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# REPRESENTATIONS OF THE SECOND KIND FOR THE SOLUTIONS TO THE FIRST ORDER GENERAL LINEAR ELLIPTIC SYSTEM IN THE SIMPLY CONNECTED PLANE DOMAIN

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Abstract. In this paper, representations of the second kind for solutions to the linear general uniform elliptic system of the first order in the simply connected plane domain $G$ are under consideration. In complex notation, we write system as $\mathcal{D}w \equiv \partial_{\bar z}w + q_1(z)\partial_z w + q_2(z)\partial_{\bar z}\overline{w} + A(z)w + B(z)\overline{w} = R(z)$, where $w = w(z) = u(z) + iv(z)$ is the desired complex function, $\partial_{\bar z} = 1/2(\partial/\partial x + i\partial/\partial y)$, $\partial_z = 1/2(\partial/\partial x - i\partial/\partial y)$, stand for Sobolev's derivatives, $q_1(z)$ and $q_2(z)$ are given measurable complex functions satisfying the condition of uniform ellipticity of the system: $|q_1(z)|+|q_2(z)| \leq q_0 = \text{const} < 1$, $z \in \overline{G}$, and $A(z)$, $B(z)$, $R(z) \in L_p(\overline{G})$, $p > 2$, are also given complex functions. The representation of the second kind is based on the well–known Pompeiu's formula: if $w(z) \in W_p^1(\overline{G})$, $p > 2$, then $w(z) = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \frac{w(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\iint\limits_{G} \frac{\partial w}{\partial \bar z} \cdot \frac{d\xi d\eta}{\zeta - z}$, $\mathcal{L} = \partial G$, where $w(z) \in W_p^1(\overline{G})$, $p > 2$. From here for the solution $w(z)$ we can write the representation $\Omega(w) = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \frac{w(\zeta)}{\zeta - z}d\zeta + TR(z)$ where $\Omega(w) \equiv w(z) + T(q_1(z)\partial_z w + q_2(z)\partial_{\bar z}\overline{w} + A(z)w + B(z)\overline{w})$, $Tf(\zeta) = -\frac{1}{\pi}\iint\limits_{G} \frac{f(\tau)}{\tau - \zeta}dxdy$. With the appropriate assumptions about $\mathcal{L}$ and the coefficients we prove that $\Omega$ is the isomorphism of the space $C_\alpha^k(\overline{G})$ $k \geq 1$, $0 < \alpha < 1$, or the space $W_p^1(\overline{G})$, $p > 2$. This result develops and supplements B.V. Boyarsky's works where representations of the first kind where obtained. Also this work supplements author's results on representations of the second kind when $G$ is a unit disk. As a corollary, we have the next a priori estimate. For an arbitrary function $w(z) \in C_\alpha^{k+1}(\overline{G})$, $k \geq 0$, we have: $\|w\|_{C_\alpha^{k+1}(\overline{G})} \leq \text{const}\left\{\|\mathcal{D}w\|_{C_\alpha^k(\overline{G})} + \|w\|_{C_\alpha^{k+1}(\mathcal{L})}\right\}$, where const depends only on $k$, $\alpha$ and the norms in $C_\alpha^k(\overline{G})$ of the coefficients of the operator $\mathcal{D}$.

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## 1. Introduction

Denote by $D \equiv D_z = \{z : |z| < 1\}$ the unit disk of the complex $z$-plane $E$, $z = x + iy$, $i^2 = -1$; $\Gamma = \partial D$ is the boundary of $D$; and $\overline{D} = D \cup \Gamma$; $G$ is the simply connected bounded domain of the complex $\zeta$-plane; $\partial G = \mathcal{L}$; $\overline{G} = G \cup \mathcal{L}$.

We use the following functional spaces with the standard norms: $L_p(\overline{D})$ is the space of functions integrable to the power $p \geq 1$ in $\overline{D}$; $W_p^k(\overline{D})$, $k = 0, 1, \ldots, p \geq 1$, is the class of functions having in $\overline{D}$ weak derivatives in the sense of Sobolev up to order $k$ integrable to the power $p$, $W_p^0(\overline{D}) \equiv L_p(\overline{D})$; $C_\alpha^k(\overline{D})$, $k = 0, 1, \ldots$, $0 \leq \alpha \leq 1$, is the space of functions having continuous partial derivatives up to order $k$ in $\overline{D}$ that are Hölder continuous with exponent $\alpha$, $C_0^k(\overline{D}) \equiv C^k(\overline{D})$, $C_\alpha^0(\overline{D}) \equiv C_\alpha(\overline{D})$; $L_p(\Gamma)$ and $C_\alpha^k(\Gamma)$ are defined similarly but for functions defined on $\Gamma$.

The notation $C_\alpha^k(\overline{G})$, $L_p(\overline{G})$, $W_p^k(\overline{G})$, $C_\alpha^k(\mathcal{L})$ has the similar sense. The detailed definitions of these spaces and norms can be found in [12].

We say the simple closed curve $\mathcal{L}$ belongs to $C_\alpha^k$ and write $\mathcal{L} \in C_\alpha^k$ if the curve $\mathcal{L}$ is the homeomorphic regular image of the circle $\Gamma$ : $\mathcal{L} = \{z : z = z(t) \equiv z(s) = x(s) + iy(s)\}$, $t = e^{is}$, $z'(s) \neq 0$, $z(t) \in C_\alpha^k(\Gamma)$, $k \geq 1$. In this case we say $\Gamma$ and $\mathcal{L}$ are $C_\alpha^k$–diffeomorphic.

Denote by $A_\alpha^k(\overline{D}) = A_\alpha^k \subset C_\alpha^k(\overline{D})$, $0 < \alpha < 1$, (respectively $A_p^k(\overline{D}) = A_p^k \subset W_p^k(\overline{D})$, $p > 2$) the closed subspace of holomorphic functions. The notation $A_\alpha^k(\overline{G})$ ($A_p^k(\overline{G})$) has the similar sense.

Following [1, Ch. 5] we denote by $W_p^{k-\frac{1}{p}}(\mathcal{L})$ the set of boundary values of functions from $W_p^k(\overline{G})$ (see too [9, p. 267]). Later we'll define in $W_p^{k-\frac{1}{p}}(\mathcal{L})$ the norm of a Banach space.

We consider in $\overline{G}$ the general linear elliptic first-order system in complex notation

$$\mathcal{D}w \equiv \partial_{\bar{\zeta}} w + q_1(\zeta)\partial_\zeta w + q_2(\zeta)\partial_{\bar{\zeta}}\overline{w} + A(\zeta)w + B(\zeta)\overline{w} = R(\zeta), \tag{1.1}$$

where $\zeta = \xi + i\eta$, $w = w(\zeta) = u(\zeta) + iv(\zeta)$ is the desired complex function, $\partial_{\bar{\zeta}} = 1/2(\partial/\partial\xi + i\partial/\partial\eta)$, $\partial_\zeta = 1/2(\partial/\partial\xi - i\partial/\partial\eta)$, stand for Sobolev's derivatives, $q_1(\zeta)$ and $q_2(\zeta)$ are given measurable complex functions satisfying the condition of uniform ellipticity of System (1.1)

$$|q_1(\zeta)| + |q_2(\zeta)| \leq q_0 = \text{const} < 1,\ \zeta \in \overline{G}, \tag{1.2}$$

and $A(\zeta)$, $B(\zeta)$, $R(\zeta) \in L_p(\overline{G})$, $p > 2$, are also given complex functions.

The representations for the solutions to System (1.1) are very useful and establish the correspondence between solutions and holomorphic functions. In the case $q_1(\zeta) = q_2(\zeta) \equiv 0$ the representations of two kinds for continuous in $\overline{G}$ solutions where established by different authors (see [12]).

The representation of the first kind for the solution to the homogeneous system is

$$w(\zeta) = \Phi(\zeta) \exp\left\{-T\left(A + B\frac{\overline{w}}{w}\right)\right\}, \tag{1.3}$$

where $\Phi(\zeta)$ is some holomorphic function and

$$Tf(\zeta) = T_G f(\zeta) = -\frac{1}{\pi}\iint\limits_G \frac{f(\tau)}{\tau - \zeta} dxdy, \quad \tau = x + iy,$$
$$\partial_{\bar\zeta} Tf(\zeta) = f(\zeta). \tag{1.4}$$

The representation of the second kind is

$$w(\zeta) + T(Aw + B\overline{w}) = \Phi(\zeta) + TR(\zeta), \tag{1.5}$$

where $\Phi(\zeta)$ is also some holomorphic function and $w(\zeta)$ is the solution to the non-homogeneous system in general.

Naturally it is interesting when these representations may be inverted, i. e. when the solution $w(\zeta)$ is uniquely determined by the holomorphic function $\Phi(\zeta)$.

For the representation (1.3) the reversibility in $W_p^1(\overline{G})$, $p > 2$, for $\Phi(z) \in A_p^1(\overline{G})$ is proved in [12, Ch. 3, §7]. If the coefficients $A(\zeta)$, $B(\zeta)$ and $\Phi(\zeta)$ are smooth enough in $\overline{G}$ we can assert only $w(\zeta) \in W_p^1(\overline{G})$ with $p > 2$ arbitrary large.

There is more complete picture for the representation (1.5). Under appropriate assumptions about $A$, $B$ and the boundary $\mathcal{L}$ the operator in the left–hand side of (1.5) is an isomorphism of the Banach space $W_p^k(\overline{G})$ or $C_\alpha^k(\overline{G})$ (see [12, Ch. 3, §5; Ch. 4]).

The first fundamental research of the solutions to the general system (1.1) belongs to B. Bojarski [2]. He received the generalization of representation of the first kind (1.3). Also he established in [2] the reversibility of this representation. These results have previous defects. Moreover if the holomorphic function $\Phi(\zeta)$ is good enough, we can assert only $w(\zeta) \in W_s^1(\overline{G})$, where $s > 2$ is close to 2.

The representation of the second kind (1.5) is based on the well–known Pompeiu's formula [12, pp. 41, 57]:

$$w(\zeta) = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \frac{w(\tau)}{\tau - \zeta} d\tau - \frac{1}{\pi}\iint\limits_G \frac{\partial w}{\partial \bar\tau} \cdot \frac{dxdy}{\tau - \zeta}, \tag{1.6}$$

where $w(\zeta) \in W_p^1(\overline{G})$, $p > 2$.

For the given solution $w(\zeta) \in W_p^1(\overline{G})$, $p > 2$, to the general equation (1.1), where $A(\zeta)$, $B(\zeta)$, $R(\zeta) \in L_p(\overline{G})$, we can write the representation of the second kind

$$\Omega(w) = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \frac{w(\tau)}{\tau - \zeta} d\tau + TR(\zeta), \tag{1.7}$$

where

$$\Omega(w) \equiv w(\zeta) + T(q_1 \partial_\tau w + q_2 \partial_{\bar\tau} \overline{w} + Aw + B\overline{w})(\zeta). \tag{1.8}$$

Of course, it is interesting when the operator $\Omega$ is invertible.

This problem is solved in [4] in the special case when $G$ is the unit disk $D$. In [4] one can find a more detailed overview of the works on this topic.

The next assertions are the main results of this paper.

**Theorem 1.1.** *If $q_1(\zeta)$, $q_2(\zeta) \in C(\overline{G})$, $A(\zeta)$, $B(\zeta) \in L_p(\overline{G})$, $p > 2$, $\partial G = \mathcal{L} \in C_\alpha^1$, $0 < \alpha < 1$, then $\Omega$ is a (real) linear isomorphism of the Banach space $W_p^1(\overline{G})$.*

**Theorem 1.2.** *If* $q_1(\zeta)$, $q_2(\zeta)$, $A(\zeta)$, $B(\zeta) \in C^k_\alpha(\overline{G})$, $k \geq 0$, $0 < \alpha < 1$, $\partial G = \mathcal{L} \in C^{k+1}_\alpha$, *then* $\Omega$ *is a (real) linear isomorphism of the Banach space* $C^{k+1}_\alpha(\overline{G})$.

The proofs of these theorems use the results on the special cases from [4].

Here are two simple but important corollaries of Theorems 1.1 and 1.2.

**Theorem 1.3.** *In assumptions of Theorem 1.1 for an arbitrary function* $w(\zeta) \in W^1_p(\overline{G})$ *we have a priori estimate:*

$$\|w\|_{W^1_p(\overline{G})} \leq \text{const}\left\{\|\mathcal{D}w\|_{L_p(\overline{G})} + \|w\|_{W_p^{1-\frac{1}{p}}(\mathcal{L})}\right\},$$

*where* const *depends only on* $p$, $G$, *and the norms* $\|q_1\|_{C(\overline{G})}$, $\|q_2\|_{C(\overline{G})}$, $\|A\|_{L_p(\overline{G})}$, $\|B\|_{L_p(\overline{G})}$.

**Theorem 1.4.** *In assumptions of Theorem 1.2 for an arbitrary function* $w(\zeta) \in C^{k+1}_\alpha(\overline{G})$, $k \geq 0$, *we have a priori estimate:*

$$\|w\|_{C^{k+1}_\alpha(\overline{G})} \leq \text{const}\left\{\|\mathcal{D}w\|_{C^k_\alpha(\overline{G})} + \|w\|_{C^{k+1}_\alpha(\mathcal{L})}\right\},$$

*where* const *depends only on* $k$, $\alpha$, $G$ *and the norms in* $C^k_\alpha(\overline{G})$ *of the coefficients of the operator* $\mathcal{D}$.

## 2. Auxiliary statements

**2.1. Operators $T$ and $\Pi$.** Denote by

$$\Pi f(\zeta) = \partial_\zeta T f(\zeta) = -\frac{1}{\pi}\iint\limits_G \frac{f(\tau)}{(\tau-\zeta)^2} d\xi d\eta,\ \tau = \xi + i\eta,$$

the singular two–dimensional integral.

**Lemma 2.1.** *The singular operator* $\Pi$ *is bounded in* $C^k_\alpha(\overline{G})$, $k \geq 0$, $0 < \alpha < 1$, *and* $W^k_p(\overline{G})$, $k \geq 0$, $p > 2$.

*We have* $\|\Pi\|_{L_2} = 1$ *and for arbitrary* $q_0 : 0 < q_0 < 1$ *there is* $s_0 = s_0(q_0) > 2$ *such that* $q_0\|\Pi\|_{L_s} < 1$ *when* $2 < s \leq s_0$.

The proof of this assertions one can find in [12, Ch. 1, §8, §9] and [5].

From (1.4) and Lemma 2.1 we have

**Lemma 2.2.** *The operator* $T$ *continuously maps* $C^k_\alpha(\overline{G})$, $k \geq 0$, $0 < \alpha < 1$, *into* $C^{k+1}_\alpha(\overline{G})$, *and* $W^k_p(\overline{G})$, $k \geq 0$, $p > 2$, *into* $W^{k+1}_p(\overline{G})$.

The similar assertion takes place for the operator

$$\overline{T} f(\zeta) = \overline{\left(T\overline{f(\zeta)}\right)},\ \ \partial_\zeta \overline{T} f(\zeta) = f(\zeta).$$

**2.2. Shifts.** Let $\mathcal{L} \in C^k_\alpha$, $k \geq 1$, $0 \leq \alpha \leq 1$, is a regular curve; $\zeta = f(t)$ is a diffeomorphism between the circle $\Gamma$ and $\mathcal{L}$; $s$ is an arc on $\Gamma$, and $\sigma$ is an arc on $\mathcal{L}$.

We have the next evident relationships:

$$\zeta_t(t) \neq 0,\ \ \zeta_t = \zeta_s \cdot s_t = \zeta_s \cdot \overline{t}_s,$$

where $t$ is an affix on $\Gamma$; if $z=\zeta^{-1}(\tau)$, then

$$z_\tau(\tau)\neq 0,\ \ z_\tau=z_\sigma\cdot\sigma_\tau=z_\sigma\cdot\overline{\tau}_\sigma,$$

where $\tau$ is an affix on $\mathcal{L}$.

Generalizing [7, p. 33] we introduce the operator $\mathcal{W}\varphi(t)=\varphi(\zeta(t))$, where $\varphi(\zeta)$ is the function defined on $\mathcal{L}$, and $\zeta=\zeta(t)\in C^k_\alpha(\Gamma)$, $k\geq 1$, $0\leq\alpha\leq 1$, is a diffeomorphism between $\Gamma$ and $\mathcal{L}\in C^k_\alpha$.

It is evident that $\mathcal{W}$ is linear, bounded, continuously invertible operator. It maps $C^k_\alpha(\mathcal{L})$ into $C^k_\alpha(\Gamma)$.

Denote by

$$S_\Gamma\varphi(t)=\frac{1}{\pi i}\int\limits_\Gamma\frac{\varphi(\tau)}{\tau-t}d\tau,\quad t\in\Gamma, \tag{2.1}$$

the one–dimensional singular integral operator. Analogously the operator $S_\mathcal{L}$ is defined.

Here we need the next properties of the composition

$$\Psi\varphi(t)=\left(\mathcal{W}S_\mathcal{L}\mathcal{W}^{-1}-S_\Gamma\right)\varphi(t)=\frac{1}{\pi i}\int\limits_\Gamma\left[\frac{\zeta(\tau)}{\zeta(\tau)-\zeta(t)}-\frac{1}{\tau-t}\right]\varphi(\tau)d\tau,$$

which have been established in [6].

**Theorem 2.3.** *Let* $\zeta(t)\in C^1_\alpha(\Gamma)$, $0<\alpha\leq 1$, $\varphi(t)\in C_\beta(\Gamma)$, $0<\beta\leq 1$, $\mu=\alpha+\beta\leq 2$.

*If* $\mu<1$ *then* $\Psi\varphi(t)\in C_\mu(\Gamma)$ *and*

$$\|\Psi\varphi(t)\|_{C_\mu(\Gamma)}\leq\text{const}\|\varphi(t)\|_{C_\beta(\Gamma)}, \tag{2.2}$$

*where* const *depends only on* $\|\zeta\|_{C^1_\alpha(\Gamma)}$.

*If* $\mu=1$ *then* $\Psi\varphi(t)\in C_{\mu-\varepsilon}(\Gamma)$ *for* $\forall\varepsilon:0<\varepsilon<\mu$ *and the estimate similar to* (2.2) *takes place.*

*If* $\mu>1$ *then* $\Psi\varphi(t)\in C^1_{\mu-1}(\Gamma)$ *and*

$$\|\Psi\varphi(t)\|_{C^1_{\mu-1}(\Gamma)}\leq\text{const}\|\varphi(t)\|_{C_\beta(\Gamma)}, \tag{2.3}$$

*where* const *depends only on* $\|\zeta\|_{C^1_\alpha(\Gamma)}$.

**Corollary 2.4.** *If* $\zeta(t)\in C^1_\alpha(\Gamma)$, $0<\alpha\leq 1$, $\varphi(t)\in C^1_\beta(\Gamma)$, $0<\beta\leq 1$, *then* $\Psi\varphi(t)\in C^1_\alpha(\Gamma)$ *and*

$$\|\Psi\varphi(t)\|_{C^1_\alpha(\Gamma)}\leq\text{const}\|\varphi(t)\|_{C^1_\beta(\Gamma)}, \tag{2.4}$$

*where* const *depends only on* $\|\zeta\|_{C^1_\alpha(\Gamma)}$.

*Remark* 2.5. In [6] $\mathcal{L}$ is a circle but it is not used anywhere.

Obviously, in theorem 2.3 we can take $\mathcal{L}$ as a circle and $\Gamma$ as its diffeomorphic image. About change of variables in the one–dimensional singular integral see [3, p. 31].

**2.3. The Space $W_p^{k-\frac{1}{p}}(\mathcal{L})$.** We assume $\partial G = \mathcal{L} \in C_\alpha^k$, $k \geq 1$, $0 < \alpha < 1$. Following [1, Ch. 5], we denote by $W_p^{k-\frac{1}{p}}(\mathcal{L})$, $p > 2$, the set of boundary values of functions from $W_p^k(\overline{G})$ (see also [9, p. 267]).

Let $f(\tau) \in W_p^{k-\frac{1}{p}}(\mathcal{L})$ and $F(z) \in W_p^k(\overline{G})$ is the harmonic function such as $F(\tau) = f(\tau)$, $\tau \in \mathcal{L}$. The norm of $f(\tau) \in W_p^{k-\frac{1}{p}}(\mathcal{L})$ we define as $\|F(z)\|_{W_p^k(\overline{G})}$. This is a Banach norm and the singular operator $S_\mathcal{L}$ is bounded in $W_p^{k-\frac{1}{p}}(\mathcal{L})$ (see [9, Ch. 6, § 1]).

Because by imbedding theorem $W_p^k(\overline{G}) \subset C_\beta^{k-1}(\overline{G})$, $\beta = \frac{p-2}{p}$, we have

$$W_p^{k-\frac{1}{p}}(\mathcal{L}) \subset C_\beta^{k-1}(\mathcal{L}), \ \ k \geq 1, \ p > 2. \tag{2.5}$$

Denote by

$$\mathcal{K}f(\zeta) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{f(\tau)}{\tau - \zeta} d\tau, \ \ \zeta \in G, \tag{2.6}$$

the Cauchy–type integral.

The inequality

$$\|\mathcal{K}f(\zeta)\|_{W_p^k(\overline{G})} \leq \text{const} \|f(t)\|_{W_p^{k-\frac{1}{p}}(\mathcal{L})}, \ \ k \geq 1, \ p > 2, \tag{2.7}$$

takes place, where the constant does not depend on $f$ (see [9, Ch. 6, § 1]).

**2.4. One property of the Cauchy–type integral.**

**Lemma 2.6.** *If $\Gamma = \partial D$ is the unit circle $|t| = 1$, $f(z) \in A_\alpha^1(\overline{D})$, $0 < \alpha < 1$, $w(t) \in C_\alpha(\Gamma)$, $\mathcal{K}(fw)(z) \in A_\alpha^1(\overline{D})$, $f(z) \neq 0$, $z \in \overline{D}$, then $\mathcal{K}w(z) \in A_\alpha^1(\overline{D})$.*

*Proof.* Consider the holomorphic in $D$ function

$$f(z)\mathcal{K}w(z) - \mathcal{K}(fw)(z) = P(z) \in C_\alpha(\overline{D}).$$

Differentiating it with respect to $z \in D$, we obtain:

$$\begin{aligned} P(z) = \frac{f(z)}{2\pi i} \int\limits_\Gamma \frac{w(t)dt}{t - z} + \frac{1}{2\pi i} \int\limits_\Gamma \frac{f(z) - f(t)}{t - z} \cdot \frac{w(t)dt}{t - z} = \\ = I_1(z) + I_2(z). \end{aligned} \tag{2.8}$$

By the properties of the Cauchy–type integral we have $I_1(z) \in C_\alpha(\overline{D})$ (see [12, Ch. 1, §3]).

Denote by

$$\varphi(z,t) = \frac{f(z) - f(t)}{t - z} \cdot w(t), \ \ t \in \Gamma, \ \ z \in \overline{D}.$$

If we put $\dfrac{f(z) - f(t)}{t - z} = -f(t)$ for $t = z$, then $\varphi(z,t)$ belongs to $C_\alpha$ with respect to $z$ and to $t$ (see [10, Ch. 1, §7]).

We rewrite $I_2(z)$ as

$$I_2(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\varphi(z,t)-\varphi(z,z)}{t-z}dt+$$

$$+\frac{\varphi(z,z)}{2\pi i}\int\limits_{\Gamma} \frac{dt}{t-z} = I_{21}(z) + I_{22}(z).$$

Similar to the arguments from [3, §4, sec. 1] we get $I_{21}(z) \to I_{21}(\tau)$ when $z \to \tau \in \Gamma$.

Thus, we have

$$I_2(z) \to I_2^+(\tau) = \frac{1}{2}\varphi(\tau,\tau) + \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau,t)dt}{t-\tau},$$

when $z \to \tau \in \Gamma$.

By Sokhotski–Plemelj formulas we get (see [10, §16])

$$I_1(z) \to I_1^+(\tau) = \frac{1}{2}f(\tau)w(\tau) + \frac{f(\tau)}{2\pi i}\int\limits_{\Gamma} \frac{w(t)dt}{t-z},$$

when $z \to \tau \in \Gamma$.

Combining this relations we get

$$P(z) \to P(\tau) = \frac{f(\tau)}{2\pi i}\int\limits_{\Gamma} \frac{w(t)dt}{t-\tau}+$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau,t)dt}{t-\tau} = I_{31}(\tau) + I_{32}(\tau).$$

By the properties of the singular operator (2.1) we have $I_{31}(\tau) \in C_\alpha(\Gamma)$ (see [10, Ch. 1, §19]).

Since $\varphi(z,t)$ belongs to $C_\alpha$ with respect to $z$ and to $t$, we have $I_{32}(\tau) \in C_\alpha(\Gamma)$ [10, Ch. 1, §18, sec. 4]. So, the boundary values $P(\tau)$ of the holomorphic function $P(z)$ belong to $C_\alpha(\Gamma)$. This fact implies $P(z) \in C_\alpha(\overline{D})$, $P(z) \in A^1_\alpha(\overline{D})$ and $\mathcal{K}w(z) \in A^1_\alpha(\overline{D})$. □

### 2.5. The uniqueness of solutions.

**Lemma 2.7.** *If in the elliptic equation* (1.1) *the coefficients* $q_1(\zeta)$, $q_2(\zeta)$ *are measurable functions,* $A(\zeta)$, $B(\zeta) \in L_p(\overline{G})$, $p > 2$, $R(\zeta) \equiv 0$, $\mathcal{L} \in C^1_\alpha$, $0 < \alpha < 1$, *and the boundary values of the solution* $w(\zeta) \in W^1_s(\overline{G})$, $s > 2$, *equal zero on a set* $l \subset \mathcal{L}$, $\operatorname{mes} l > 0$, *then* $w(\zeta) \equiv 0$.

This assertion follows immediately from theorem 4.4 of [2] (see also [12, Ch. 3, §17]).

**2.6. Two–dimensional integrals.**

**Lemma 2.8.** *Let $\mathcal{L} = \partial G \in C^1_\alpha$, $0 < \alpha < 1$, $D$ be the unit disk $|z| \leq 1$, $\partial D = \Gamma$, and $\zeta = \zeta(z) : D \to G$ be a conformal (one-to-one) mapping. If $w(\zeta) \in W^1_s(\overline{G})$, $s > 2$, then we have*

$$\iint\limits_D \left[\frac{\zeta(t)}{\zeta(t)-\zeta(z)} - \frac{1}{t-z}\right] \partial_{\bar t} w(\zeta(t)) dxdy =$$

$$= \frac{1}{2i} \int\limits_\Gamma \left[\frac{\zeta(t)}{\zeta(t)-\zeta(z)} - \frac{1}{t-z}\right] w(\zeta(t)) dt, \quad t = x + iy. \tag{2.9}$$

*Proof.* From (1.6) we get the next formulas:

$$w(\zeta) = -\frac{1}{\pi} \iint\limits_G \frac{w_{\bar\tau}(\tau)}{\tau - \zeta} d\xi d\eta + \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{w(\tau)}{\tau-\zeta} d\tau, \quad \tau = \xi + i\eta, \tag{2.10}$$

$$w(\zeta(z)) = -\frac{1}{\pi} \iint\limits_D \frac{w_{\bar t}(\zeta(t))}{t-z} dxdy + \frac{1}{2\pi i} \int\limits_\Gamma \frac{w(\zeta(t))}{t-z} dt, \quad t = x + iy. \tag{2.11}$$

Note that by Kellog's theorem we have $\zeta(z) \in C^1_\alpha(\overline{D})$ (see [12, Ch. 1, §2]).

By changing of variable $\tau = \zeta(t)$ and replacing $\zeta$ by $\zeta(z)$ in (2.10), we get:

$$w(\zeta(z)) = -\frac{1}{\pi} \iint\limits_D \frac{w_{\bar t}(\zeta(t)) \cdot \zeta(t)}{\zeta(t)-\zeta(z)} dxdy + \frac{1}{2\pi i} \int\limits_\Gamma \frac{w(\zeta(t)) \cdot \zeta(t)}{\zeta(t)-\zeta(z)} dt.$$

Now (2.9) follows from (2.11) and the last equality. □

We denote by $Nw(z)$ the left–hand side of (2.9).

**Lemma 2.9.** *In the assumptions of Lemma 2.8 and when $\alpha s < 2$ we have $Nw(z) \in A^1_q(\overline{D})$, where $q = \dfrac{2s}{2-\alpha s}$.*

*Proof.* From (2.9) it is evident that $\partial_{\bar z} Nw(z) \equiv 0$. So, it is sufficient to prove that $\partial_z Nw(z) \in L_q(\overline{D})$.

We differentiate (2.10) with respect to $\zeta$, and (2.11) with respect to $z$:

$$\partial_\zeta w(\zeta) = -\frac{1}{\pi} \iint\limits_G \frac{w_{\bar\tau}(\tau)}{(\tau-\zeta)^2} d\xi d\eta + \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{w(\tau)}{(\tau-\zeta)^2} d\tau, \tag{2.12}$$

$$\partial_z w(\zeta(z)) = -\frac{1}{\pi} \iint\limits_D \frac{w_{\bar t}(\zeta(t))}{(t-z)^2} dxdy + \frac{1}{2\pi i} \int\limits_\Gamma \frac{w(\zeta(t))}{(t-z)^2} dt. \tag{2.13}$$

We mean the two–dimensional integrals in the right–hand sides of these formulas in the Cauchy sense. All terms in the right–hand sides of these formulas belong to $L_s(\overline{G})$ and to $L_s(\overline{D})$ respectively [12, Ch. 1, §§ 8, 9].

Because $\zeta = \zeta(z)$ is a conformal mapping, we have a usual rule of change of variables in the two–dimensional singular integral in (2.12) (see [8, Ch. 2, §5, Sec. 4,5]). Using this fact, we pass in (2.12) to the variables $t$, $z$:

$$\partial_z w(\zeta(z)) = -\frac{1}{\pi}\iint\limits_D \frac{w_{\bar t}(\zeta(t))\zeta(t)\zeta(z)}{(\zeta(t)-\zeta(z))^2}dxdy + \frac{1}{2\pi i}\int\limits_\Gamma \frac{w(\zeta(t))\zeta(t)\zeta'(z)}{(\zeta(t)-\zeta(z))^2}dt. \quad (2.14)$$

Subtracting (2.14) and (2.13), we get:

$$\partial_z Nw(z) = -\frac{1}{\pi}\iint\limits_D \left[\frac{\zeta(t)\zeta(z)}{(\zeta(t)-\zeta(z))^2} - \frac{1}{(t-z)^2}\right] w_{\bar t}(\zeta(t))dxdy \in L_s(\overline{D}).$$

Further, we have:

$$\left|\frac{\zeta(t)\zeta(z)}{(\zeta(t)-\zeta(z))^2} - \frac{1}{(t-z)^2}\right| \le \frac{\text{const}}{|t-z|^{2-\alpha}},$$

where the constant depends only on $\|\zeta(z)\|_{C^1_\alpha(\Gamma)}$ (see [6]).

From here and $\partial_{\bar z} w(\zeta(z)) \in L_s(\overline{D})$ we get $\partial_z Nw(z) \in L_q(\overline{D})$ (see [11, Ch. 5, Sec. 1.2]). □

## 3. Proofs of the main results

**3.1. Proof of Theorem 1.1.** For the beginning, we give a scheme of the proof. At first, we prove a unique solvability of the equation

$$\Omega(w) = F \in W^1_p(\overline{G}), \quad p > 2, \quad (3.1)$$

in $W^1_s(\overline{G})$, where $s : 2 < s \le p$, and $s$ is sufficiently close to 2. Further, we prove that the unique solution to Equation (3.1) $w(\zeta) \in W^1_s(\overline{G})$ actually belongs to $W^1_p(\overline{G})$. We use here a special case of Theorem 1.1 when $G$ is the unit disk $D$. This special case has been proved in [4]. Since the operator $\Omega$ is bounded in $W^1_p(\overline{G})$ (see Lemma 2.2), now it is sufficient to use the theorem of Banach.

**Lemma 3.1.** *If $q_1(\zeta)$ and $q_2(\zeta)$ are the measurable functions, satisfying* (1.2), *$A(\zeta)$, $B(\zeta) \in L_s(\overline{G})$, $s > 2$, then the equation*

$$\Omega(w) = 0 \quad (3.2)$$

*has only zero solution in $W^1_s(\overline{G})$.*

*Proof.* Let $w(\zeta) \in W^1_s(\overline{G}) \subset C_\beta(\overline{G})$, $\beta = \dfrac{s-2}{s}$, be a non–zero solution to Equation (3.2). It is known, that if $f(\zeta) \in L_s(\overline{G})$, then the function $Tf(\zeta) \in C_\beta(E)$, is holomorphic outside of $\overline{G}$, and $Tf(\infty) = 0$ [12, p. 54–58]. Thus, by (3.2) we get $w(\zeta) \in C(E)$, and $w(\zeta)$ is holomorphic outside of $\overline{G}$, and $w(\infty) = 0$.

From the other hand, by differentiation (3.2) with respect to $\bar\zeta$ we get that $w(\zeta)$ satisfies to the homogeneous equation (1.1) (under the condition $R(\zeta) \equiv 0$).

It follows from here and (1.6) that $w(\zeta)$ satisfies the equation

$$\Omega(w) = \Phi(\zeta), \text{ where } \Phi(\zeta) = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \frac{w(\tau)}{\tau-\zeta}d\tau. \quad (3.3)$$

So, the Cauchy type integral $\Phi(\zeta)$ in (3.3) is continues in the complex plane $E$ and $\Phi(\infty) = 0$. It's possible only when $w(\tau) \equiv 0$, $\tau \in \mathcal{L}$ (see [3, p. 39]).

Thus, $w(\zeta) \in C_\beta(\overline{G})$ is a solution to the homogeneous equation (1.1), and $w(\tau) \equiv 0$, $\tau \in \mathcal{L}$. But this implies $w(\zeta) \equiv 0$, $\zeta \in \overline{G}$ (see Lemma 2.7). □

**Lemma 3.2.** *If $q_1(\zeta)$ and $q_2(\zeta)$ are measurable functions, satisfying* (1.2)*, $A(\zeta)$, $B(\zeta) \in L_p(\overline{G})$, $p > 2$, then there exists a unique solution $w(\zeta) \in W^1_s(\overline{G})$ to Equation* (3.1)*, where $s : 2 < s \le p$, and $s$ is sufficiently close to 2.*

*Proof.* The uniqueness follows from Lemma 3.1. Let us show that the operator

$$\Omega_1(w) = w + T_G(q_1 \partial_\zeta w + q_2 \partial_{\bar\zeta} \overline{w}) \tag{3.4}$$

has bounded inverse one in $W^1_s(\overline{G})$, if $2 < s \le p$, and $s$ is sufficiently close to 2.

Consider the equations

$$\Omega_1(w) = \omega \in W^1_s(\overline{G}), \tag{3.5}$$

$$\lambda + \Pi(q_1\lambda + q_2\overline{\lambda}) \equiv \lambda + \sigma\lambda = \partial_\zeta \omega. \tag{3.6}$$

We obtained Equation (3.6) by differentiating (3.5) with respect to $\zeta$ and replacing $\partial_\zeta w$ by $\lambda(\zeta)$. By Lemma 2.1 there exists the number $s : 2 < s \le s_0(q_0) \le p$ such that $q_0\|\Pi\|_{L_s} < 1$.

By contracting–mapping principle Equation (3.6) is uniquely solvable in $L_s(\overline{G})$:

$$\lambda(\zeta) = (I + \sigma)^{-1} \partial_\zeta \omega(\zeta), \tag{3.7}$$

and a norm of the linear operator $(I + \sigma)^{-1} : L_s(\overline{G}) \to L_s(\overline{G})$ is bounded by the constant, which depends only on $q_0$.

Let us find a solution to Equation (3.5) in the form

$$w(\zeta) = \overline{T\overline{\lambda}} + \Psi(\bar\zeta), \tag{3.8}$$

where $\Psi(\zeta) \in A^1_s(\overline{G})$ is a desired holomorphic function; $\lambda$ is defined in (3.7).

Substituting (3.8) in (3.5), we get:

$$\Psi(\bar\zeta) = \omega(\zeta) - \overline{T\overline{\lambda}} - T(q_1\lambda + q_2\overline{\lambda}). \tag{3.9}$$

If $f(\zeta) \in L_s(\overline{G})$, then $Tf(\zeta) \in W^1_s(\overline{G})$ (see [12, Ch. 1, §6]). Hence, by (1.4), (3.6) we have $\partial_\zeta \Psi(\bar\zeta) = 0$, i. e. the function $\Psi(\zeta)$, defined by (3.9) is holomorphic and belongs to $W^1_s(\overline{G})$.

So, the formula

$$w(\zeta) = \omega(\zeta) - T\left(q_1(\zeta)(I + \sigma)^{-1}\partial_\zeta \omega + q_2(\zeta)\overline{(I + \sigma)^{-1}\partial_\zeta \omega}\right) \tag{3.10}$$

defines a solution to Equation (3.5) from $W^1_s(\overline{G})$. From Lemma 2.2 we have that the operator $\Omega_1 : W^1_s(\overline{G}) \to W^1_s(\overline{G})$ is continuous. Hence, by the Banach theorem the inverse operator $\Omega_1^{-1} : W^1_s(\overline{G}) \to W^1_s(\overline{G})$ is continuous too.

*Remark* 3.3. From (3.10) and the properties of $T$ [12, Ch. 1, §6] it follows, that a norm of the linear operator $\Omega_1^{-1} : W^1_s(\overline{G}) \to W^1_s(\overline{G})$ is bounded by the number, which depends only on $q_0$.

Let us rewrite Equation (3.1) in the form

$$w + \Omega_1^{-1} \circ Pw = \Omega_1^{-1} F, \tag{3.11}$$

where $Pw = T(Aw + B\overline{w})$. Since the operator $P$ is completely continuous in $C(\overline{G})$ and maps $C(\overline{G})$ into $W_p^1(\overline{G})$ [12, Ch. 1, §6], the operator $\Omega_1^{-1} \circ P$ is completely continuous in $C(\overline{G})$ and maps $C(\overline{G})$ into $W_s^1(\overline{G})$.

Owing to the properties of the operators in (3.11), a continuous in $\overline{G}$ solution to the homogenous equation (3.11) belongs to $W_s^1(\overline{G})$. By Lemma 3.1 this solution equals to zero.

Thus, by Fredholm's theorem Equation (3.11) is uniquely solvable in $C(\overline{G})$ and its solution $w(\zeta) \in W_s^1(\overline{G})$. □

By Lemma 3.2 there exists a unique solution $w(\zeta) \in W_s^1(\overline{G})$, $s : 2 < s \leq p$, to the equation

$$\Omega(w) = F \in W_p^1(\overline{G}). \tag{3.12}$$

Let us show that this solution $w(\zeta) \in W_p^1(\overline{G})$.

Denote by $\zeta = \zeta(z)$ a conformal schlicht mapping of the closed disk $\overline{D}$ onto the closed domain $\overline{G}$. It is known that $\zeta(z) \in C_\alpha^1(\overline{D})$ [12, Ch. 1, §2], and the invers mapping $z = z(\zeta) \in C_\alpha^1(\overline{G})$. Denote by $w(z)$ the composition $w(\zeta(z)) \in W_s^1(\overline{D})$.

The function $w(z)$ satisfies to the differential equation

$$\partial_{\bar z} w(z) + \tilde q_1(z) \partial_z w + \tilde q_2(z) \partial_{\bar z} \overline{w}(z) + \tilde A(z) w(z) + \tilde B(z) \overline{w}(z) = \tilde F(z), \tag{3.13}$$

where

$$\tilde q_1(z) = q_1(\zeta(z)) \cdot \frac{\overline{\zeta_z}}{\zeta_z}, \quad \tilde q_2(z) = q_2(\zeta(z)),$$
$$\tilde A(z) = A(\zeta(z)) \cdot \overline{\zeta_z}, \quad \tilde B(z) = B(\zeta(z)) \cdot \overline{\zeta_z}, \quad \tilde F(z) = \partial_{\bar z} F(\zeta(z));$$

$\tilde q_1(z)$, $\tilde q_2(z) \in C(\overline{D})$, $\tilde A(z)$, $\tilde B(z)$, $\tilde F(z) \in L_p(\overline{D})$.

It follows from (3.13) and (1.6), that $w(z)$ satisfies to the integro–differential equation

$$\tilde\Omega(w) = T_D \tilde F(z) + \tilde\Phi(z), \tag{3.14}$$

where

$$\tilde\Omega(w) = w(z) + T_D(\tilde q_1 \partial_z w + \tilde q_2 \partial_{\bar z} \overline{w} + \tilde A w + \tilde B \overline{w}),$$
$$\tilde\Phi(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{w(t)dt}{t - z} \in A_s^1(\overline{D}),$$

$T_D \tilde F(z) \in W_p^1(\overline{D})$ (see. (2.7)).

It is known from [4], that the operator $\tilde\Omega$ is a (real) linear isomorphism of the Banach space $W_p^1(\overline{D})$. Thus, to complete the proof of Theorem 1.1 we have to show that $\tilde\Phi(z) \in A_p^1(\overline{D})$. Let us do it.

First we mark, that by (1.6)

$$F(\zeta) = T_G \partial_{\bar\zeta} F + \Phi(\zeta), \tag{3.15}$$

where

$$\Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{w(\tau)d\tau}{\tau - \zeta} \in A_p^1(\overline{G}). \tag{3.16}$$

Now we suppose $s < p$ and $\alpha s < 2$. By (2.9) and Lemma 2.9 we have $\Phi(\zeta(z)) - \tilde{\Phi}(z) \in A^1_q(\overline{D})$, where $q = \min\left[p, s + \dfrac{\alpha s}{2 - \alpha s}\right] > s$.

If $q \geq p$, from here and (3.16) we have $\tilde{\Phi}(z) \in A^1_p(\overline{D})$ and the proof is completed.

If $q < p$ and $\alpha q < 2$, changing $s$ by $s_1 = q$ we repeat the last argument and get $\Phi(\zeta(z)) - \tilde{\Phi}(z) \in A^1_{q_1}(\overline{D})$, where $q_1 = \min\left[p, s_1 + \dfrac{\alpha s_1}{2 - \alpha s_1}\right] > q$.

If $\alpha s \geq 2$, in the first reasoning we can decrease $s$ such, that there will be $\alpha s < 2$ and $q \geq p$. Analogously, we can do it in the reiterated reasoning for $s_1$.

It is clear, that at the next step with number $n$ we get $q_n \geq p$ and complete the proof of Theorem 1.1.

**3.2. Proof of Theorem 1.2.** We use the method of mathematical induction for $k \geq 0$.

**3.2.1.** *The base case* $k = 0$. To prove this special case we slightly modify the arguments from Sec. 3.1.

Consider the equation

$$\Omega(w) = F(\zeta) \in C^1_\alpha(\overline{G}). \tag{3.17}$$

By Theorem 1.1 this equation has a unique solution $w(\zeta) \in W^1_s(\overline{G}) \subset C_\beta(\overline{G})$, where $s > 2$ is arbitrary large; hence $\beta = \dfrac{s-2}{s}$ is arbitrary close to 1. As in Sec. 3.1, owing to Banachs theorem we have to show that $w(\zeta) \in C^1_\alpha(\overline{G})$.

As in (3.15) we have

$$F(\zeta) = T_G \partial_{\bar{\zeta}} F(\zeta) + \Phi(\zeta),$$

where the holomorphic function $\Phi(\zeta)$ is represented by (3.16), and $\Phi(\zeta) \in A^1_\alpha(\overline{G})$.

As in Sec. 3.1 we consider the function $w(z) = w(\zeta(z)) \in W^1_s(\overline{D})$, satisfying to Equation (3.13). We note, that now the coefficients and the constant term of Equation (3.13) belong to $C_\alpha(\overline{D})$.

Also the function $w(z)$ satisfies to Equation (3.14), where $\tilde{\Phi}(z) \in C_\beta(\overline{D})$, $T_D \tilde{F}(z) \in C^1_\alpha(\overline{D})$.

Consider a holomorphic in $D$ function

$$\Psi w(z) = \frac{1}{2\pi i} \int\limits_\Gamma \left[\frac{\zeta(t)}{\zeta(t) - \zeta(z)} - \frac{1}{t - z}\right] w(t)dt = \Phi(\zeta(z)) - \tilde{\Phi}(z).$$

From Sokhotski–Plemelj formulas [3, Ch.1, §4, Sec.4.2] it is easy to see, that

$$\lim_{z \to t \in \Gamma} \Psi w(z) = \Psi w(t).$$

Because $w(z) \in C_\beta(\overline{D})$, where $\beta : 0 < \beta < 1$ is arbitrary close to 1, from Theorem 2.3 we get $\Psi w(t) \in C^1_\mu(\Gamma)$, $\mu = \alpha + \beta - 1$, $0 < \mu < 1$; hence, $\Psi w(z) \in A^1_\mu(\overline{D})$.

Since $\Phi(\zeta) \in A^1_\alpha(\overline{G})$, from here we get $\tilde{\Phi}(z) \in A^1_\mu(\overline{D})$.

Further, since the operator $\tilde{\Omega}$ is a linear isomorphism of the Banach space $C^1_\alpha(\overline{D})$ [4], we get $w(z) \in C^1_\mu(\overline{D})$, $w(\zeta) \in C^1_\mu(\overline{G})$.

Now, using Corollary 2.4 of Theorem 2.3 we get $\tilde{\Phi}(z) \in C^1_\alpha(\overline{D})$; from here we have $w(z) \in C^1_\alpha(\overline{D})$ and $w(\zeta) \in C^1_\alpha(\overline{G})$.

The case $k = 0$ is exhausted.

**3.2.2.** *The step case* $k \geq 1$. Assume, that the statement of theorem holds for $k = m \geq 0$. Using this assumption we must prove the statement for $k = m + 1$.

Consider the equation

$$\Omega(w) = F(\zeta) \in C^{m+2}_\alpha(\overline{G}),$$

where $q_1(\zeta)$, $q_2(\zeta)$, $A(\zeta)$, $B(\zeta) \in C^{m+1}_\alpha(\overline{G})$. By virtue of inductive hypothesis this equation has a unique solution $w(\zeta) \in C^{m+1}_\alpha(\overline{G})$. Analogously to the previous arguments we have to show that $w(\zeta) \in C^{m+2}_\alpha(\overline{G})$ and then to use the Banach theorem. As above, we reduce the proof to the special case investigated in [4], when the domain $G$ is a unit disk $D$.

As above, we represent the function $F(\zeta)$ by the formula (3.15); in this formula now we have $T_G \partial_{\bar{\zeta}} F \in C^{m+2}_\alpha(\overline{G})$, $\Phi(\zeta) \in A^{m+2}_\alpha(\overline{G})$. Differentiating $\Phi(\zeta)$ with respect to $\zeta$ $(m+1)$ times, we obtain (see [12, Ch. 1, §3]):

$$\Phi^{(m+1)}(\zeta) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{w_{m+1}(\tau)}{\tau - \zeta} d\tau \in A^1_\alpha(\overline{G}),$$

where

$$w_{m+1}(\tau) = \overline{\tau_\sigma} \frac{d}{d\sigma} \left( \overline{\tau_\sigma} \frac{d}{d\sigma} (\ldots) \right) w(\tau); \tag{3.18}$$

$\tau = \tau(\sigma) \in C^{m+2}_\alpha$ is the equation of the contour $\mathcal{L}$; $\sigma$ is a length of arc on $\mathcal{L}$; the operator $\overline{\tau_\sigma} \dfrac{d}{d\sigma}$ is repeated in (3.18) $(m+1)$ times.

Because $w(\zeta) \in C^{m+1}_\alpha(\overline{G})$, opening brackets in (3.18) we can rewrite the function $w_{m+1}(\tau)$ in the form:

$$w_{m+1}(\tau) = W_1(\tau) + W_2(\tau), \tag{3.19}$$

where

$$W_1(\tau) = (\overline{\tau_\sigma})^{m+1} \frac{d^{m+1}}{d\sigma^{m+1}} w(\tau) \in C_\alpha(\mathcal{L}),$$

and $W_2(\tau) \in C^1_\alpha(\mathcal{L})$.

Further, we have:

$$\Phi^{(m+1)}(\zeta) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{W_1(\tau)}{\tau - \zeta} d\tau + \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{W_2(\tau)}{\tau - \zeta} d\tau \in A^1_\alpha(\overline{G}). \tag{3.20}$$

By the properties of the Cauchy–type integral we also have (see [12, Ch. 1, §3]):

$$\Phi_2(\zeta) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{W_2(\tau)}{\tau - \zeta} d\tau \in A^1_\alpha(\overline{G}).$$

From here, we get:

$$\Phi_1(\zeta) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{W_1(\tau)}{\tau - \zeta} d\tau \in A^1_\alpha(\overline{G}), \tag{3.21}$$

and by (3.21) we obtain:

$$\Phi_1(\zeta(z)) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\tilde{W}_1(t)\zeta_t(t)}{\zeta(t)-\zeta(z)}dt \in A^1_\alpha(\overline{D}),$$

where

$$\tilde{W}_1(t) = (\zeta_t)^{-m-1}(\overline{t_s})^{m+1}\frac{d^{m+1}}{ds^{m+1}}w(t) \in C_\alpha(\Gamma), \quad t = e^{is},$$

$w(t) = w(\tau(t))$.

Consider the function

$$\Phi^*_1(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\tilde{W}_1(t)}{t-z}dt \in C_\alpha(\overline{D}),$$

and the difference

$$\Psi\tilde{W}_1(z) = \Phi_1(\zeta(z)) - \Phi^*_1(z) = \frac{1}{2\pi i}\int\limits_\Gamma \left[\frac{\zeta(t)}{\zeta(t)-\zeta(z)} - \frac{1}{t-z}\right]\tilde{W}_1(t)dt.$$

Since $\zeta(z) \in C^2_\alpha(\overline{D}) \subset C^1_1(\overline{D})$, using theorem 2.3 we get $\Psi\tilde{W}_1(z) \in C^1_\alpha(\overline{D})$; hence, $\Phi^*_1(z) \in A^1_\alpha(\overline{D})$.

Further, because $\zeta_z \in A^1_\alpha(\overline{D})$ and $\zeta_z \neq 0$, from here and Lemma 2.6 we obtain

$$\tilde{\Phi}_1(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{(\overline{t_s})^{m+1}\dfrac{d^{m+1}}{ds^{m+1}}w(t)}{t-z}dt \in A^1_\alpha(\overline{D}). \tag{3.22}$$

Now, consider the function

$$\tilde{\Phi}(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{w(\zeta(t))}{t-z}dt.$$

Because $w(\zeta(z)) \in C^{m+1}_\alpha(\overline{D})$, we have $\tilde{\Phi}(z) \in A^{m+1}_\alpha(\overline{D})$.

Analogously to (3.20), we obtain:

$$\tilde{\Phi}^{(m+1)}(z) = \tilde{\Phi}_1(z) + \tilde{\Phi}_2(z),$$

where $\tilde{\Phi}_2(z) \in A^1_\alpha(\overline{D})$. From here and (3.22), we get $\tilde{\Phi}(z) \in A^{m+2}_\alpha(\overline{D})$.

The function $w(z) = w(\zeta(z))$ satisfies to Equation (3.14), where $T_D\tilde{F}(z) \in C^{m+2}_\alpha(\overline{D})$, and $\tilde{\Omega}$ is a linear isomorphism of the Banach space $C^{m+2}_\alpha(\overline{D})$ [4]. From here we get $w(z) \in C^{m+2}_\alpha(\overline{D})$, and $w(\zeta) \in C^{m+2}_\alpha(\overline{G})$.

Theorem 1.2 is proved.

## References


[1] Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, *Comm. Pure Appl. Math.* **XII** (1959) 623–727.

[2] Bojarski, B.: Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients, *Mat Sb.* **43** (1957) 451-503. [in Russian]

[3] Gakhov, F.D.: *Boundary value problems*, Pergamon Press, Oxford, 1966.

[4] Klimentov, S.B.: About isomorphism of some functional spaces under the action of the integro–differential operators, *Ufa Math. Journal* **11** N1 (2019). [in Russian]

[5] Klimentov, S.B.: Another version of Kellogg's theorem, *Complex Variables and Elliptic Equations* **60** (2015) 1647–1657.
[6] Klimentov, S.B.: On combinations of the circle shifts and some one–dimensional integral operators, *Vladikavkaz Mathematical Journal* **19** N1 (2017) 30–40. [in Russian]
[7] Litvinchuk, G.S.: *Boundary–value problems and singular integral equations with shift*, Nauka, Moscow, 1968. [in Russian]
[8] Mikhlin, S.G.: *Multidimensional singular integrals and integral equations*, Pergamon Press, Oxford, 1965.
[9] Monakhov, V.N.: *Boundary value problems with free boundaries for elliptic systems of equations. Translations of Mathematical Monographs, Vol. 57*, Amer Math Soc, Providence, R.I., 1983.
[10] Muskhelishvili, N.I.: *Singular integral equations*, P. Noordhoff, Groningen, 1953.
[11] Stein, E.M.: *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, New Jersey, 1970.
[12] Vekua, I.N.: *Generalized Analytic Functions*, Pergamon Press, Oxford/London/New York/Paris, 1962.

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