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SOME RESULTS ON $\psi-$HILFER MIXED FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the existence and uniqueness of solutions for $\psi-$Hilfer mixed fractional integrodifferential equation. Also, we study the Ulam-Hyers and Ulam-Hyers-Rassias stability via successive approximation method. Further, we investigate the dependence of solutions on the initial conditions and uniqueness via $\epsilon-$approximated solution.

1. Introduction

In the present paper, we study the global existence and uniqueness of solution, and Ulam-Hyers stability for the $\psi-$Hilfer mixed fractional integrodifferential equations (MFIDE) of the following type

$$^{H}D_{a+}^{\alpha,\beta;\psi}w(t) = f\left(t, w(t), \int_a^t K(t,s)w(s)ds, \int_a^b H(t,s)w(s)ds\right), \tag{1.1}$$

$$I_{a+}^{1-\rho;\psi}w(a) = w_a, \tag{1.2}$$

where $t \in [a,b],\ 0 < \alpha < 1,\ 0 \leq \beta \leq 1,\ ^{H}D_{a+}^{\alpha,\beta;\psi}(.)$ is the (left-sided) $\psi-$Hilfer fractional derivative of order α and type $\beta, I_{a+}^{1-\rho;\psi}$ is (left-sided) fractional integral of order $1-\rho$ with respect to another function ψ in Riemann-Liouville sense and $f : [a,b] \times R \times R \times R \to R$ is a given function that will be specified later.

Recently, several researchers have studied the results such as existence, uniqueness, stabilities and other properties of solutions for the fractional differential and fractional integrodifferential equations by different techniques, see [1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 14, 17, 18, 19, 20, 21] and the detailed literature for fractional calculus can be found in [4, 11, 15, 16].

The paper is organized as follows: some basic definitions and results concerning $\psi-$Hilfer fractional derivative are presented in section 2. Section 3 deals with global existence and uniqueness of solutions of the problem (1.1)-(1.2). In section 4, we discuss Ulam-Hyers (HU) and Ulam-Hyers-Rassias (HUR) stability of $\psi-$Hilfer MFIDE (1.1) via successive approximations. Section 5 concerned with $\epsilon-$approximate solution of the $\psi-$Hilfer MFIDE (1.1).

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2. Preliminaries

Here we present some definitions, notations and results from ([11, 18, 19]) which are used throughout this paper.

Let $0 < a < b < \infty$, $\Delta = [a,b] \subset R_+ = [0,\infty)$, $0 \le \rho < 1$ and $\psi \in C^1(\Delta, R)$ be an increasing function such that $\psi'(x) \neq 0, \forall x \in \Delta$. The weighted spaces $C_{1-\rho;\psi}(\Delta, R)$, $C^{\rho}_{1-\rho;\psi}(\Delta, R)$ and , $C^{\alpha,\beta}_{1-\rho;\psi}(\Delta, R)$ of functions are defined as follows:

(i) $C_{1-\rho;\psi}(\Delta, R) = \{h : (a,b] \to R : (\psi(t) - \psi(a))^{1-\rho} h(t) \in C(\Delta, R)\}$, with the norm $\|h\|_{C_{1-\rho;\psi}} = \max_{t\in\Delta} |(\psi(t) - \psi(a))^{1-\rho} h(t)|$,

(ii) $C^{\rho}_{1-\rho;\psi}(\Delta, R) = \{h \in C_{1-\rho;\psi}(\Delta, R) : D^{\rho}_{a^+} h(t) \in C_{1-\rho;\psi}(\Delta, R)\}$,

(iii) $C^{\alpha,\beta}_{1-\rho;\psi}(\Delta, R) = \{h \in C_{1-\rho;\psi}(\Delta, R) : {}^H D^{\alpha,\beta}_{a^+} h(t) \in C_{1-\rho;\psi}(\Delta, R)\}$.

Definition 2.1. ([11], [15]) The $\psi-$Riemann fractional integral of order $\alpha > 0$ of the function h is given by

$$I^{\alpha;\psi}_{a^+} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{L}^{\alpha}_{\psi}(t,\eta) h(\eta) d\eta,$$

where $\mathcal{L}^{\alpha}_{\psi}(t,\eta) = \psi'(\eta)(\psi(t) - \psi(\eta))^{\alpha-1}$.

Lemma 2.2. *([11]) Let $\alpha > 0$, $\beta > 0$ and $\delta > 0$. Then*

(i) $I^{\alpha;\psi}_{a^+} I^{\beta;\psi}_{a^+} h(t) = I^{\alpha+\beta;\psi}_{a^+} h(t)$

(ii) *If $h(t) = (\psi(t) - \psi(\eta))^{\delta-1}$, then $I^{\alpha;\psi}_{a^+} h(t) = \frac{\Gamma(\delta)}{\Gamma(\alpha+\delta)} (\psi(t) - \psi(\eta))^{\alpha+\delta-1}$.*

We need following results [**?**, **?**] which are useful in the subsequent analysis of the paper.

Lemma 2.3. *([19]) If $\alpha > 0$ and $0 \le \rho < 1$, then $I^{\alpha;\psi}_{a^+}$ is bounded from $C_{\rho;\psi}(\Delta, R)$ to $C_{\rho;\psi}(\Delta, R)$. Also, if $\rho \le \alpha$, then $I^{\alpha;\psi}_{a^+}$ is bounded from $C_{\rho;\psi}(\Delta, R)$ to $C(\Delta, R)$.*

Definition 2.4. ([18]) The $\psi-$Hilfer fractional derivative of a function h of order $0 < \alpha < 1$ and type $0 \le \beta \le 1$, is defined by

$${}^H D^{\alpha,\beta}_{a^+} h(t) = I^{\beta(1-\alpha);\psi}_{a^+} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) I^{(1-\beta)(1-\alpha);\psi}_{a^+} h(t).$$

Lemma 2.5. *([18]) If $h \in C^1(\Delta, R)$, $0 < \alpha < 1$ and $0 \le \beta \le 1$, then*

(i) $I^{\alpha;\psi}_{a^+} {}^H D^{\alpha,\beta}_{a^+} h(t) = h(t) - \Omega^{\rho}_{\psi}(t,a) I^{(1-\beta)(1-\alpha);\psi}_{a^+} h(a)$
where $\Omega^{\rho}_{\psi}(t,a) = \frac{(\psi(t)-\psi(a))^{\rho-1}}{\Gamma(\rho)}$

(ii) ${}^H D^{\alpha,\beta}_{a^+} I^{\alpha;\psi}_{a^+} h(t) = h(t)$.

Definition 2.6. ([11]) Let $\alpha > 0$, $\beta > 0$. The one parameter Mittag-Leffler function is defined as $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, and the two parameter Mittag-Leffler function is defined as $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}$.

3. Existence and Uniqueness results

In this section, we will study the existence and uniqueness results of the Cauchy-type problem (1.1)-(1.2) by applying the following modified version of contraction principle.

Lemma 3.1. [16] *Let χ be a Banach space and let T be an operator which maps the elements of χ into itself for which T^r is a contraction, where r is a positive integer then T has a unique fixed point.*

Theorem 3.2. *Let $0<\alpha<1,\ 0\leq\beta\leq 1$ and $\rho=\alpha+\beta-\alpha\beta$. Let $f:(a,b]\times R\times R\to R$ be a function such that $f\left(t,w(t),\int_a^t K(t,s)w(s)ds,\int_a^b H(t,s)w(s)ds\right)\in C_{1-\rho;\psi}(\Delta,R)$ for any $w\in C_{1-\rho;\psi}(\Delta,R)$, and let f satisfies the Lipschitz condition*

$$|f(t,w_1,z_1,x_1)-f(t,w_2,z_2,x_2)|\leq L[\ |w_1-w_2|+|z_1-z_2|+|x_1-x_2|\ ], \quad (3.1)$$

for all $t\in(a,b]$ and for all $w_1,w_2,z_1,z_2,x_1,x_2\in R$, where $L>0$ is Lipschitz constant. Then the Cauchy problem (1.1)-(1.2) has unique solution in $C_{1-\rho;\psi}(\Delta,R)$.

Proof. The equivalent fractional integral to the initial value problem (1.1)-(1.2) is given by [18], for $t\in(a,b]$,

$$\begin{aligned} w(t)&=\Omega_\psi^\rho(t,a)w_a+I_{a^+}^{\alpha;\psi}f\left(t,w(t),\int_a^t K(t,s)w(s)ds,\int_a^b H(t,s)w(s)ds\right)\\ &=\Omega_\psi^\rho(t,a)w_a+\frac{1}{\Gamma(\alpha)}\int_a^t \mathcal{L}_\psi^\alpha(t,\eta)\\ &\quad\times f\left(\eta,w(\eta),\int_a^\eta K(\eta,\sigma)w(\sigma)d\sigma,\int_a^b H(\eta,\sigma)w(\sigma)d\sigma\right)d\eta. \end{aligned}\quad (3.2)$$

Our aim is to prove that the fractional integral (3.2) has a solution in the weighted space
$C_{1-\rho;\psi}(\Delta,R)$.
Consider the operator T defined on $C_{1-\rho;\psi}(\Delta,R)$ by

$$\begin{aligned} (Tw)(t)&=\Omega_\psi^\rho(t,a)w_a+\frac{1}{\Gamma(\alpha)}\int_a^t \mathcal{L}_\psi^\alpha(t,\eta)\\ &\quad\times f\left(\eta,w(\eta),\int_a^\eta K(\eta,\sigma)w(\sigma)d\sigma,\int_a^b H(\eta,\sigma)w(\sigma)d\sigma\right)d\eta. \end{aligned}\quad (3.3)$$

By Lemma 2.3, it follows that $I_{a^+}^{\alpha;\psi}f\left(t,w(t),\int_a^t K(t,s)w(s)ds,\int_a^b H(t,s)w(s)ds\right)\in C_{1-\rho;\psi}(\Delta,R)$. Clearly, $w_a\Omega_\psi^\rho(t,a)\in C_{1-\rho;\psi}(\Delta,R)$. Therefore, from (3.3), we have $Tw\in C_{1-\rho;\psi}(\Delta,R)$ for any $w\in C_{1-\rho;\psi}(\Delta,R)$. This proves T maps $C_{1-\rho;\psi}(\Delta,R)$ into itself. Note that the fractional integral equation (3.3) can be written as fixed point operator equation $w=Tw,\ w\in C_{1-\rho;\psi}(\Delta,R)$.
We prove that the operator T has fixed point which will act as a solution for the problem (1.1)-(1.2). For any $t\in(a,b]$, consider the space $C_{t;\psi}=C_{1-\rho;\psi}([a,t],R)$ with the norm defined by,

$$\|w\|_{C_{t;\psi}}=\max_{\omega\in[a,t]}\left|(\psi(\omega)-\psi(a))^{1-\rho}w(\omega)\right|.$$

Using mathematical induction for any $w_1, w_2 \in C_{t;\psi}$ and $t \in (a, b]$, we prove that for $j \in N$,

$$\|T^j w_1 - T^j w_2\|_{C_{t;\psi}} \leq \Gamma(\rho) \frac{(L(1 + (b-a)k_b + (b-a)h_b)(\psi(t) - \psi(a))^{\alpha})^j}{\Gamma(j\alpha + \rho)} \times \|w_1 - w_2\|_{C_{t;\psi}}, \tag{3.4}$$

where $k_b = \sup\{|K(t,s)| : a < t, s \leq b\}$ and $h_b = \sup\{|H(t,s)| : a < t, s \leq b\}$. Let any $w_1, w_2 \in C_{t;\psi}$. Then from the definition of operator T given in (3.3) and using Lipschitz condition on f, we have

$$\begin{aligned} &\|Tw_1 - Tw_2\|_{C_{t;\psi}} \\ &\leq \frac{L(1 + (b-a)k_b + (b-a)h_b)(\psi(t) - \psi(a))^{1-\rho}}{\Gamma(\alpha)} \|w_1 - w_2\|_{C_{t;\psi}} \\ &\quad \times \int_a^t \mathcal{L}_{\psi}^{\alpha}(t, \eta)\ (\psi(\eta) - \psi(a))^{\rho-1} d\eta \\ &\leq \frac{L(1 + (b-a)k_b + (b-a)h_b)(\psi(t) - \psi(a))^{\alpha}\Gamma(\rho)}{\Gamma(\alpha + \rho)} \|w_1 - w_2\|_{C_{t;\psi}} \end{aligned}$$

Thus the inequality (3.4) holds for $j = 1$. Let us suppose that the inequality (3.4) holds for $j = r \in N$, i.e. suppose

$$\|T^r w_1 - T^r w_2\|_{C_{t;\psi}} \leq \Gamma(\rho) \frac{(L(1 + (b-a)k_b + (b-a)h_b)(\psi(t) - \psi(a))^{\alpha})^r}{\Gamma(r\alpha + \rho)} \times \|w_1 - w_2\|_{C_{t;\psi}}, \tag{3.5}$$

holds. Next we prove that (3.4) holds for $j = r + 1$. Let $w_1, w_2 \in C_{t;\psi}$ and denote $w_1^* = T^r w_1$ and $w_2^* = T^r w_2$. Then using the definition of operator T and Lipschitz condition on f, we get

$$\begin{aligned} \|T^{r+1} w_1 - T^{r+1} w_2\|_{C_{t;\psi}} &= \|Tw_1^* - Tw_2^*\|_{C_{t;\psi}} \\ &\leq L(1 + (b-a)k_b + (b-a)h_b) \max_{\omega \in [a,t]} \Big|(\psi(\omega) - \psi(a))^{1-\rho} \\ &\quad \times \frac{1}{\Gamma(\alpha)} \int_a^{\omega} \mathcal{L}_{\psi}^{\alpha}(\omega, \eta) |w_1^*(\eta) - w_2^*(\eta)| d\eta \Big| \\ &\leq \frac{L(1 + (b-a)k_b + (b-a)h_b)(\psi(t) - \psi(a))^{1-\rho}}{\Gamma(\alpha)} \\ &\quad \times \int_a^t \mathcal{L}_{\psi}^{\alpha}(t, \eta)\ (\psi(\eta) - \psi(a))^{\rho-1} \|w_1^* - w_2^*\|_{C_{t;\psi}} d\eta \end{aligned}$$

From (3.5), we have

$$\|w_1^* - w_2^*\|_{C_{t;\psi}} \leq \Gamma(\rho) \frac{(L(1 + (b-a)k_b + (b-a)h_b)(\psi(t) - \psi(a))^{\alpha})^r}{\Gamma(r\alpha + \rho)} \times \|w_1 - w_2\|_{C_{t;\psi}}.$$

Therefore,

$$\begin{aligned}
&\|T^{r+1}w_1 - T^{r+1}w_2\|_{C_{t;\psi}} \\
&\leq \frac{L(1+(b-a)k_b+(b-a)h_b)(\psi(t)-\psi(a))^{1-\rho}}{\Gamma(\alpha)} \int_a^t \mathcal{L}_\psi^\alpha(t,\eta)\ (\psi(\eta)-\psi(a))^{\rho-1} \\
&\times \Gamma(\rho)\frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(\eta)-\psi(a))^\alpha)^r}{\Gamma(r\alpha+\rho)}\|w_1-w_2\|_{C_{t;\psi}} d\eta \\
&\leq \frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(t)-\psi(a))^\alpha)^{r+1}\Gamma(\rho)}{\Gamma((r+1)\alpha+\rho)}\|w_1-w_2\|_{C_{t;\psi}}
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\|T^{r+1}w_1 - T^{r+1}w_2\|_{C_{t;\psi}} \\
&\qquad \leq \frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(t)-\psi(a))^\alpha)^{r+1}\Gamma(\rho)}{\Gamma((r+1)\alpha+\rho)}\|w_1-w_2\|_{C_{t;\psi}}.
\end{aligned}$$

Therefore, by principle of mathematical induction the inequality (3.4) holds for all $j \in N$ and for every t in Δ. As a consequence we find on the fundamental interval Δ,

$$\begin{aligned}
&\|T^j w_1 - T^j w_2\|_{C_{1-\rho;\psi}(\Delta,R)} \\
&\leq \Gamma(\rho)\frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(t)-\psi(a))^\alpha)^j}{\Gamma(j\alpha+\rho)}\|w_1-w_2\|_{C_{1-\rho;\psi}(\Delta,R)}.
\end{aligned} \tag{3.6}$$

By definition of two parameter Mittag-Leffler function, we have

$$\begin{aligned}
&E_{\alpha,\rho}(L(1+(b-a)k_b+(b-a)h_b)(\psi(b)-\psi(a))^\alpha) \\
&\qquad = \sum_{j=0}^{\infty}\frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(b)-\psi(a))^\alpha)^j}{\Gamma(j\alpha+\rho)}.
\end{aligned}$$

Note that $\dfrac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(b)-\psi(a))^\alpha)^j}{\Gamma(j\alpha+\rho)}$ is the j^{th} term of the convergent series of real numbers. Therefore,

$$\lim_{j\to\infty}\frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(b)-\psi(a))^\alpha)^j}{\Gamma(j\alpha+\rho)} = 0.$$

Thus we can choose $j \in N$ such that

$$\Gamma(\rho)\frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(b)-\psi(a))^\alpha)^j}{\Gamma(j\alpha+\rho)} < 1,$$

so that T^j is a contraction. Therefore, by Lemma 3.1, T has a unique fixed point w^* in $C_{1-\rho;\psi}(\Delta, R)$, which is a unique solution of the Cauchy-type problem (1.1)-(1.2). $\square$

Remark 3.3. The existence result proved above with no restriction on the interval $\Delta = [a,b]$, and hence solution w^* of (1.1)-(1.2) exists for any $a, b(0 < a < b < \infty)$. Thus the Theorem 3.2 guarantees global unique solution in $C_{1-\rho;\psi}(\Delta, R)$.

4. Ulam-Hyers stability

To discuss HU and HUR stability of (1.1), we adopt the approach of [14, 21]. For $t \in \Delta,\ \epsilon > 0$ and continuous function $\phi : \Delta \to [0, \infty)$, we consider the following inequalities :

$$\left| {}^H D_{a+}^{\alpha,\beta;\psi} w^*(t) - f\left(t, w^*(t), \int_a^t K(t,s)y^*(s)ds, \int_a^b H(t,s)w^*(s)ds\right)\right| \le \epsilon, \quad (4.1)$$

$$\left| {}^H D_{a+}^{\alpha,\beta;\psi} w^*(t) - f\left(t, w^*(t), \int_a^t K(t,s)y^*(s)ds, \int_a^b H(t,s)w^*(s)ds\right)\right| \le \phi(t), \quad (4.2)$$

$$\left| {}^H D_{a+}^{\alpha,\beta;\psi} w^*(t) - f\left(t, w^*(t), \int_a^t K(t,s)y^*(s)ds, \int_a^b H(t,s)w^*(s)ds\right)\right| \le \epsilon\phi(t), \quad (4.3)$$

Definition 4.1. The problem (1.1) has HU stability if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $y^* \in C_{1-\rho;\psi}(\Delta, R)$ of the inequation (4.1) there exists a solution $y \in C_{1-\rho;\psi}(\Delta, R)$ of (1.1) with

$$\|y^* - y\|_{C_{1-\rho;\psi}(\Delta,R)} \le C_f \epsilon.$$

Definition 4.2. The problem (1.1) has generalized HU stability if there exists a function $C_f \in ([0,\infty), [0,\infty))$ with $C_f(0) = 0$ such that for each solution $w^* \in C_{1-\rho;\psi}(\Delta, R)$ of the inequation (4.1) there exists a solution $w \in C_{1-\rho;\psi}(\Delta, R)$ of (1.1) with $\|w^* - w\|_{C_{1-\rho;\psi}(\Delta,R)} \le C_f(\epsilon)$.

Definition 4.3. The problem (1.1) has HUR stability with respect to a function ϕ if there exists a real number $C_{f,\phi} > 0$ such that for each solution $y^* \in C_{1-\rho;\psi}(\Delta, R)$ of the inequation (4.3) there exists a solution $w \in C_{1-\rho;\psi}(\Delta, R)$ of (1.1) with

$$|(\psi(t) - \psi(a))^{1-\rho}(w^*(t) - w(t))| \le C_{f,\phi}\epsilon\phi(t),\ t \in (\Delta, R).$$

Definition 4.4. The problem (1.1) has generalized HUR stability with respect to a function ϕ if there exists a real number $C_{f,\phi} > 0$ such that for each solution $w^* \in C_{1-\rho;\psi}(\Delta, R)$ of the inequation (4.2) there exists a solution $y \in C_{1-\rho;\psi}(\Delta, R)$ of (1.1) with

$$|(\psi(t) - \psi(a))^{1-\rho}(w^*(t) - w(t))| \le C_{f,\phi}\phi(t),\ t \in \Delta.$$

In the next theorem we will make use of the successive approximation method to prove that the $\psi-$Hilfer FDE (1.1) is HU stable.

Theorem 4.5. *Let $f : (a,b] \times R \times R \to R$ be a function such that* $f\left(t, w(t), \int_a^t K(t,s)w(s)ds, \int_a^b H(t,s)w(s)ds\right) \in C_{1-\rho;\psi}(\Delta, R)$, *for any $w \in C_{1-\rho;\psi}(\Delta, R)$, and that satisfies the Lipschitz condition*

$$|f(t, w_1, z_1, x_1) - f(t, w_2, z_2, x_2)| \le L[\ |w_1 - w_2| + |z_1 - z_2| + |x_1 - x_2|\ ],$$

where $t \in (a,b]$, $w_1, w_2, z_1, z_2, x_1, x_2 \in R$ *and* $L > 0$ *is Lipschitz constant.*
For every $\epsilon > 0$, *if* $w^* \in C_{1-\rho;\psi}(\Delta, R)$ *satisfies*

$$\left| {}^H D_{a+}^{\alpha,\beta;\psi} w^*(t) - f\left(t, w^*(t), \int_a^t K(t,s)y^*(s)ds, \int_a^b H(t,s)w^*(s)ds\right)\right| \leq \epsilon,\ t \in \Delta,$$

then there exists a solution w *of equation (1.1) in* $C_{1-\rho;\psi}(\Delta, R)$ *with* $I_{a^+}^{1-\rho;\psi} w^*(a) = I_{a^+}^{1-\rho;\psi} w(a)$ *such that for* $t \in \Delta$,

$$\begin{aligned}
&\|w^* - w\|_{C_{1-\rho;\psi}(\Delta,R)} \\
&\leq \left[\frac{(E_\alpha(L(1+(b-a)k_b+(b-a)h_b)(\psi(b)-\psi(a))^\alpha)-1)}{L(1+(b-a)k_b+(b-a)h_b)}(\psi(b)-\psi(a))^{1-\rho}\right]\epsilon.
\end{aligned}$$

Proof. Fix any $\epsilon > 0$, let $w^* \in C_{1-\rho;\psi}(\Delta, R)$ satisfies

$$\left| {}^H D_{a+}^{\alpha,\beta;\psi} w^*(t) - f\left(t, w^*(t), \int_a^t K(t,s)w^*(s)ds, {}_a{}^b H(t,s)w^*(s)ds\right)\right| \leq \epsilon,\ t \in \Delta. \tag{4.4}$$

Then there exists a function $\sigma_{w^*} \in C_{1-\rho;\psi}(\Delta, R)$ (depending on w^*) such that $|\sigma_{w^*}(t)| \leq \epsilon,\ t \in \Delta$ and

$${}^H D_{a+}^{\alpha,\beta;\psi} w^*(t) = f\left(t, w^*(t), \int_a^t K(t,s)w^*(s)ds, \int_a^b H(t,s)w^*(s)ds\right) + \sigma_{w^*}(t). \tag{4.5}$$

If $w^*(t)$ satisfies (4.5) then it satisfies equivalent fractional integral equation

$$\begin{aligned}
w^*(t) &= \Omega_\psi^\rho(t,a) I_{a^+}^{1-\rho} w^*(a) + \frac{1}{\Gamma(\alpha)}\int_a^t \mathcal{L}_\psi^\alpha(t,\eta) \\
&\quad \times f\left(\eta, w^*(\eta), \int_a^\eta K(\eta,\sigma)w^*(\sigma)d\sigma, \int_a^b H(\eta,\sigma)w^*(\sigma)d\sigma\right) d\eta \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t \mathcal{L}_\psi^\alpha(t,\eta)\sigma_{w^*}(\eta)d\eta.
\end{aligned} \tag{4.6}$$

Define

$$w_0(t) = w^*(t),\ t \in \Delta, \tag{4.7}$$

and consider the sequence $\{w_n\}_{n=1}^\infty \subseteq C_{1-\rho;\psi}(\Delta, R)$ defined by

$$\begin{aligned}
w_n(t) &= \Omega_\psi^\rho(t,a) I_{a^+}^{1-\rho} w^*(a) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t \mathcal{L}_\psi^\alpha(t,\eta) f\bigg(\eta, w_{n-1}(\eta), \int_a^\eta K(\eta,\sigma)w_{n-1}(\sigma)d\sigma, \\
&\quad\quad \int_a^b K(\eta,\sigma)w_{n-1}(\sigma)d\sigma\bigg) d\eta.
\end{aligned} \tag{4.8}$$

Using mathematical induction firstly we prove that for every $t \in \Delta,\ j \in N$ and $w_j \in C_{1-\rho;\psi}[a,t] = C_{t;\psi}$

$$\begin{aligned}
&\|w_j - w_{j-1}\|_{C_{t;\psi}} \\
&\leq \frac{\epsilon}{L(1+(b-a)k_b+(b-a)h_b)} \\
&\times \ \frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(t)-\psi(a))^{\alpha})^j}{\Gamma(j\alpha+1)}(\psi(t)-\psi(a))^{1-\rho}. \qquad (4.9)
\end{aligned}$$

By definition of successive approximations and using (4.6) we have

$$\begin{aligned}
&\|w_1 - w_0\|_{C_{t;\psi}} \\
&= \max_{\omega\in[a,t]} \left|(\psi(\omega)-\psi(a))^{1-\rho}\frac{1}{\Gamma(\alpha)}\int_a^{\omega}\mathcal{L}_{\psi}^{\alpha}(\omega,\eta)\sigma_{w_1}(\eta)d\eta\right| \\
&\leq \epsilon \max_{\omega\in[a,t]}\left[(\psi(\omega)-\psi(a))^{1-\rho}\frac{1}{\Gamma(\alpha)}\int_a^{\omega}\mathcal{L}_{\psi}^{\alpha}(\omega,\eta)d\eta\right] \\
&\leq \frac{\epsilon}{L(1+(b-a)k_b+(b-a)h_b)} \ \frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(t)-\psi(a))^{\alpha})}{\Gamma(\alpha+1)} \\
&\quad \times \ (\psi(t)-\psi(a))^{1-\rho}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|w_1 - w_0\|_{C_{t;\psi}} \leq &\frac{\epsilon}{L(1+(b-a)k_b+(b-a)h_b)} \\
&\frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(t)-\psi(a))^{\alpha})}{\Gamma(\alpha+1)} \ (\psi(t)-\psi(a))^{1-\rho},
\end{aligned}$$

which proves the inequality (4.9) for $j = 1$. Let us suppose that the inequality (4.9) holds for $j = r \in N$, we prove it for $j = r+1$. By definition of successive approximations and Lipschitz condition on f, we obtain

$$\begin{aligned}
\|w_{r+1} - w_r\|_{C_{t;\psi}} &= \max_{\omega\in[0,t]}\left|(\psi(\omega)-\psi(a))^{1-\rho}\{w_{r+1}(\omega)-w_r(\omega)\}\right| \\
&\leq L(1+(b-a)k_b+(b-a)h_b)\max_{\omega\in[a,t]}\Big[(\psi(\omega)-\psi(a))^{1-\rho} \\
&\frac{1}{\Gamma(\alpha)}\int_a^{\omega}\mathcal{L}_{\psi}^{\alpha}(\omega,\eta)|w_r(\eta)-w_{r-1}(\eta)|d\eta\Big] \\
&\leq \frac{L(1+(b-a)k_b+(b-a)h_b)(\psi(t)-\psi(a))^{1-\rho}}{\Gamma(\alpha)} \\
&\times \int_a^t \mathcal{L}_{\psi}^{\alpha}(t,\eta)(\psi(\eta)-\psi(a))^{\rho-1}\left\|w_r - w_{r-1}\right\|_{C_{\eta;\psi}} d\eta
\end{aligned}$$

Using the inequality (4.9) for $j = r$, we have

$$\begin{aligned}
&\|w_{r+1} - w_r\|_{C_{t;\psi}} \\
&\leq \frac{\epsilon}{L(1+(b-a)k_b+(b-a)h_b)} \frac{(L(1+(b-a)k_b+(b-a)h_b))^{r+1}}{\Gamma(r\alpha+1)} \\
&\times \ (\psi(t)-\psi(a))^{1-\rho} I_{a^+}^{\alpha;\psi} (\psi(t)-\psi(a))^{r\alpha} \\
&\leq \frac{\epsilon}{L(1+(b-a)k_b+(b-a)h_b)} \frac{(L(1+(b-a)k_b+(b-a)h_b))^{r+1}}{\Gamma(r\alpha+1)} \\
&\times (\psi(t)-\psi(a))^{1-\rho} \frac{\Gamma(r\alpha+1)}{\Gamma((r+1)\alpha+1)} (\psi(t)-\psi(a))^{(r+1)\alpha}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|w_{r+1} - w_r\|_{C_{t;\psi}} \\
&\leq \frac{\epsilon}{L(1+(b-a)k_b+(b-a)h_b)} \frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(t)-\psi(a))^{\alpha})^{r+1}}{\Gamma((r+1)\alpha+1)} \\
&\times \ (\psi(t)-\psi(a))^{1-\rho},
\end{aligned}$$

which is the inequality (4.9) for $j = r+1$. Using the principle of mathematical induction the inequality (4.9) holds for every $j \in N$ and every $t \in \Delta$.
Therefore,

$$\begin{aligned}
&\|w_j - w_{j-1}\|_{C_{1-\rho;\psi}(\Delta,R)} \\
&\leq \frac{\epsilon}{L(1+(b-a)k_b+(b-a)h_b)} \frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(t)-\psi(a))^{\alpha})^{j}}{\Gamma(j\alpha+1)} \\
&\times \ (\psi(t)-\psi(a))^{1-\rho}.
\end{aligned}$$

Now using this estimation we have

$$\begin{aligned}
\sum_{j=1}^{\infty} \|w_j - w_{j-1}\|_{C_{1-\rho;\psi}(\Delta,R)} &\leq \frac{\epsilon}{L(1+(b-a)k_b+(b-a)h_b)} \ (\psi(t)-\psi(a))^{1-\rho} \\
&\times \ \sum_{j=1}^{\infty} \frac{(L(1+(b-a)k_b+(b-a)h_b)(\psi(t)-\psi(a))^{\alpha})^{j}}{\Gamma(j\alpha+1)}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\sum_{j=1}^{\infty} \|w_j - w_{j-1}\|_{C_{1-\rho;\psi}(\Delta,R)} &\leq \frac{\epsilon}{L(1+(b-a)k_b+(b-a)h_b)} \ (\psi(t)-\psi(a))^{1-\rho} \\
&(E_{\alpha}(L(1+(b-a)k_b+(b-a)h_b)(\psi(b)-\psi(a))^{\alpha})-1)
\end{aligned} \tag{4.10}$$

Hence the series

$$w_0 + \sum_{j=1}^{\infty} (w_j - w_{j-1}) \tag{4.11}$$

converges in the weighted space $C_{1-\rho;\psi}(\Delta, R)$. Let $w \in C_{1-\rho;\psi}(\Delta, R)$ such that

$$w = w_0 + \sum_{j=1}^{\infty}(w_j - w_{j-1}), \tag{4.12}$$

Noting that $w_n = w_0 + \sum_{j=1}^{n}(w_j - w_{j-1})$, is the n^{th} partial sum of the series (4.11), we have $\|w_n - w\|_{C_{1-\rho;\psi}(\Delta,R)} \to 0$ as $n \to \infty$.

Next, we prove that this limit function w is the solution of fractional integral equation with $I_{a^+}^{1-\rho;\psi}w^*(a) = I_{a^+}^{1-\rho;\psi}w(a)$. Therefore, by the definition of successive approximation, for any $t \in \Delta$, we have

$$\begin{aligned}
&\Bigg|(\psi(t)-\psi(a))^{1-\rho}\Bigg(w(t) - \Omega_\psi^\rho(t,a)I_{a^+}^{1-\rho}w(a) \\
&-\frac{1}{\Gamma(\alpha)}\int_a^t \mathcal{L}_\psi^\alpha(t,\eta)f\left(\eta, w(\eta), \int_a^\eta K(\eta,\sigma)w(\sigma)d\sigma, \int_a^b H(\eta,\sigma)w(\sigma)d\sigma\right)d\eta\Bigg)\Bigg| \\
&\le \|w - w_n\|_{C_{1-\rho;\psi}[a,b]} + L(1 + (b-a)k_b + (b-a)h_b) \\
&\quad \times \left[(\psi(t)-\psi(a))^{1-\rho}\frac{1}{\Gamma(\alpha)}\int_a^t \mathcal{L}_\psi^\alpha(t,\eta)|w_{n-1}(\eta) - w(\eta)|d\eta\right] \\
&\le \|w - w_n\|_{C_{1-\rho;\psi}[a,b]} + \left(\frac{L(1 + (b-a)k_b + (b-a)h_b)\Gamma(\rho)}{\Gamma(\alpha+\rho)}(\psi(t)-\psi(a))^\alpha\right) \\
&\quad \times \|w_{n-1} - w\|_{C_{1-\rho;\psi}[a,b]}, \ \forall n \in N
\end{aligned}$$

By taking limit as $n \to \infty$ in the above inequality, for all $t \in [a, b]$, we obtain

$$\begin{aligned}
&\Bigg|(\psi(t)-\psi(a))^{1-\rho}\Bigg(w(t) - \Omega_\psi^\rho(t,a)I_{a^+}^{1-\rho}w(a) - \frac{1}{\Gamma(\alpha)}\int_a^t \mathcal{L}_\psi^\alpha(t,\eta) \\
&\qquad \times f\left(\eta, w(\eta), \int_a^\eta K(\eta,\sigma)w(\sigma)d\sigma, \int_a^b H(\eta,\sigma)w(\sigma)d\sigma\right)d\eta\Bigg)\Bigg| = 0.
\end{aligned}$$

Since, $(\psi(t)-\psi(a))^{1-\rho} \ne 0$ for all $t \in \Delta$, we have

$$\begin{aligned}
w(t) &= \Omega_\psi^\rho(t,a)I_{a^+}^{1-\rho}w(a) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t \mathcal{L}_\psi^\alpha(t,\eta)f\left(\eta, w(\eta), \int_a^\eta K(\eta,\sigma)w(\sigma)d\sigma, \int_a^b H(\eta,\sigma)w(\sigma)d\sigma\right)d\eta.
\end{aligned} \tag{4.13}$$

This proves that w is the solution of (1.1)-(1.2) in $C_{1-\rho;\psi}(\Delta, R)$. Further, for the solution w^* of inequation (4.4) and the solution w of the equation (1.1), using (4.7) and (4.12), for any $t \in \Delta$, we have

$$\begin{aligned}
&\left|(\psi(t)-\psi(a))^{1-\rho}(w^*(t) - w(t))\right| \\
&\le \sum_{j=1}^{\infty}\left|(\psi(t)-\psi(a))^{1-\rho}\ (w_j(t) - w_{j-1}(t))\right| \le \sum_{j=1}^{\infty}\|w_j - w_{j-1}\|_{C_{1-\rho}[a,b]} \\
&\le \frac{\epsilon}{L(1 + (b-a)k_b + (b-a)h_b)}\ (\psi(b)-\psi(a))^{1-\rho} \\
&\quad \times\ (E_\alpha(L(1 + (b-a)k_b + (b-a)h_b)(\psi(b)-\psi(a))^\alpha) - 1)
\end{aligned}$$

Therefore,

$$\|w^* - w\|_{C_{1-\rho;\psi}[a,b]} \leq \Bigg(\frac{(E_\alpha(L(1+(b-a)k_b+(b-a)h_b)(\psi(b)-\psi(a))^\alpha)-1)}{L(1+(b-a)k_b+(b-a)h_b)} \times (\psi(b)-\psi(a))^{1-\rho}\Bigg)\ \epsilon.$$

This proves the equation (1.1) is HU stable. □

Corollary 4.6. *Suppose that the function f satisfies the assumptions of Theorem 4.5. Then the problem (1.1) is generalized HU stable.*

Proof. Set

$$\psi_f(\epsilon) = \left(\frac{(E_\alpha(L(1+(b-a)k_b+(b-a)h_b)(\psi(b)-\psi(a))^\alpha)-1)}{L(1+(b-a)k_b+(b-a)h_b)}\ (\psi(b)-\psi(a))^{1-\rho}\right)\epsilon,$$

in the proof of Theorem 4.5. Then $\psi_f(0) = 0$ and for each $w^* \in C_{1-\rho;\psi}(\Delta, R)$ that satisfies the inequality

$$\left| {}^H D^{\alpha,\beta;\psi}_{a+} w^*(t) - f\left(t, w^*(t), \int_a^t K(t,s)w^*(s)ds, \int_a^b H(t,s)w^*(s)ds\right)\right| \leq \epsilon,\ t \in \Delta,$$

there exists a solution w of equation (1.1) in $C_{1-\rho;\psi}(\Delta, R)$ with $I^{1-\rho;\psi}_{a^+} w^*(a) = I^{1-\rho;\psi}_{a^+} w(a)$ such that

$$\|w^* - w\|_{C_{1-\rho;\psi}[a,b]} \leq \psi_f(\epsilon),\ t \in \Delta.$$

Hence mixed fractional integrodifferential equation (1.1) is generalized HU stable. □

Theorem 4.7. *Let $f : (a, b] \times R \times R \to R$ be a function such that $f\left(t, w(t), \int_a^t K(t,s)w(s)ds, \int_a^b H(t,s)w(s)ds\right) \in C_{1-\rho;\psi}(\Delta, R)$, for any $w \in C_{1-\rho;\psi}(\Delta, R)$, and that satisfies the Lipschitz condition*

$$|f(t, w_1, z_1, x_1) - f(t, w_2, z_2, x_2)| \leq L[\ |w_1 - w_2| + |z_1 - z_2| + |x_1 - x_2|\ ],$$

where $t \in (a, b],\ w_1, w_2, z_1, z_2, x_1, x_2 \in R$ and $L > 0$ is Lipschitz constant. For every $\epsilon > 0,\ t \in \Delta,$ if $w^* \in C_{1-\rho;\psi}(\Delta, R)$ satisfies

$$\left| {}^H D^{\alpha,\beta;\psi}_{a^+} w^*(t) - f\left(t, w^*(t), \int_a^t K(t,s)w^*(s)ds, \int_a^b H(t,s)w^*(s)ds\right)\right| \leq \epsilon\phi(t),$$

where $\phi \in C(\Delta, R_+)$ is a non-decreasing function such that

$$|I^{\alpha;\psi}_{a^+}\phi(t)| \leq \lambda\phi(t),\ t \in \Delta$$

and $\lambda > 0$ is a constant satisfying $0 < \lambda L(1 + (b-a)k_b + (b-a)h_b) < 1$. Then, there exists a solution $w \in C_{1-\rho;\psi}(\Delta, R)$ of equation (1.1) with $I^{1-\rho;\psi}_{a^+} w^*(a) = I^{1-\rho;\psi}_{a^+} w(a)$ such that

$$\begin{aligned}&\left|(\psi(t)-\psi(a))^{1-\rho}(w^*(t)-w(t))\right| \\ &\leq \left[\frac{\lambda}{1-\lambda L(1+(b-a)k_b+(b-a)h_b)}(\psi(b)-\psi(a))^{1-\rho}\right]\epsilon\phi(t),\ t \in \Delta.\end{aligned}$$

Proof. For every $\epsilon > 0$, let $w^* \in C_{1-\rho;\psi}(\Delta, R)$ satisfies

$$\left| {}^H D_{a^+}^{\alpha,\beta;\psi} w^*(t) - f\left(t, w^*(t), \int_a^t K(t,s)w^*(s)ds, \int_a^b H(t,s)w^*(s)ds\right)\right| \leq \epsilon\phi(t).$$

Proceeding as in the proof of Theorem 4.5 there exists a function $\sigma w^* \in C_{1-\rho;\psi}(\Delta, R)$ (depending on w^*) such that

$$\begin{aligned} w^*(t) &= \Omega_\psi^\rho(t,a) I_{a^+}^{1-\rho;\psi} w^*(a) \\ &\quad + I_{a^+}^{\alpha;\psi} f\left(t, w^*(t), \int_a^t K(t,\sigma)w^*(\sigma)d\sigma, \int_a^b H(t,\sigma)w^*(\sigma)d\sigma\right) + I_{a^+}^{\alpha;\psi}\sigma_{w^*}(t). \end{aligned}$$

Further, using mathematical induction, one can prove that the sequence of successive approximations $\{w_n\}_{n=1}^\infty \subset C_{1-\rho;\psi}(\Delta, R)$ defined by

$$\begin{aligned} w_n(t) &= \Omega_\psi^\rho(t,a) I_{a^+}^{1-\rho} w^*(a) + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{L}_\psi^\alpha(t,\eta) \\ &\quad \times f\left(\eta, w_{n-1}(\eta), \int_a^\eta K(\eta,\sigma)w_{n-1}(\sigma)d\sigma, \int_a^b H(\eta,\sigma)w_{n-1}(\sigma)d\sigma\right) d\eta. \end{aligned} \tag{4.14}$$

satisfy the inequality

$$\begin{aligned} \|w_j - w_{j-1}\|_{C_{t;\psi}} \leq & \frac{\epsilon}{L(1+(b-a)k_b+(b-a)h_b)} \left(\lambda L(1+(b-a)k_b+(b-a)h_b)\right)^j \\ & \times (\psi(t)-\psi(a))^{1-\rho}\phi(t), \ j \in N. \end{aligned} \tag{4.15}$$

Using the inequation (4.15), we obtain

$$\begin{aligned} \sum_{j=1}^\infty \|w_j - w_{j-1}\|_{C_{t;\psi}} &\leq \frac{\epsilon}{L(1+(b-a)k_b+(b-a)h_b)} \\ &\quad \times \left(\sum_{j=1}^\infty (\lambda L(1+(b-a)k_b+(b-a)h_b))^j\right) (\psi(t)-\psi(a))^{1-\rho}\phi(t). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{j=1}^\infty \|w_j - w_{j-1}\|_{C_{t;\psi}} &\leq \epsilon \left(\frac{\lambda}{1-\lambda L(1+(b-a)k_b)+(b-a)h_b)}\right) \\ &\quad \times \ (\psi(t)-\psi(a))^{1-\rho}\phi(t). \end{aligned} \tag{4.16}$$

Following the steps as in the proof of the Theorem 4.5 there exists $w \in C_{1-\rho;\psi}(\Delta, R)$ such that $\|w_n - w\|_{C_{1-\rho;\psi}(\Delta,R)} \to 0$ as $n \to \infty$. This w is the solution of the problem (1.1)-(1.2) with $I_{a^+}^{1-\rho;\psi} w(a) = I_{a^+}^{1-\rho;\psi} w^*(a)$, and we have $w = w_0 + \sum_{j=1}^\infty (w_j - w_{j-1})$. Further, for the solution w^* of inequation and the solution w of the equation (1.1), for any $t \in \Delta$,

$$\begin{aligned}
&\left|(\psi(t)-\psi(a))^{1-\rho}(w^*(t)-w(t))\right| \\
&\leq \sum_{j=1}^{\infty}\left|(\psi(t)-\psi(a))^{1-\rho}\ (w_j(t)-w_{j-1}(t))\right| \\
&\leq \sum_{j=1}^{\infty}\|w_j-w_{j-1}\|_{C_{t,\psi}} \\
&\leq \epsilon\ \left(\frac{\lambda}{1-\lambda L(1+(b-a)k_b+(b-a)h_b)}\right)\ (\psi(t)-\psi(a))^{1-\rho}\phi(t).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left|(\psi(t)-\psi(a))^{1-\rho}(w^*(t)-w(t))\right| \\
&\quad\leq \left(\frac{\lambda}{1-\lambda L(1+(b-a)k_b+(b-a)h_b)}\ (\psi(b)-\psi(a))^{1-\rho}\ \right)\epsilon\ \phi(t).
\end{aligned}$$

This proves the equation (1.1) is HUR stable. □

Corollary 4.8. *Suppose that the function f satisfies the assumptions of Theorem 4.7. Then, the problem (1.1) is generalized HUR stable.*

Proof. Set $\epsilon = 1$ and $C_{f,\phi} = \left(\frac{\lambda}{1-\lambda L(1+(b-a)k_b+(b-a)h_b)}\ (\psi(b)-\psi(a))^{1-\rho}\ \right)$ in the proof of Theorem 4.7. Then for each solution $w^* \in C_{1-\rho;\psi}(\Delta, R)$ that satisfies the inequality

$$\left|{}^{H}D_{a^+}^{\alpha,\beta;\psi}w^*(t) - f\left(t, y^*(t), \int_a^t K(t,s)y^*(s)ds, \int_a^b H(t,s)y^*(s)ds\right)\right| \leq \phi(t),$$

there exists a solution w of equation (1.1) in $C_{1-\rho;\psi}(\Delta, R)$ with $I_{a^+}^{1-\rho;\psi}w^*(a) = I_{a^+}^{1-\rho;\psi}w(a)$ such that

$$\left|(\psi(t)-\psi(a))^{1-\rho}(w^*(t)-w(t))\right| \leq C_{f,\phi}\phi(t),\ t \in \Delta.$$

Hence the fractional integrodifferential equation (1.1) is generalized HUR stable. □

5. $\epsilon-$Approximate solutions to Hilfer MFIDE

Definition 5.1. A function $w^* \in C_{1-\rho;\psi}(\Delta, R)$ that satisfy the fractional integrodifferential inequality

$$\left|{}^{H}D_{a^+}^{\alpha,\beta;\psi}w^*(t) - f\left(t, w^*(t), \int_a^t K(t,s)w^*(s)ds, \int_a^b H(t,s)y^*(s)ds\right)\right| \leq \epsilon,\ t \in \Delta,$$

is called an $\epsilon-$approximate solutions of $\psi-$Hilfer MFIDE (1.1).

Theorem 5.2. *([18]) Let u, v be two integrable, non negative functions and g be a continuous, nonnegative, nondecreasing function with domain Δ. If*

$$u(t) \leq v(t) + g(t) \int_a^t \mathcal{L}_\psi^\alpha(\tau, s) u(\tau) d\tau,$$

then

$$u(t) \leq v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \mathcal{L}_\psi^{\alpha k}(\tau, s) v(\tau) d\tau, \ \forall t \in \Delta. \tag{5.1}$$

Theorem 5.3. *Let $f : (a, b] \times R \times R \to R$ be a function which satisfies the Lipschitz condition $|f(t, w_1, z_1, x_1) - f(t, w_2, z_2, x_2)| \leq L[\ |w_1 - w_2| + |z_1 - z_2| + |x_1 - x_2|\ ]$, where $t \in (a, b],\ w_1, w_2, z_1, z_2, x_1, x_2 \in R$ and $L > 0$ is Lipschitz constant.*
Let $w_i^* \in C_{1-\rho;\psi}(\Delta, R),\ (i = 1, 2)$ be an ϵ_i-approximate solutions of MFIDE (1.1) corresponding to $I_{a^+}^{1-\rho;\psi} w_i^*(a) = w_a^{(i)} \in R$, respectively. Then

$$\begin{aligned}
&\|w_1^* - w_2^*\|_{C_{1-\rho;\psi}(\Delta,R)} \\
&\leq (\epsilon_1 + \epsilon_2) \Bigg(\frac{(\psi(b) - \psi(a))^{\alpha-\rho+1}}{\Gamma(\alpha + 1)} \\
&+ \sum_{k=1}^{\infty} \frac{(L(1 + (b - a)k_b + (b - a)h_b))^k}{\Gamma((k + 1)\alpha - \rho + 1)} (\psi(b) - \psi(a))^{(k+1)\alpha} \Bigg) \\
&+ |w_a^{(1)} - w_a^{(2)}| \left(\frac{1}{\Gamma(\rho)} + \sum_{k=1}^{\infty} \frac{(L(1 + (b - a)k_b + (b - a)h_b))^k}{\Gamma(\rho + k\alpha)} (\psi(b) - \psi(a))^{k\alpha} \right).
\end{aligned} \tag{5.2}$$

Proof. Let $w_i^* \in C_{1-\rho;\psi}(\Delta, R),\ (i = 1, 2)$ be an ϵ_i-approximate solutions of MFIDE (1.1) that satisfy the initial condition $I_{a^+}^{1-\rho;\psi} w_i^*(a) = w_a^{(i)} \in R$. Then

$$\left| {}^H D_{a^+}^{\alpha,\beta;\psi} w_i^*(t) - f\left(t, w_i^*(t), \int_a^t K(t, s) w_i^*(s) ds, \int_a^b H(t, s) w_i^*(s) ds\right) \right| \leq \epsilon_i. \tag{5.3}$$

Operating $I_{a^+}^{\alpha;\psi}$ on both the sides of the above inequation and using the Lemma2.5, we get

$$\begin{aligned}
&I_{a^+}^{\alpha;\psi} \epsilon_i \\
&\geq I_{a^+}^{\alpha;\psi} \left| {}^H D_{a^+}^{\alpha,\beta;\psi} w_i^*(t) - f\left(t, w_i^*(t), \int_a^t K(t, s) w_i^*(s) ds, \int_a^b H(t, s) w_i^*(s) ds\right) \right| \\
&\geq \left| I_{a^+}^{\alpha;\psi} {}^H D_{a^+}^{\alpha,\beta;\psi} w_i^*(t) - I_{a^+}^{\alpha;\psi} f\left(t, w_i^*(t), \int_a^t K(t, s) w_i^*(s) ds, \int_a^b H(t, s) w_i^*(s) ds\right) \right| \\
&\geq \Bigg| w_i^*(t) - I_{a^+}^{1-\rho;\psi} w_i^*(a) \Omega_\psi^\rho(t, a) \\
&\qquad - I_{a^+}^{\alpha;\psi} f\left(t, w_i^*(t), \int_a^t K(t, s) w_i^*(s) ds, \int_a^b H(t, s) w_i^*(s) ds\right) \Bigg|.
\end{aligned}$$

Therefore,

$$\frac{\epsilon_i}{\Gamma(\alpha+1)}\,(\psi(t)-\psi(a))^{\alpha} \geq \Big|w_i^*(t) - w_a^{(i)}\Omega_\psi^\rho(t,a)$$
$$-I_{a^+}^{\alpha;\psi} f\left(t, w_i^*(t), \int_a^t K(t,s)w_i^*(s)ds, \int_a^b H(t,s)w_i^*(s)ds\right)\Bigg|,\ i=1,2. \tag{5.4}$$

Using the following inequalities

$$|x-w| \leq |x| + |w| \text{ and } |x| - |w| \leq |x-w|,\ x, w \in R,$$

from the inequation (5.4), for any $t \in \Delta$, we have

$$\frac{\epsilon_1+\epsilon_2}{\Gamma(\alpha+1)}\,(\psi(t)-\psi(a))^{\alpha}$$
$$\geq |(w_1^*(t) - w_2^*(t))| - \left|(w_a^{(1)} - w_a^{(2)})\Omega_\psi^\rho(t,a)\right|$$
$$-\left|I_{a^+}^{\alpha}\left[f\left(t, w_1^*(t), \int_a^t K(t,s)w_1^*(s)ds\right) - f\left(t, w_2^*(t), \int_a^t K(t,s)w_2^*(s)ds\right)\right]\right|$$

Therefore,

$$|(w_1^*(t) - w_2^*(t))| \leq \frac{\epsilon_1+\epsilon_2}{\Gamma(\alpha+1)}\,(\psi(t)-\psi(a))^{\alpha} + \left|(w_a^{(1)} - w_a^{(2)})\Omega_\psi^\rho(t,a)\right|$$
$$+\frac{L(1+(b-a)k_b+(b-a)h_b)}{\Gamma(\alpha)}$$
$$\times \int_0^t \mathcal{L}_\psi^\alpha(t,\eta)\,|(w_1^*(\eta) - w_2^*(\eta))|\,d\eta$$

Applying Theorem 5.2 with

$$u(t) = |(w_1^*(t) - w_2^*(t))|$$
$$v(t) = \frac{\epsilon_1+\epsilon_2}{\Gamma(\alpha+1)}\,(\psi(t)-\psi(a))^{\alpha} + \left|(w_a^{(1)} - w_a^{(2)})\Omega_\psi^\rho(t,a)\right|$$
$$g(t) = \frac{L(1+(b-a)k_b+(b-a)h_b)}{\Gamma(\alpha)},$$

we obtain

$$|(w_1^*(t) - w_2^*(t))|$$
$$\leq (\epsilon_1+\epsilon_2)\left(\frac{(\psi(t)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{(L(1+(b-a)k_b+(b-a)h_b))^k}{\Gamma((k+1)\alpha-\rho+1)}(\psi(t)-\psi(a))^{(k+1)\alpha}\right)$$
$$+\left|w_a^{(1)} - w_a^{(2)}\right|\left(\frac{(\psi(t)-\psi(a))^{\rho-1}}{\Gamma(\rho)}\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{(L(1+(b-a)k_b+(b-a)h_b))^k}{\Gamma(\rho+k\alpha)}(\psi(t)-\psi(a))^{k\alpha+\rho-1}\right)$$

Thus for every $t \in \Delta$, we have

$$
\begin{aligned}
&(\psi(t)-\psi(a))^{1-\rho}\left|(w_1^*(t)-w_2^*(t))\right| \\
&\leq (\epsilon_1+\epsilon_2)\Bigg(\frac{(\psi(t)-\psi(a))^{\alpha-\rho+1}}{\Gamma(\alpha+1)} \\
&\qquad +\sum_{k=1}^{\infty}\frac{(L(1+(b-a)k_b+(b-a)h_b))^k}{\Gamma((k+1)\alpha-\rho+1)}(\psi(t)-\psi(a))^{(k+1)\alpha}\Bigg) \\
&\qquad +\left|w_a^{(1)}-w_a^{(2)}\right|\left(\frac{1}{\Gamma(\rho)}+\sum_{k=1}^{\infty}\frac{(L(1+(b-a)k_b+(b-a)h_b))^k}{\Gamma(\rho+k\alpha)}(\psi(t)-\psi(a))^{k\alpha}\right) \\
&\leq (\epsilon_1+\epsilon_2)\Bigg(\frac{(\psi(b)-\psi(a))^{\alpha-\rho+1}}{\Gamma(\alpha+1)} \\
&\qquad +\sum_{k=1}^{\infty}\frac{(L(1+(b-a)k_b+(b-a)h_b))^k}{\Gamma((k+1)\alpha-\rho+1)}(\psi(b)-\psi(a))^{(k+1)\alpha}\Bigg) \\
&\qquad +\left|w_a^{(1)}-w_a^{(2)}\right|\left(\frac{1}{\Gamma(\rho)}+\sum_{k=1}^{\infty}\frac{(L(1+(b-a)k_b+(b-a)h_b))^k}{\Gamma(\rho+k\alpha)}(\psi(b)-\psi(a))^{k\alpha}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|w_1^*-w_2^*\|_{C_{1-\rho;\psi}(\Delta,R)} \\
&\leq (\epsilon_1+\epsilon_2)\Bigg(\frac{(\psi(b)-\psi(a))^{\alpha-\rho+1}}{\Gamma(\alpha+1)} \\
&\qquad +\sum_{k=1}^{\infty}\frac{(L(1+(b-a)k_b+(b-a)h_b))^k}{\Gamma((k+1)\alpha-\rho+1)}(\psi(b)-\psi(a))^{(k+1)\alpha}\Bigg) \\
&\qquad +\left|w_a^{(1)}-w_a^{(2)}\right|\left(\frac{1}{\Gamma(\rho)}+\sum_{k=1}^{\infty}\frac{(L(1+(b-a)k_b+(b-a)h_b))^k}{\Gamma(\rho+k\alpha)}(\psi(b)-\psi(a))^{k\alpha}\right)
\end{aligned}
$$

which is the desired inequality. □

Remark 5.4. If $\epsilon_1=\epsilon_2=0$ in the inequality (5.3) then w_1^* and w_2^* are the solutions of Cauchy problem (1.1)-(1.2) in the space $C_{1-\rho;\psi}[a,b]$. Further, for $\epsilon_1=\epsilon_2=0$ the inequality takes the form

$$
\begin{aligned}
\|w_1^*-w_2^*\|_{C_{1-\rho;\psi}(\Delta,R)} \leq \left|w_a^{(1)}-w_a^{(2)}\right|\Bigg(&\frac{1}{\Gamma(\rho)} \\
&+\sum_{k=1}^{\infty}\frac{(L(1+(b-a)k_b+(b-a)h_b))^k}{\Gamma(\rho+k\alpha)}(\psi(b)-\psi(a))^{k\alpha}\Bigg),
\end{aligned}
$$

which provides the information regarding continuous dependance of the solution of the problem (1.1)-(1.2) on initial condition. In addition, if $w_a^{(1)}=w_a^{(2)}$ we have $\|w_1^*-w_2^*\|_{C_{1-\rho;\psi}(\Delta,R)}=0$, which gives the uniqueness of solution of the problem (1.1)-(1.2).

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References

1. Abbas, S., Benchohra, M., Petrusel, A.: Ulam stability for Hilfer type fractional differential inclusions via the weakly Picard operators theory, *Fractional Calc. Appl. Anal.,* **20** (2017) 383–398.
2. Abbas, S., Benchohra, M., Lagreg, J. E., Alsaedi, A., Zhou, Y.: Existence and Ulam stability for fractional differential equations of Hilfer-Hadamard type, *Adv. Diff. Equa.,* **2017** (1) (2017):180/DOI 10.1186/s13662-017-1231-1.
3. Benchohra, M., Lagreg, J. E.: Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative, *Studia Universitatis Babes-Bolyai, Mathematica,* **62** (1) (2017) 27–38.
4. Hilfer, R.: Applications of fractional calculus in Physics, *World Scientific, Singapore,* (2000).
5. Huang, J., Li, Y.: Hyers-Ulam stability of delay differential equations of first order, *Math. Nachr.,* **289** (1) (2016) 60–66.
6. Furati, K. M., Kassim, M. D., Tatar, N. E.: Existence and uniqueness for a problem involving Hilfer fractional derivative, *Comput. Math. Appl.,* **64** (6) (2012) 1616–1626.
7. Furati, K. M., Kassim, M. D., Tatar, N. E.: Non-existence of global solutions for a differential equation involving Hilfer fractional derivative, *Electron. J. Diff. Eqn.,* (235) (2013) 1–10.
8. Kendre, S. D., Kharat, V. V.: On nonlinear mixed fractional integrodifferential equations with nonlocal condition in Banach spaces, *J. App. Anal.,* **20** (2) (2014) 167–175.
9. Kharat, V. V.: On existence and uniqueness of fractional integrodifferential equations with an integral fractional boundary condition, *Malaya J. Matematik,* **6** (3) (2018) 485–491.
10. Kharat, V. V., Hasabe, D. R., Dhaigude, D. B.: On existence of solution to mixed nonlinear fractional integro differential equations, *Appl. Math. Sci.,* **11** (45) (2017) 2237–2248.
11. Kilbas, A. A., Srivastava, H. M., Trujillo, J. J.: Theory and applications of fractional differential equations, [in:] *North-Holland Mathematics Studies, vol.204, Elsevier Science B. V., Amsterdam,* 2006.
12. Kucche, K. D., Kharade, J. P.: Global existence and Ulam-Hyers stability of $\psi-$Hilfer fractional differential equations, *arXiv:1807.10105v2*
13. Rassians, T. M.: On the stability of linear mappings in Banach spaces, *Proc. Amer. Math. Soc.,* **72** (1978) 297–300.
14. Rus, I. A.:Ulam stability of ordinary differential equations, *Studia "BABES-BOLYAI", Mathematica,* **54** (4) (2009) 125–133.
15. Samko, S. G., Kilbas, A. A., Marichev, O. I.: Fractional Integrals and Derivatives, Theory and Applications, *Gordon and Breach, New York,* 1983.
16. Siddiqi, A. H.: Functional analysis with applications, *Tata McGraw-Hill Publishing Ltd, New Delhi,* (1986).
17. Ulam,S. M.: A Collection of the Mathematical Problems, *Interscience Publishers, New York,* (1960).
18. Sousa J. Vanterler da C., Oliveria E. Capelas de: On the $\psi-$Hilfer fractional derivative, *Commun. Nonlinear Sci. Numer. Simulat.,* **60** (2018) 72–91.
19. Sousa J. Vanterler da C., Oliveria E. Capelas de: A Gronwall inequality and the Cauchy-type problem by means of $\psi-$Hilfer operator, *arXiv:1709.03634,* (2017).
20. Sousa, J. Vanterler da C, J., Oliveria E. Capelas de: On the Ulam-Hyers-Rassias stability for nonlinear fractional differential equations using the $\psi-$Hilfer operator, *Fixed Point Theory Appl.,* (2018) 20:96 https://doi.org/10.1007/s11784-018-0587-5.
21. Wang, J., Lv, L., Zhou, Y.: Ulam stability and data dependence for fractional differential equations with Caputo derivative, *Electron. J. Qual. Theory Differ. Equ.,* (63) (2011) 1–10.

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