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SINGLE SERVER MARKOVIAN QUEUEING NETWORK WITH FEEDBACK

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Abstract. In this study we analyze M/M/1 queue with feedback. We consider single server Markovian queue, the customer can enter in to the first node, after getting the service, they can go for next nodes with two options with probability p (enter in to the second node) and 1-p (enter in to the third node) respectively. After completing the service at the second node the customer can either leave the system with probability q or enter in to the third node with probability 1-q. After getting the service at the third node the customer leave the system with probability r or enter in to the node 2 with probability 1-r (feedback). After getting the service in the second node the customer behaves as either enters node 3 or leave the system with probability 1-q and q respectively. The numerical examples are given to test the performance of the system.

Keywords: Performance measures-Queueing network-Feedback-Markovian Queue.

1. Introduction

Queueing theory is the mathematical study of waiting lines. It is introduced by the Danish Engineer A.K. Erlang in 1909. Queueing theory originated in telephony with the work of Erlang [1]. The basic features which characterize a queueing system are (i) Arrival Pattern (ii) Service Pattern (iii) Number of servers (iv)Queue discipline (v) Capacity of the system (vi) Calling source. D.G. Kendall [6] has introduced a set of notations which have become standard in the literature of queueing models in 1953. A general queueing system is denoted by (a/b/c): (d/e/f) where the first and second symbols denote the type of distributions of inter arrival times and of inter service times respectively. Third symbol specifies the number of servers; whereas fourth symbol stands for the queue discipline, the fifth symbol denote the capacity of the system and the last symbol denotes the calling source or population.

Queue network can be regarded as a group of interconnected nodes, where each node represents a service facility of some kind with servers at each node. The Queueing networks were first identified by James. R. Jackson in 1957. An earlier product- form solution was found by Jackson [3] for tandem queues. Jackson [4] has also explained the network of waiting lines. The most significant contribution in queueing network is Jackson's network. Queueing networks can be classified as open, closed and mixed networks. In an open network customers enter from outside, receive service at systems and leave the network. In closed network new customers never enter in to and the existing customers never depart from the system. In mixed network, the network may be open for some classes of customers and closed for some other classes. Queueing network models have various applications in many areas, such as service centers, computer networks, communication networks, production and flexible manufacturing systems, airport terminals, hospitals, machine shop, automobiles, and supermarkets etc.

Feedback queues play an important role in real life service system, where tasks may require repeated services. The queueing systems which include the possibility for a customer to return to the service because of unsatisfied service or for requirement of additional service are called queues with feedback. The phenomenon of feedback reflects in many practical situations for example reworking in the production system, communication networks and super markets etc. Takacs [12] has introduced queues with feedback mechanism in 1963.Santhakumaran and Shamugasundaram [9] have studied preparatory work on arrival customers with a single server feedback mechanism. Sreekala and Manoharan [11] have focused on a queueing network model with feedback and its application in healthcare. Lasse Leskela and Jacques Resing [7] have analyzed a tandem queueing network with feedback admission control. Vander Mei et.al [13] have studied the response times in a two node queueing network with feedback. Jonathan Brandon and Uri Yechiali [5] have examined a tandem Jackson network with feedback to the first node that is applied mainly in a manufacturing process. Erol A.Pekoz and Nitindra Joglekar [2] have discussed about the traffic flow in a general feedback queue. Raghavendran et.al [8] has proposed a two node tandem communication network with feedback.

Shanmugasundaram and Vanitha [10] have analyzed an open queueing network system in healthcare. In this paper we consider single server Markovian queueing network with feedback.

2. DESCRIPTION OF THE MODEL

We consider an open queueing network consisting of three nodes. The customers arrive to the system according to a Poisson process and get service with a general service time distribution. Each node follows an $M/M/1$ schedule. The service rates for node 1, node 2 and node 3 are exponentially distributed with the rates μ_1, μ_2, and μ_3 respectively. The customer first enter in to the node 1 after getting the service they can either go to node 2 or node 3 with probability p and with probability $1-p$ respectively. After getting the service at node 2 the customers can either leave the system with probability q or they go to node 3 with probability $1-q$. After completing the service at node 3 the customer may leave the system with probability r or they may enter in to the node 2 with feedback corresponding to the probability $1-r$. After completing the service at node 2, they may leave the system or they may again enter in to the node 3 with probability q and $1-q$ respectively. Figure 1 represents the system.

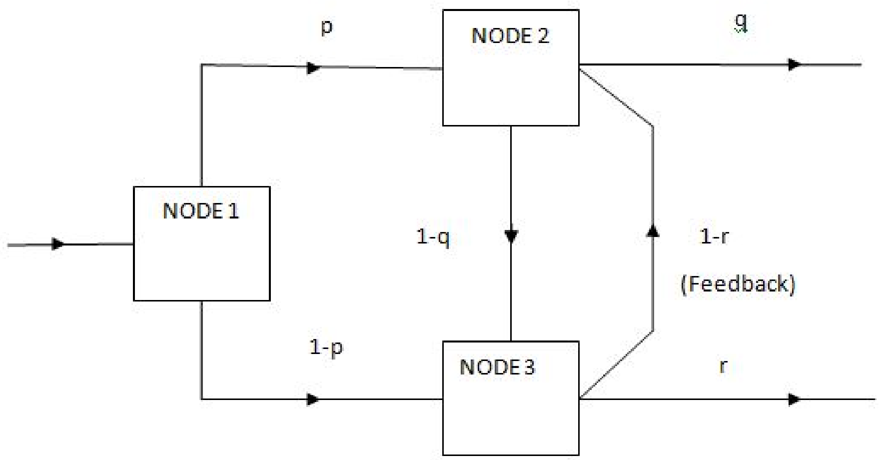

Figure 1

3. BALANCE EQUATIONS

We define λ_i where $(i = 1, 2, 3)$ is the arrival rate to each nodes. The balance equations for this model are obtained as given below:

(1) $$\lambda_1 = \lambda$$

(2) $$\lambda_2 = \lambda p + \lambda_3(1-r)$$

(3) $$\lambda_3 = \lambda(1-p) + \lambda_2(1-q)$$

Solving these equations we get,

(4) $$\lambda_2 = (\lambda(1-r+pr))/(r+q-rq)$$

and

(5) $$\lambda_3 = (\lambda(1-pq))/(r+q-rq)$$

If n_1, n_2, n_3 are the number of customers at node 1, node 2, node 3 respectively then using Jackson network the steady state probability is denoted by $P(n_1, n_2, n_3)$. The steady state probability for n_1, n_2, n_3 customers at the three nodes respectively is

$$P(n_1,n_2,n_3) = (1-\rho_1)\rho_1^{n_1}(1-\rho_2)\rho_2^{n_2}(1-\rho_3)\rho_3^{n_3} \tag{6}$$

Where,

$$\rho_1 = \frac{\lambda_1}{\mu_1}, \rho_2 = \frac{\lambda_2}{\mu_2}, \rho_3 = \frac{\lambda_3}{\mu_3}$$

$$\begin{aligned} P(n_1,n_2,n_3) &= \left(1-\frac{\lambda_1}{\mu_1}\right)\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left(1-\frac{\lambda_2}{\mu_2}\right)\left(\frac{\lambda_2}{\mu_2}\right)^{n_2}\left(1-\frac{\lambda_3}{\mu_3}\right)\left(\frac{\lambda_3}{\mu_3}\right)^{n_3} \\ &= \left(1-\frac{\lambda}{\mu_1}\right)\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left(1-\frac{\lambda(1-r+pr)}{\mu_2(r+q-rq)}\right)\left(\frac{\lambda(1-r+pr)}{\mu_2(r+q-rq)}\right)^{n_2} \\ &\quad \left(1-\frac{\lambda(1-pq)}{\mu_3(r+q-rq)}\right)\left(\frac{\lambda(1-pq)}{\mu_3(r+q-rq)}\right)^{n_3} \end{aligned} \tag{7}$$

Average number of customers in the system:
Let N_i (i=1,2,3) be the number of customers in the node i.
We have

$$\begin{aligned} N_1 &= \frac{\rho_1}{1-\rho_1} \\ &= \frac{\lambda}{\mu_1-\lambda} \end{aligned} \tag{8}$$

$$\begin{aligned} N_2 &= \frac{\rho_2}{1-\rho_2} \\ &= \frac{\lambda_2}{\mu_2-\lambda_2} \\ &= \frac{\lambda(1-r+pr)}{\mu_2(r+q-rq)-\lambda(1-r+pr)} \end{aligned} \tag{9}$$

$$\begin{aligned} N_3 &= \frac{\rho_3}{1-\rho_3} \\ &= \frac{\lambda_3}{\mu_3-\lambda_3} \\ &= \frac{\lambda(1-pq)}{\mu_3(r+q-rq)-\lambda(1-pq)} \end{aligned} \tag{10}$$

Average number of customers in the overall system,

$$L_s = N_1+N_2+N_3 \tag{11}$$

$$= \frac{\lambda}{\mu_1-\lambda} + \frac{\lambda(1-r+pr)}{\mu_2(r+q-rq)-\lambda(1-r+pr)} + \frac{\lambda(1-pq)}{\mu_3(r+q-rq)-\lambda(1-pq)} \tag{12}$$

Average waiting time of a customer in the system,

$$W_s = \frac{L_s}{\lambda} \tag{13}$$

$$= \frac{1}{\mu_1 - \lambda} + \frac{1 - r + pr}{\mu_2(r + q - rq) - \lambda(1 - r + pr)} + \frac{1 - pq}{\mu_3(r + q - rq) - \lambda(1 - pq)} \tag{14}$$

Average number of customers in all the three queues

$$L_q = L_s - \frac{\lambda}{\mu} \tag{15}$$

$$= \frac{\lambda}{\mu_1 - \lambda} + \frac{\lambda(1 - r + pr)}{\mu_2(r + q - rq) - \lambda(1 - r + pr)} + \frac{\lambda(1 - pq)}{\mu_3(r + q - rq) - \lambda(1 - pq)} - \frac{\lambda}{\mu} \tag{16}$$

Average waiting time of a customer in all the three queues

$$W_q = \frac{L_q}{\lambda} \tag{17}$$

$$= \frac{1}{\mu_1 - \lambda} + \frac{1 - r + pr}{\mu_2(r + q - rq) - \lambda(1 - r + pr)} + \frac{1 - pq}{\mu_3(r + q - rq) - \lambda(1 - pq)} - \frac{1}{\mu} \tag{18}$$

4. NUMERICAL EXAMPLES

In this section we investigate the steady state probability and the performance measures.
For $\lambda = 0.5$, $\mu_1 = 2.1$, $\mu_2 = 3.2$, $\mu_3 = 4.2$, $p = 0.2$, $q = 0.3$, $r = 0.4$.
Average number of customers in the overall system $L_s = 0.7758 \cong 1$
Average waiting time of a customer in the system $W_s = 1.5517 mts$
Average number of customers in the queue $L_q = 0.6179 mts$
Average waiting time of a customer in the queue $W_q = 1.2359 \cong 1$
The steady state probability values for n_1, n_2, n_3 customers at the three nodes respectively are given in Table 1.

Table 1

(n_1, n_2, n_3)	$P(n_1, n_2, n_3)$	(n_1, n_2, n_3)	$P(n_1, n_2, n_3)$	(n_1, n_2, n_3)	$P(n_1, n_2, n_3)$	(n_1, n_2, n_3)	$P(n_1, n_2, n_3)$
0 0 0	5.02E-01	1 3 0	7.35E-04	3 0 0	6.78E-03	4 3 0	9.92E-06
0 0 1	9.69E-02	1 3 1	1.42E-04	3 0 1	1.31E-03	4 3 1	1.91E-06
0 0 2	1.87E-02	1 3 2	2.74E-05	3 0 2	2.52E-04	4 3 2	3.69E-07
0 0 3	3.61E-03	1 3 3	5.28E-06	3 0 3	4.87E-05	4 3 3	7.13E-08
0 0 4	6.96E-04	1 3 4	1.02E-06	3 0 4	9.39E-06	4 3 4	1.38E-08
0 0 5	1.34E-04	1 3 5	1.97E-07	3 0 5	1.81E-06	4 3 5	2.65E-09
0 1 0	9.20E-02	1 4 0	1.35E-04	3 1 0	1.24E-03	4 4 0	1.82E-06
0 1 1	1.78E-02	1 4 1	2.60E-05	3 1 1	2.40E-04	4 4 1	3.51E-07
0 1 2	3.43E-03	1 4 2	5.01E-06	3 1 2	4.62E-05	4 4 2	6.77E-08
0 1 3	6.61E-04	1 4 3	9.67E-07	3 1 3	8.92E-06	4 4 3	1.31E-08
0 1 4	1.28E-04	1 4 4	1.87E-07	3 1 4	1.72E-06	4 4 4	2.52E-09
0 1 5	2.46E-05	1 4 5	3.60E-08	3 1 5	3.32E-07	4 4 5	4.86E-10
0 2 0	1.69E-02	1 5 0	2.47E-05	3 2 0	2.28E-04	4 5 0	3.33E-07

(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$	(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$	(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$	(n_1,n_2,n_3)	$P(n_1,n_2,n_3)$
0 2 1	3.25E-03	1 5 1	4.76E-06	3 2 1	4.39E-05	4 5 1	6.42E-08
0 2 2	6.27E-04	1 5 2	9.18E-07	3 2 2	8.47E-06	4 5 2	1.24E-08
0 2 3	1.21E-04	1 5 3	1.77E-07	3 2 3	1.63E-06	4 5 3	2.39E-09
0 2 4	2.34E-05	1 5 4	3.42E-08	3 2 4	3.15E-07	4 5 4	4.61E-10
0 2 5	4.51E-06	1 5 5	6.70E-09	3 2 5	6.08E-08	4 5 5	8.90E-11
0 3 0	3.09E-03	2 0 0	2.85E-02	3 3 0	4.17E-0	5 0 0	3.84E-04
0 3 1	5.96E-04	2 0 1	5.49E-03	3 3 1	8.04E-06	5 0 1	7.41E-05
0 3 2	1.15E-04	2 0 2	1.06E-03	3 3 2	1.55E-06	5 0 2	1.43E-05
0 3 3	2.22E-05	2 0 3	2.04E-04	3 3 3	2.99E-07	5 0 3	2.76E-06
0 3 4	4.28E-06	2 0 4	3.95E-05	3 3 4	5.78E-08	5 0 4	5.33E-07
0 3 5	8.26E-07	2 0 5	7.61E-06	3 3 5	1.11E-08	5 0 5	1.03E-07
0 4 0	5.66E-04	2 1 0	5.22E-03	3 4 0	7.63E-06	5 1 0	7.04E-05
0 4 1	1.09E-04	2 1 1	1.01E-03	3 4 1	1.47E-06	5 1 1	1.36E-05
0 4 2	2.11E-05	2 1 2	1.94E-04	3 4 2	2.84E-07	5 1 2	2.62E-06
0 4 3	4.06E-06	2 1 3	3.75E-05	3 4 3	5.48E-08	5 1 3	5.06E-07
0 4 4	7.84E-07	2 1 4	7.23E-06	3 4 4	1.06E-08	5 1 4	9.76E-08
0 4 5	1.51E-07	2 1 5	1.39E-06	3 4 5	2.04E-09	5 1 5	1.88E-08
0 5 0	1.04E-04	2 2 0	9.56E-04	3 5 0	1.40E-06	5 2 0	1.29E-05
0 5 1	2.00E-05	2 2 1	1.84E-04	3 5 1	2.70E-07	5 2 1	2.49E-06
0 5 2	3.86E-06	2 2 2	3.56E-05	3 5 2	5.21E-08	5 2 2	4.80E-07
0 5 3	7.44E-07	2 2 3	6.86E-06	3 5 3	1.00E-08	5 2 3	9.26E-08
0 5 4	1.44E-07	2 2 4	1.32E-06	3 5 4	1.94E-09	5 2 4	1.79E-08
0 5 5	2.77E-08	2 2 5	2.55E-07	3 5 5	3.74E-10	5 2 5	3.45E-09
1 0 0	1.20E-01	2 3 0	1.75E-04	4 0 0	1.61E-03	5 3 0	2.36E-06
1 0 1	2.31E-02	2 3 1	3.38E-05	4 0 1	3.11E-04	5 3 1	4.56E-07
1 0 2	4.45E-03	2 3 2	6.52E-06	4 0 2	6.01E-05	5 3 2	8.79E-08
1 0 3	8.59E-04	2 3 3	1.26E-06	4 0 3	1.16E-05	5 3 3	1.70E-08
1 0 4	1.66E-04	2 3 4	2.43E-07	4 0 4	2.24E-06	5 3 4	3.27E-09
1 0 5	3.20E-05	2 3 5	4.68E-08	4 0 5	4.32E-07	5 3 5	6.32E-10
1 1 0	2.19E-02	2 4 0	3.21E-05	4 1 0	2.96E-04	5 4 0	4.33E-07
1 1 1	4.23E-03	2 4 1	6.19E-06	4 1 1	5.70E-05	5 4 1	8.35E-08
1 1 2	8.15E-04	2 4 2	1.19E-06	4 1 2	1.10E-05	5 4 2	1.61E-08
1 1 3	1.57E-04	2 4 3	2.30E-07	4 1 3	2.12E-06	5 4 3	3.11E-09
1 1 4	3.04E-05	2 4 4	4.44E-08	4 1 4	4.10E-07	5 4 4	6.00E-10
1 1 5	5.86E-06	2 4 5	8.57E-09	4 1 5	7.91E-08	5 4 5	1.16E-10
1 2 0	4.01E-03	2 5 0	5.87E-06	4 2 0	5.42E-05	5 5 0	7.93E-08
1 2 1	7.74E-04	2 5 1	1.13E-06	4 2 1	1.05E-05	5 5 1	1.53E-08
1 2 2	1.49E-04	2 5 2	2.19E-07	4 2 2	2.02E-06	5 5 2	2.95E-09
1 2 3	2.88E-05	2 5 3	4.22E-08	4 2 3	3.89E-07	5 5 3	5.69E-10
1 2 4	5.56E-06	2 5 4	8.14E-09	4 2 4	7.51E-08	5 5 4	1.10E-10
1 2 5	1.07E-06	2 5 5	1.57E-09	4 2 5	1.45E-08	5 5 5	2.12E-11

For the arrival rate λ from 0.5 to 0.9 and service rate from 2.1 to 2.5 the average number of customers in the system and the average waiting time in the system are calculated in Table 2 and Table 3. From Figure 2 and Figure 3 it is clear that as the arrival rate increases the number of customers in the system and the waiting time in the system increases.

Table 2

λ/μ	2.1	2.2	2.3	2.4	2.5
0.5	0.7758	0.7575	0.7411	0.7265	0.7133
0.6	0.9831	0.9581	0.9360	0.9164	0.8988
0.7	1.2150	1.1817	1.1525	1.1268	1.1039
0.8	1.4766	1.4326	1.3945	1.3612	1.3318
0.9	1.7740	1.7163	1.6669	1.6240	1.5865

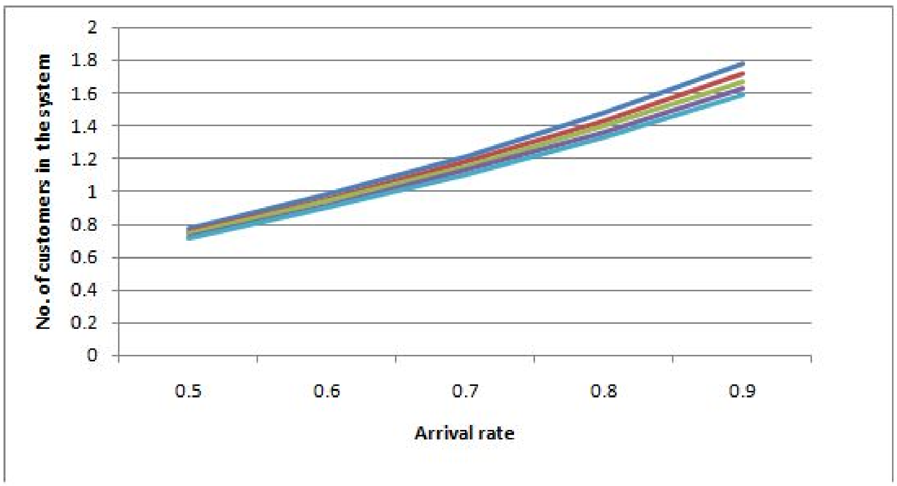

Figure 2

Table 3

λ/μ	2.1	2.2	2.3	2.4	2.5
0.5	1.5517	1.5149	1.4822	1.4530	1.4267
0.6	1.6384	1.5968	1.5600	1.5273	1.4981
0.7	1.7357	1.6881	1.6464	1.6097	1.5770
0.8	1.8457	1.7908	1.7432	1.7015	1.6647
0.9	1.9712	1.9070	1.8521	1.8045	1.7628

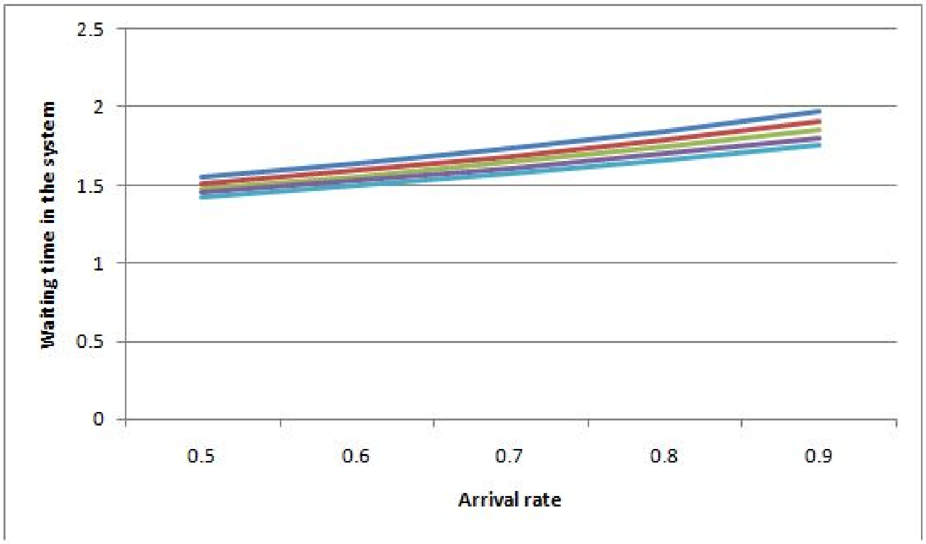

Figure 3

For the service rate from 2.1 to 2.5 and the arrival rate λ from 0.5 to 0.9 the average number of customers in the queue and the average waiting time of customers in the queue are calculated in Table 4 and Table 5. From Figure 4 and Figure 5 it is clear that as the service rate increases the number of customers in the queue and the waiting time in the queue decreases.

Table 4

μ/λ	0.5	0.6	0.7	0.8	0.9
2.1	0.6179	0.7936	0.9940	1.2239	1.4898
2.2	0.5996	0.7686	0.9606	1.1800	1.4321
2.3	0.5832	0.7465	0.9315	1.1419	1.3827
2.4	0.5686	0.7269	0.9057	1.1086	1.3398
2.5	0.5554	0.7094	0.8828	1.0791	1.3023

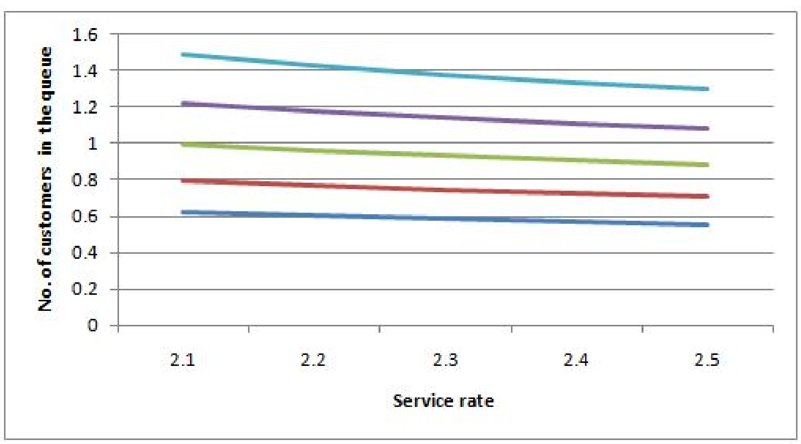

Figure 4

Table 5

μ/λ	0.5	0.6	0.7	0.8	0.9
2.1	1.2359	1.3226	1.4199	1.5299	1.6554
2.2	1.1991	1.2810	1.3723	1.4750	1.5913
2.3	1.1664	1.2442	1.3307	1.4274	1.5363
2.4	1.1372	1.2115	1.2939	1.3857	1.4887
2.5	1.1109	1.1823	1.2612	1.3489	1.4470

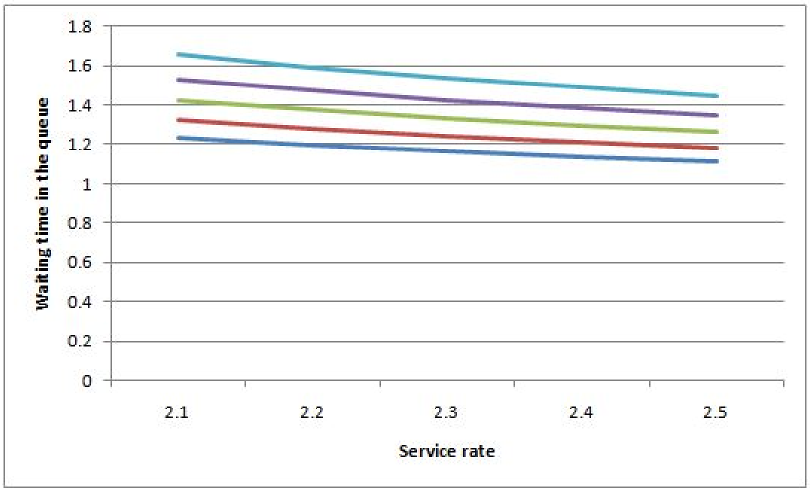

Figure 5

5. CONCLUSION

Numerical examples show that as the arrival rate increases, queue length and waiting time in the system increases. Also as the service rate increases, queue length and waiting time in the queue decreases. It shows the feasibility of the model. The various number of customers in all the nodes and their corresponding probabilities are calculated to test the correctness of the model.

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