t) = (t, \xi) f(t)\,\forall\, t \in \mathcal{G},\, \xi \in \hat{\mathcal{G}}.$$

The fourier transform $\hat{f}(\xi)$ of $f \in L^1(\mathcal{G})$ such that

$$\hat{f}(\xi) = \int_{\mathcal{G}} (-t, \xi) f(t) dt;\, t \in \mathcal{G},\, \xi \in \hat{\mathcal{G}}.$$

The operators of the form $T_x M_\xi$ or $M_\xi T_x$ are called Time-Frequency shifts.We observe that the canonical commutation relations

$$T_x M_\xi = (-x, \xi) M_\xi T_x\ , \forall\, x \in \mathcal{G}\, \xi \in \hat{\mathcal{G}}.$$

$$\begin{aligned}
T_x M_\xi(t) &= M_\xi f(t-x) \\
&= (t-x, \xi) f(t-x) \\
&= (-x, \xi)(t, \xi) f(t-x) \\
&= (-x, \xi)(t, \xi) T_x f(t) \\
&= (-x, \xi) M_\xi T_x f(t) \\
\Rightarrow T_x M_\xi &= (-x, \xi) M_\xi T_x\ , \forall\, x \in \mathcal{G}\, , \xi \in \hat{\mathcal{G}}.
\end{aligned}$$

We denote the Time-Frequency shift operator by $\pi(\gamma)$ i.e.

$$\pi(\gamma) = M_\xi T_x,\, \gamma = (x, \xi) \in \mathcal{G} \times \hat{\mathcal{G}}$$

and also we denote the phase space $\mathcal{G} \times \hat{\mathcal{G}}$ by Γ i.e. any element $\gamma \in \Gamma$ is the form $\gamma = (x, \xi) \in \mathcal{G} \times \hat{\mathcal{G}} = \Gamma$ where $x \in \mathcal{G},\, \xi \in \hat{\mathcal{G}}$.

It is know that Γ is unimodular.We shall use following laws and computation rules [6 ,page 152]which is originally introduced by Feichtinger and Kozek " Quantinzation of TF lattice-invariant operators on elementary LCA groups,in Gabor Analysis and Alogrithm, (Birkhäuser 1998) "

$$\begin{aligned}
\gamma_1.\gamma_2 &= (x_1, \xi_1)(x_2, \xi_2) &= (x_1 + x_2, \xi_1 \xi_2), \\
\gamma_1^{-1} &= (x, \xi)^{-1} \quad = (-x, \xi) &= (x, -\xi),
\end{aligned}$$

$\pi(\gamma) = M_\xi T_x$, we easily see that

$$\pi^*(\gamma) = \overline{(x, \xi)} \pi(-\gamma) = (-x, \xi) \pi(-\gamma)$$

and

$$\pi(\gamma_1)\pi(\gamma_2) = \overline{(x_1, \xi_2)} \pi(\gamma_1 + \gamma_2) = (-x_1, \xi_2) \pi(\gamma_1 + \gamma_2).$$

2.1. Weighted Banach space on $\mathcal{G} \times \hat{\mathcal{G}} = \Gamma$. Let $L^p_\omega \mathcal{G}), 1 \leq p < \infty$, the space of functions given by

$$L^p_\omega \mathcal{G} = \{f :\| f \|_{p,\omega} \;=\; (\int_{\mathcal{G}} \mid f(x) \mid^p \omega^p(x) dx)^{1/p} < \infty\}. \qquad (2.1.1)$$

The space $L^p_\omega(\mathcal{G})$ is Banach space under the norm (2.1.1).In case $p = \infty$ the space$L^p_\omega(\mathcal{G})$ denotes the space of all measurable functions such that

$$\| f \|_{\infty,\omega} \;=\; ess\ sup_{x\in\mathcal{G}}\{\mid f(x) \mid \omega(x) < \infty\}. \qquad (2.1.2)$$

It is well know that the space $L^p_\omega(\mathcal{G})$ is translation invariant ,$L^p_\omega(\mathcal{G}), 1 \leq p < \infty$. is a reflexive Banach space and $L^1_w(\mathcal{G})$ is commutative Banach algebra with respect to convolutions, which is well know as Beurling algebra.Also $L^p_w(\mathcal{G})$ is a convolution module with respect to $L^1_w(\mathcal{G})$.i.e. the following properties are satisfied:

$$L^p_\omega * L^1_\omega \subseteq L^p_\omega \qquad (2.1.3)$$

and

$$\| (g * f) \|_{p,\omega} \;\leq\; \| f \|_{p,\omega} \| g \|_{1,\omega}$$

$$\text{for all } f \in L^p_\omega(\mathcal{G}) \text{ and } g \in L^1_\omega(\mathcal{G}).$$

Through out this paper we assume that ω is m-moderate and satisfies (BD) condition (2.1).We define $L^p_\omega(\Gamma), 1 \leq p < \infty$, given by

$$L^p_\omega(\Gamma) \;=\; \{F :\| F \|_{p,\omega} = (\int_{\Gamma} (\mid F(x,\xi) \mid^p \omega^p(x) dx) d\xi)^{1/p} < \infty\}. \qquad (2.1.4)$$

The space $L^p_\omega(\Gamma)$is weighted Banach space under the norm (2.1.4). In case $p = \infty$ we define the space $L^\infty_\omega(\Gamma)$ as the space of all measurable function F on Γ such that

$$\| F \|_{\infty,\omega} \;=\; ess\ \ supp_{x\in\mathcal{G},\xi\in\hat{\mathcal{G}}}\{\mid \Gamma(x,\xi) \mid \omega(x) < \infty.\} \qquad (2.1.5)$$

The unimodularity of Γ, it is clear that the left and right translation operators given by

$$L_\gamma F(\gamma') \;=\; F(\gamma^{-1}\gamma')$$

and

$$R_\gamma F(\gamma') \;=\; F(\gamma'\gamma); \gamma, \gamma' \in \Gamma,$$

act isometrically on the weighted Banach spaces $L^p_w(\Gamma), 1 \leq p \leq \infty$.It is well know that the space $L^p_\omega(\Gamma)$ is translation invariant, $L^1_\omega(\Gamma)$ is Banach algebra under Convolutions, not necessarily commutative. And also the space $L^p_\omega(\Gamma)$ Banach convolution module over $L^1_\omega(\Gamma)$ i.e.

$$L^p_\omega(\Gamma) * L^1_\omega(\Gamma) \subseteq L^p_\omega(\Gamma) \quad 1 \leq p \leq \infty, \qquad (2.1.6)$$

and

$$\| (F * G) \|_{P,\omega} \;\leq\; \| F \|_{P,\omega} \| G \|_{P,\omega}$$

$$\text{for all } F \in L^p_\omega(\Gamma) \text{ and } G \in L^1_\omega(\Gamma).$$

3. The Space $\mathcal{H}^p_\omega(\mathcal{G})$ With Gabor Transform

The Gabor transform of a function $f \in L^2(\mathcal{G})$ with respect to window function g is given by

$$\begin{aligned} V_g f(\gamma) &= \int_{\mathcal{G}} f(t)\bar{g}(t-x)(-t,\xi)dt \\ &= \langle f, M_\xi T_x g\rangle \\ &= \langle f, \pi(\gamma) g\rangle \\ &= \langle f, M_\xi T_x g\rangle \\ &= (f.T_x\bar{g})^\wedge(\xi) \\ &= \langle \hat{f}, T_\xi M_{-x}\hat{g}\rangle. \end{aligned}$$

Where $\bar{g}$ is the conjugate function of g, $\hat{f}$ is fourier Transform of the function $f \in L^1(\mathcal{G})$ is given in section 2. Now we denotes $A^1_\omega(\mathcal{G}), 1 \leq p < \infty,$ the class of analyzing vectors given by [3, P 317]such that

$$A^1_\omega(\mathcal{G}) = \{g \in L^2(\mathcal{G}), V_g g \in L^1_\omega(\Gamma)\}. \tag{3.1}$$

For a fixed an arbitrary non-zero element $g \in A^1_\omega(\mathcal{G})$,the space $\mathcal{H}^p_\omega(\mathcal{G})$ is defined by

$$\mathcal{H}^p_\omega(\mathcal{G}) = \{f \in L^2(\mathcal{G}), V_g f \in L^p_\omega(\Gamma)\}. \tag{3.2}$$

and endow it with the norm

$$\|f\|_{\mathcal{H}^p_\omega(\mathcal{G})} = \|V_g f\|_{L^p_\omega(\Gamma)} \quad 1 \leq p < \infty. \tag{3.3}$$

In case $p = 1$ and ω a constant the space $\mathcal{H}^p_\omega(\mathcal{G})$ reduces to the well-know Feichtinger algebra $\mathcal{S}_0(\mathcal{G})$. The above definitions imply that the continuous embeddings

$$\mathcal{H}^p_\omega(\mathcal{G}) \hookrightarrow L^2(\mathcal{G}) \hookrightarrow \mathcal{H}^p_\omega\ (\mathcal{G}).$$

Where $\mathcal{H}^p_\omega(\mathcal{G})$ is the space of all continuous conjugate liner functionals on $\mathcal{H}^p_\omega(\mathcal{G})$. [3] the definitions of $\mathcal{H}^p_\omega(\mathcal{G})$ is independent of choice of $g \in A'_\omega(\mathcal{G})$.Since the space$\mathcal{H}^p_\omega(\mathcal{G})$ is analogous to the modulation spaces so we call it modulation-type spaces.

Theorem 3.1. *The space$\mathcal{H}^p_\omega(\mathcal{G})$ is Banach space under the norm*

$$\|f\|_{\mathcal{H}^p_\omega(\mathcal{G})} = \|V_g f\|_{L^p_\omega(\Gamma)} \quad 1 \leq p < \infty.$$

Proof. it is sufficient to show that $\mathcal{H}^p_\omega(\mathcal{G})$ is complete i.e. every cauchy sequence in $\mathcal{H}^p_\omega(\mathcal{G})$ is convergent in $\mathcal{H}^p_\omega(\mathcal{G})$.Suppose $\{f_n\}$ is a cauchy sequence in $\mathcal{H}^p_\omega(\mathcal{G})$,this implies that $\{V_g f_n\}$ is cauchy sequence in $L^p_\omega(\Gamma)$, since $L^p_\omega(\Gamma)$ is Banach space $\{V_g f_n\}$ converges to a function h in $L^p_\omega(\Gamma)$ this implies that there exists a subsequence $\{V_g f_{nk}\}$ of $\{V_g f_n\}$ which pointwise convergent to h almost everywhere .Hence, for any given $\epsilon > 0, \exists\ f \in L^1_w(\mathcal{G})$ and $n_0 \in N$ such that

$$\|f_n - f\|_{1,w} < \frac{\epsilon}{\|\bar{g}\|_{1,w}}$$

for all $n \geq n_0$.

If we apply the lemma 3.1.1 in [5 page 39] and holder's inequality, we see that

$$\begin{aligned} |V_g f_n(x,\xi) - V_g f(x,\xi)| &= |V_g(f_n - f)(x,\xi)| \\ &= |\langle (f_n - f), \pi(\gamma)g\rangle| \\ &\leq \|(f_n - f)\|_\infty \|\overline{\pi(\gamma)g}\|_1 \\ &= \|(f_n - f)\|_\infty \ \ \|\overline{g}\|_1 \\ &\leq \|(f_n - f)\|_1 \ \ \|\overline{g}\|_1 \\ &\leq \|f_n - f\|_{1,w}\|\overline{g}\|_{1,w} \quad \text{for } \ w \geq 1 \\ &< \frac{\epsilon}{\|\overline{g}\|_1, w}\|\overline{g}\|_1, w = \epsilon. \quad \text{since } L^1\mathcal{G} \text{ is a Banach space.} \end{aligned}$$

Hence the sequence $\{V_g f_n\}$ is point-wise convergent to $V_g\ f$.
Also we have

$$\begin{aligned} |V_g f_{n_k}(\gamma) - V_g f(\gamma)| &\leq |V_g f_{n_k}(\gamma) - V_g f_n(\gamma)| + |V_g f_n(\gamma) - V_g f(\gamma)| \\ &\leq \|f_{n_k} - f_n\|_{1,w}\ \|\overline{g}\|_{1,w} + \|f_n - f\|_{1,w}\ \|\overline{g}\|_{1,w} \\ &< 2\epsilon. \end{aligned}$$

$\Rightarrow$ The subsequenc $\{V_g f_{n_k}\}$ is pointwise convergent to $V_g f$. From above we see that

$$\begin{aligned} |V_g f(\gamma) - h(\gamma)| &\leq |V_g f(\gamma) - V_g f_{n_k}(\gamma)| + |V_g f_{n_k}(\gamma) - h(\gamma)| \\ &\leq 2\epsilon + 2\epsilon = 4\epsilon. \end{aligned}$$

$$V_g f(\gamma) = h(\gamma) \ \text{ a.e. } \ \ \epsilon \to 0$$

Where $\gamma = (x,\xi) \in \mathcal{G} \times \hat{\mathcal{G}} = \Gamma$, $\quad x \in \mathcal{G},\ \xi \in \hat{\mathcal{G}}$.

$$\begin{aligned} \|f_n - f\|_{\mathcal{H}^p_w} &= \|V_g(f_n - f)\|_{p,w} \\ &= \|V_g f_n - V_g f\|_{p,w} \\ &< \epsilon. \quad \text{for all } n > n_0 \end{aligned}$$

Therefore $\mathcal{H}^p_\omega(\mathcal{G})$ is a Banach space. ☐

Lemma 3.2. *The space $\mathcal{H}^p_w(\mathcal{G})$ is Translation invariant and translation operator is continuous in $\mathcal{H}^p_w(\mathcal{G})$ for $x \in \mathcal{G}$*

Proof. Suppose $f \in \mathcal{H}^p_w(\mathcal{G})$ and $x \in \mathcal{G}$. We knows that $\|T_{(x,\xi)}V_g f\|_{p,w} = \|V_g f\|_{p,w}$. It is also know that convariance property lemma 3.1.3 in [5 page no. 41]

$$|V_g(T_x M_\xi f)(\mu,\vartheta)| = |V_g f(\mu - x, \vartheta - \xi)| = |T_{(x,\xi)}V_g f(\mu,\vartheta)|.$$

from above we have

$$\|V_g(T_x M_\xi f)\|_{p,w} = \|T_{(x,\xi)}V_g f\|_{p,w} = \|V_g f\|_{p,w} = \|f\|_{\mathcal{H}^p_w(\mathcal{G})}.$$

from this we obtain

$$\|T_x f\|_{\mathcal{H}^p_w(\mathcal{G})} = \|V_g(T_x f)\|_{p,w} = \|T_{(x,0)}V_g f\|_{p,w} = \|V_g f\|_{p,w} = \|f\|_{\mathcal{H}^p_w(\mathcal{G})}.$$

Hence $\mathcal{H}^p_w(\mathcal{G})$ is translation invariant. Now we shall show that translation is continuous in $\mathcal{H}^p_w(\mathcal{G})$. It is knows that translation is continuous in $L^p_w(\Gamma)$ thus we see that

$$\begin{aligned} \|T_x f_n - f\|_{\mathcal{H}^p_w} &= \|V_g(T_x f_n - f)\|_{p,w} \\ &= \|V_g(T_x f_n) - V_g f\|_{p,w} \\ &< \epsilon. \end{aligned}$$

Therefore the translation is continuous in $\mathcal{H}^p_w(\mathcal{G})$.

Theorem 3.3. *$\mathcal{H}^p_\omega(\mathcal{G})$is an essential Banach convolution module over $L^1_\omega(\mathcal{G})$.*

Proof. Let $f \in \mathcal{H}^p_\omega(\mathcal{G})$ and $h \in L^1_\omega(\mathcal{G}$ and we shall make use lemma 3.1.1 in [5 page no 39].
Hence we have

$$\begin{aligned} \|(f * h)\|_{\mathcal{H}^p_w(\mathcal{G})} &= \|V_g(f * h)\|_{p,w} \\ &= \|\langle (f * h),\ \pi(\gamma)g\rangle\|_{p,w} \\ &= \|\int_{\mathcal{G}} (f * h)(y)\ \bar{g}(y - x)\ (-y, \xi)\ dy\|_{p,w} \\ &= \|\int_{\mathcal{G}} \left[\int_{\mathcal{G}} f(z)\ \ T_z\ \ h(y)dz\right]\ \bar{g}(y - x)\ (-y, \xi)\ dy\|_{p,w} \\ &= \|\int_{\mathcal{G}} f(z) \left[\int_{\mathcal{G}} T_z\ \ h(y)\bar{g}(y - x)\ (-y, \xi)\ dy\right]\ dz\|_{p,w} \\ &= \|\int_{\mathcal{G}} f(z)\langle\ T_z\ h,\ \pi(\gamma)g\rangle\ dz\|_{p,w} \\ &\leq \int_{\mathcal{G}} \|f(z)\ \ V_g\ T_z\ h(\gamma)\|_{p,w} dz \\ &\leq \|f\|_1\ \ \|V_g T_z h\|_{p,w} \\ &\leq \|f\|_{1,w}\ \ \|V_g h\|_{p,w}. \end{aligned}$$

.

since $f \in \mathcal{H}^p_w(\mathcal{G})$ and the translation operator T_x is continuous on $\mathcal{H}^p_w(\mathcal{G})$, for any given $\epsilon > 0, \exists$ a compact neighbourhood U of e in $\mathcal{G}$ such that

$$\|T_x\ f - f\|_{\mathcal{H}^p_w(\mathcal{G})} < \epsilon$$

for all $x \in U$. We suppose that $k \in L^1(\mathcal{G})$ is a non-negative continuous function with compact support such that

$$\text{supp}\ k \subset U$$

and

$$\int_{\mathcal{G}} k(z)\ dz = 1$$

Then we have

$$\begin{aligned}
\|(k*f)-f\|_{\mathcal{H}^p_w(\mathcal{G})} &= \| \left[\int_{\mathcal{G}} k(z)f(y-z)\,dz - \int_{\mathcal{G}} k(z)f(y)\,dz\right] \|_{\mathcal{H}^p_w(\mathcal{G})} \\
&\leq \int_{\mathcal{G}} \|\,[k(z)\,(f(y-z)-f(y))]\,\|_{\mathcal{H}^p_w(\mathcal{G})}\,dz \\
&\leq \int_{\mathcal{G}} |k(z)|\,\|(t_z\,f-f)\|_{\mathcal{H}^p_w(\mathcal{G})}\,dz \\
&< \epsilon \int_{\mathcal{G}} k(z)\,dz \\
&= \epsilon.
\end{aligned}$$

Since $\mathcal{H}^p_w(\mathcal{G})$ is a Banach module over $L^1_w(\mathcal{G})$ and $f*g \in L^1_w(\mathcal{G}) * \mathcal{H}^p_w(\mathcal{G})$ hence $L^1_w(\mathcal{G}) * \mathcal{H}^p_w(\mathcal{G})$ is dense in $\mathcal{H}^p_w(\mathcal{G})$ as in [7]. Thus by module factorization theorem see that that

$$\mathcal{H}^p_w(\mathcal{G}) = L^1_w(\mathcal{G}) * \mathcal{H}^p_w(\mathcal{G}).$$

Therefore $\mathcal{H}^p_\omega(\mathcal{G})$is an essential Banach convolution module over $L^1_\omega(\mathcal{G})$. □

4. A Weighted Segal Algebra on $\mathcal{G}$

We define a space $S^p_w(\mathcal{G}) = L^1_w(\mathcal{G}) \cap \mathcal{H}^p_w(\mathcal{G})$ and equip it with the norm

$$\|f\|_{S^p_w(\mathcal{G})} = \|f\|_{1,w} + \|f\|_{\mathcal{H}^p_w(\mathcal{G})}. \tag{4.1}$$

Lemma 4.1. *For $1 \leq p < \infty$, The space $S^p_w(\mathcal{G})$ is a Banach space under the norm*

$$\|f\|_{S^p_w(\mathcal{G})} = \|f\|_{1,w} + \|f\|_{\mathcal{H}^p_w(\mathcal{G})}.$$

Proof. It is enough to show that every cauchy sequence in $S^p_w(\mathcal{G})$ is convergent. Let $\{f_n\}$ be a Cauchy sequence in $S^p_w(\mathcal{G})$. This implies that $\{f_n\}$ is a Cauchy sequence in $L^1_w(\mathcal{G})$ and $\mathcal{H}^p_w(\mathcal{G})$.Since $L^1_w(\mathcal{G})$ and $\mathcal{H}^p_w(\mathcal{G})$ both are Banach spaces,$\{f_n\}$ converges to a function $f \in L^1_w(\mathcal{G})$ and from the definition of the norm (3.3),it is clear that $\{V_g\, f_n\}$ converges to $h \in L^p_w(\Gamma)$ this implies that there exists a subsequence $\{V_g\, f_{n_k}\}$ of $\{V_g\, f_n\}$ which convergent point-wise to h almost every-where.Hence, for any given $\epsilon > 0, \exists\, f \in L^1_w(\mathcal{G})$ and $n_0 \in N$ such that

$$\|f_n - f\|_{1,w} < \frac{\epsilon}{\|\bar{g}\|_{1,w}} \tag{4.2}$$

for all $n \geq n_0$.

If we apply the lemma 3.1.1 in [5 page 39] and holder's inequality, we see that

$$\begin{aligned}
|V_g\, f_n(x,\xi) - V_g\, f(x,\xi)| &= |V_g\,(f_n - f)(x,\xi)| \\
&= |\langle (f_n - f),\ \pi(\gamma)\, g\rangle| \\
&\leq \|f_n - f\|_\infty\, \|\overline{\pi(\gamma) g}\|_1 \\
&= \|f_n - f\|_\infty \|\bar{g}\|_1 \\
&\leq \|f_n - f\|_1 \|\bar{g}\|_1 \\
&\leq \|f_n - f\|_{1,w} \|\bar{g}\|_{1,w} \quad \text{for } w \geq 1 \\
&\leq \frac{\epsilon}{\|\bar{g}\|_{1,w}} \times \|\bar{g}\|_{1,w} = \epsilon.
\end{aligned}$$

This means the sequence $\{V_g\, f_n\}$ is point-wise convergent to $V_g\, f$.
Also we have

$$\begin{aligned}
|V_g\, f_{n_k}(x,\xi) - V_g\, f(x,\xi)| &\leq |V_g\, f_{n_k}(x,\xi) - V_g\, f_n(x,\xi)| + |V_g\, f_n(x,\xi) - V_g\, f(x,\xi)| \\
&\leq \|f_{n_k} - f_n\|_{1,w}\, \|\bar{g}\|_{1,w} + \|f_n - f\|_{1,w}\, \|\bar{g}\|_{1,w} \\
&= 2\epsilon.
\end{aligned}$$

$\Rightarrow \{V_g\, f_{n_k}\}$ converges point-wise to $V_g\, f$. from above we obtain

$$\begin{aligned}
|V_g\, f(\gamma) - h(\gamma)| &\leq |V_g\, f_{n_k}(\gamma) - V_g\, f(\gamma)| + |V_g\, f_{n_k}(\gamma) - h(\gamma)| \\
&< 4\epsilon \\
\Rightarrow V_g\, f(\gamma) &= h(\gamma) \quad \text{a.e.}
\end{aligned}$$

Where $\gamma = (x,\xi) \in \mathcal{G} \times \hat{\mathcal{G}} = \Gamma$, $x \in \mathcal{G}$, $\xi \in \hat{\mathcal{G}}$. Thus, for any given $\epsilon > 0, \exists\, n_1, n_2 \in N$ such that

$$\|f_n - f\|_{1,w} < \frac{\epsilon}{2}$$

and

$$\|V_g(f_n - f)\|_{p,w} = \|V_g\, f_n - V_g\, f\|_{p,w} < \frac{\epsilon}{2}$$

for all $n > n_1$ and $n > n_2$.

This implies that

$$\begin{aligned}
\|(f_n - f)|S^p_w(\mathcal{G})\| &= \|f_n - f\|_{1,w}| + \|V_g(f_n - f)\|_{p,w} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}.
\end{aligned}$$

for all $n > \max\{n_1, n_2\}$.
Therefore $S^p_w(\mathcal{G})$ is a Banach space. following Theorem show that $S^p_w(\mathcal{G})$ is Segal algebra w.r.t.$L^1_w(\mathcal{G})$. □

Theorem 4.2. *$S^p_w(\mathcal{G})$ is a Segal algebra with respect to $L^1_w(\mathcal{G})$.*

Proof. we shall first show that $S^p_w(\mathcal{G})$ is a Banach algebra under the norm (4.1) for this we shall show that $S^p_w(\mathcal{G})$ is a Banach space with respect to convolution as multiplication.

Let $f, h \in \mathcal{H}_w^p(\mathcal{G})$. Then we have

$$\begin{aligned}
\|V_g(f*h)\|_{p,w} &= \Big\| \int_{\mathcal{G}} (f*g)(y)\bar{g}(y-x)(-y,\xi)\, dy \Big\|_{p.w} \\
&= \Big\| \int_{\mathcal{G}} \Big[\int_{\mathcal{G}} f(z)\, T_z\, h(y)\, dz \Big] \bar{g}(y-x)(-y,\xi)\, dy \Big\|_{p.w} \\
&= \Big\| \int_{\mathcal{G}} f(z) \Big[\int_{\mathcal{G}} T_z\, h(y)\bar{g}(y-x)(-y,\xi)\, dy \Big]\, dz \Big\|_{p.w} \\
&= \Big\| \int_{\mathcal{G}} f(z)\langle T_z h,\ \pi(\gamma)g\rangle\, dz \Big\|_{p.w} \\
&\leq \int_{\mathcal{G}} \Big\| f(z)\langle T_z h,\ \pi(\gamma)g\rangle \Big\|_{p.w}\, dz \\
&\leq \|f\|_{1,w}\ \ \|V_g h\|_{p,w}
\end{aligned}$$

Thus

$$\begin{aligned}
\|(f*h)\|_{S_w^p(\mathcal{G})} &= \|f*h\|_{1,w} + \|(f*h)\|_{\mathcal{H}_w^p(\mathcal{G})} \\
&= \|f*h\|_{1,w} + \|V_g(f*h)\|_{p,w}. \\
&= \|f*h\|_{1,w} + \|f*V_g\ h\|_{p,w}. \\
&\leq \|f\|_{1,w}\ \|h\|_{1,w} + \|f\|_{1,w}\ \|V_g\ h\|_{p,w}. \\
&= \|f\|_{1,w}\ (\|h\|_{1,w} + \|V_g\ h\|_{p,w}). \\
&= \|f\|_{1,w}\ \|h\|_{S_w^p(\mathcal{G})} \\
&\leq \|f\|_{S_w^p}\ \|h\|_{S_w^p}.
\end{aligned}$$

it is easily verify the other conditions to make $S_w^p(\mathcal{G})$ a Banach algebra.

Now we shall prove that $S_w^p(\mathcal{G})$ is a strongly translation invariant and translation is continuous in the norm of topology of $S_w^p(\mathcal{G})$.

suppose $f \in S_w^p(\mathcal{G})$ and $x \in \mathcal{G}$. We knows that $\|T_x f\|_{1,w} = \|f\|_{1,w}$ and $\|T_{(x,\xi)}V_g f\|_{p,w} = \|V_g f\|_{p,w}$. It is also know that convariance property lemma 3.1.3 in [5 page no. 41]

$$|V_g(T_x M_\xi f)(\mu,\vartheta)| = |V_g f(\mu - x, \vartheta - \xi)| = |T_{(x,\xi)}V_g f(\mu,\vartheta)|.$$

from above we have

$$\|V_g(T_x M_\xi f)\|_{p,w} = \|T_{(x,\xi)}V_g f\|_{p,w} = \|V_g f\|_{p,w} = \|f\|_{\mathcal{H}_w^p(\mathcal{G})}.$$

from this we obtain

$$\|T_x f\|_{\mathcal{H}_w^p(\mathcal{G})} = \|V_g(T_x f)\|_{p,w} = \|T_{(x,0)}V_g f\|_{p,w} = \|V_g f\|_{p,w} = \|f\|_{\mathcal{H}_w^p(\mathcal{G})}.$$

Now let $f \in S_w^p(\mathcal{G})$ and $x \in \mathcal{G}$. Then we have

$$\begin{aligned}
\|T_x f\|_{S_w^p(\mathcal{G})} &= \|T_x f\|_{1,w} + \|T_x f\|_{\mathcal{H}_w^p(\mathcal{G})} \\
&= \|T_x f\|_{1,w} + \|T_{x,0} V_g f\|_{p,w} \\
&= \|f\|_{1,w} + \|V_g f\|_{p,w} \\
&= \|f\|_{S_w^p}.
\end{aligned}$$

This implies that $S_w^p(\mathcal{G})$ is a strongly translation invariant. Next we show that translation is continuous in the norm topology of $S_w^p(\mathcal{G})$.It is knows that translation is continuous in $L_w^p(\Gamma)$ Thus we see that

$$\begin{aligned}
\|(T_x f - f)\|_{S_w^p(\mathcal{G})} &= \|T_x f - f\|_{1,w} + \|V_g(T_x f - f)\|_{p,w} \\
&< \epsilon/2 + \|V_g(T_x f) - V_g f\|_{p,w} \\
&< \epsilon/2 + \epsilon/2.
\end{aligned}$$

Therfore the translation is continuous in the norm topology of $S_w^p(\mathcal{G})$.

Lastly we prove that $S_w^p(\mathcal{G})$ is dense in $L_w^1(\mathcal{G})$. It is known that Feichtinger algebra $S_0(\mathcal{G})$ is dense in $L^1(\mathcal{G})$. Then by the definition of $\mathcal{H}_w^p(\mathcal{G})$, it is clear that $\mathcal{H}_w^p(\mathcal{G})$ is dense in $L^1(\mathcal{G})$ This implies that $\mathcal{H}_w^p(\mathcal{G})$ is dense in $L_w^1(\mathcal{G})$.Since

$$S_w^p(\mathcal{G}) = L_w^1(\mathcal{G}) \cap \mathcal{H}_w^p(\mathcal{G}),$$

Hence, $S_w^p(\mathcal{G})$ is dense in $L_w^1(\mathcal{G})$.
Therefore $S_w^p(\mathcal{G})$ is a Segal algebra with respect to $L_w^1(\mathcal{G})$.

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Department of Mathematics & Computer science R.D.University, Jabalpur, India.
E-mail address: kartikey.deepak55@gmail.com

Department of Mathematics & Computer science R.D.University, Jabalpur, India.
E-mail address: ashu_pr2000@yahoo.com

Department of Mathematics & Computer science R.D.University, Jabalpur, India.
E-mail address: jkmrdvv@rediffmail.com