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STOCHASTIC BARENBLATT–ZHELTOV–KOCHINA MODEL ON THE INTERVAL WITH WENTZELL BOUNDARY CONDITIONS

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Abstract. In terms of the theory of relative *p-bounded operators, we study the stochastic Barenblatt–Zheltov–Kochina model, which describes dynamics of pressure of a filtered fluid in a fractured-porous medium with general Wentzell boundary conditions. In particular, we examine the relative spectrum in the one-dimensional Barenblatt–Zheltov–Kochina equation, and construct the resolving group in the stochastic Cauchy-Wentzell problem with general Wentzell boundary conditions. In the paper, these problems are solved under the assumption that the initial space is a restriction of the space* $L^2(0,1)$.

Introduction

On the interval $[0,1]$, let us consider the differential operator

$$Au(x) = u''(x), \quad x \in [0,1] \tag{0.1}$$

with the general Wentzell boundary conditions

$$Au(0) + \alpha_0 u'(0) + \alpha_1 u(0) = 0, \tag{0.2}$$

$$Au(1) + \beta_0 u'(1) + \beta_1 u(1) = 0. \tag{0.3}$$

By formulas (0.1)–(0.3), we define the linear operator $A : dom\ A \subset \mathfrak{F} \to \mathfrak{F}$. Here $\mathfrak{F}$ is the space $(L^2[0,1], dx\Big|_{(0,1)} \oplus \eta ds\big|_{\{0,1\}})$ with the norm

$$\|u\|^2_{\mathfrak{F}} = \int_0^1 |u(x)|^2 dx + \eta_0 |u(0)|^2 + \eta_1 |u(1)|^2,$$

(the full construction of the space $\mathfrak{F}$ see, for example, in [1]), where dx is the Lebesgue measure on the interval $(0,1)$; ds is the point measure at the boundary; $\eta_0 = \frac{1}{-\alpha_1}, \eta_1 = \frac{1}{\beta_1}$, where $\alpha_1 < 0 < \beta_1$ are positive weights. We consider also the linear manifold $dom\ A = \{u \in C^2[0,1] : \text{conditions } (0.2), (0.3) \text{ are fulfilled}\}$ as the domain of the operator A. Fix $\alpha, \lambda \in \mathbb{R}$ and construct the operators $L = \lambda - A$

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and $M = \alpha A$, where the operator A is taken from the considerations above. It is known (see, for example, [2]), that the operators $L, M \in \mathcal{L}(domA; \mathfrak{F})$ and the space $domA$ is densely embedded in the space $\mathfrak{F}$.

On the interval $[0, 1]$, let us consider the stochastic Barenblatt-Zheltova-Kochina equation

$$L \overset{\circ}{\eta} (\omega, t) = M\eta(\omega, t) + Nf, \quad (\omega, t) \in [0, 1] \times (0, \tau), \tag{0.4}$$

which describes dynamics of pressure of a filtered fluid in a fractured-porous medium, with the initial Cauchy condition

$$\eta(0) = \xi_0, \tag{0.5}$$

and the Wentzell boundary conditions

$$\begin{aligned} &\eta_{xx}(0, t) + \alpha_0 \eta_x(0, t) + \alpha_1 \eta(0, t) = 0, \\ &\eta_{xx}(1, t) + \beta_0 \eta_x(1, t) + \beta_1 \eta(1, t) = 0. \end{aligned} \tag{0.6}$$

Here $\eta = \eta(t)$ is a stochastic process on the interval $(0, \tau)$; $\overset{\circ}{\eta}$ is the Nelson–Gliklikh derivative of the process $\eta(t)$; f is a "white noise", which we understand the Nelson–Gliklich derivative an one-dimensional Wiener process (see, for example., [3, 4, 5]); α and λ are the material parameters characterizing the environment; the parameter $\alpha \in \mathbb{R}_+$; the operator $N \in \mathcal{L}(\mathfrak{U}, \mathfrak{F})$ is subject to further clarification.

The purpose of this work is to research the solvability of the problem (0.4) – (0.6) with Wentzell boundary conditions. Except Introduction, Conclusion and References, the article contains five sections. The space of differentiable $\mathbf{K}$–"noises" is defined in Section 1. The transition from a deterministic Sobolev type equation to a stochastic one is presented in Section 2. Stochastic Cauchy–Wentzell problem for Barenblatt-Zheltova-Kochina model is described in Section 3. The algorithm and implementation for the numerical solution to this model according to the modified Galerkin method are described in Section 4. The solution of the stochastic Cauchy–Wentzell problem for a specific example is given in Section 5.

1. The space of "noises"

Let $\Omega \equiv (\Omega, \mathcal{A}, P)$ be a full probability space; $\mathbb{R}$ be set of real numbers endowed with the Borel σ-algebra. By a *random variable* we mean measurable mapping $\xi \colon \Omega \to \mathbb{R}$. A set of random variables $\{\xi : E\xi = 0, D\xi \leq +\infty\}$, the mathematical expectation of which is equal to zero, and the dispersion is finite, forms the Hilbert space $\mathbf{L_2}$ with the scalar product $(\xi_1, \xi_2) = E\xi_1\xi_2$ and the norm $\|\xi\|^2_{\mathbf{L_2}} = D\xi$.

Consider the set $\mathcal{J} \subset \mathbb{R}$ and the following two mappings. First, $f \colon \mathcal{J} \to \mathbf{L_2}$, associates each $t \in \mathcal{J}$ with a random variable $\xi \in \mathbf{L_2}$. Second, $g \colon \mathbf{L_2} \times \Omega \to \mathbb{R}$, associates each pair (ξ, ω) with a point $\xi(\omega) \in \mathbb{R}$.

A mapping $\eta \colon \mathcal{J} \times \Omega \to \mathbb{R}$, having the form $\eta = \eta(t, \omega) = g(f(t), \omega)$, is called an *(one-dimensional) stochastic process.* For each fixed $t \in \mathcal{J}$, the value of the stochastic process $\eta = \eta(t, \cdot)$ is a random value, i.e. $\eta = \eta(t, \cdot) \in \mathbf{L_2}$, which is called *a section of a stochastic process at* $t \in \mathcal{J}$. For each fixed $\omega \in \Omega$, the function $\eta = \eta(\cdot, \omega)$ is called *a (sample) path of a stochastic process*, corresponding to the elementary event result $\omega \in \Omega$. The paths are also called realizations or sample functions of a random process.

Usually, when this does not lead to ambiguity, the dependence of $\eta(t,\omega)$ on ω is not specified and a random process is denoted by $\eta(t)$.

Let be an interval $\mathcal{J} \subset \mathbb{R}$, then the stochastic process $\eta = \eta(t), t \in \mathcal{J}$ is called *continuous*, if all its paths are almost sure continuos.

The set of continuous stochastic processes forms a Banach space, which we denote by $\mathbf{CL_2}$, where

$$\|\eta\|^2_{\mathbf{CL_2}} = \sup D\eta(t,\omega).$$

Let $\mathcal{A}_0$ be a σ-subalgebra of the σ-algebra $\mathcal{A}$. Construct the subspace $\mathbf{L_2^0} \subset \mathbf{L_2}$ of random variables measurable with respect to $\mathcal{A}_0$. Denote by $\Pi\colon \mathbf{L_2} \to \mathbf{L_2^0}$ an orthoprojector.

For any $\xi \in \mathbf{L_2}$, a random value of $\Pi\xi$ is called a*conditional expectation* of a random value of ξ with respect to $\mathcal{A}_0$ and is denoted by $\mathbf{E}(\xi|\mathcal{A}_0)$.

Fix $\eta \in \mathbf{CL_2}$ and $t \in \mathcal{J}$. Denote by $\mathbf{N}_t^\eta$ a σ-algebra generated by a random value of $\eta(t)$, and denote by $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathbf{N}_t^\eta)$ a conditional expectation with respect to $\mathbf{N}_t^\eta$.

Let $\eta \in \mathbf{CL_2}$, the *Nelson–Gliklikh derivative* $\overset{\circ}{\eta}$ *of the stochastic process* $\eta(t)$ *at the point* $t \in \mathcal{J}$ is called a random variable

$$\overset{\circ}{\eta}(t,\cdot) = \frac{1}{2}\Big\{\lim_{\Delta t \to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t+\Delta t,\cdot) - \eta(t,\cdot)}{\Delta t}\right) + \lim_{\Delta t \to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t,\cdot) - \eta(t-\Delta t,\cdot)}{\Delta t}\right)\Big\},$$

if the limits exist in the sense of the uniform metric on $\mathbb{R}$.

If the Nelson–Gliklikh derivatives $\overset{\circ}{\eta}(t,\cdot)$ of the stochastic process $\eta(t)$ exist in all (or almost all) points of the interval $\mathcal{J}$, then we say that the Nelson–Gliklikh derivative $\overset{\circ}{\eta}(t,\cdot)$ exist on $\mathcal{J}$ (almost sure on $\mathcal{J}$).

As an example, consider the Nelson–Gliklikh derivative for the Wiener process $\beta(t)$ (see, for example, [6]), describing Brownian motion in the Einstein-Smoluchowski model

$$\overset{\circ}{\beta}(t) = \frac{\beta(t)}{2t}, \quad t \in \mathbb{R}_+.$$

Note that the set of continuous stochastic processes having the derivative $\overset{\circ}{\eta}(t,\cdot)$ forms the Banach space $\mathbf{C^1L_2}$ with the norm

$$\|\eta\|^2_{\mathbf{C^1L_2}} = \sup_{\mathcal{J}}\left(D\eta(t,\omega) + D\overset{\circ}{\eta}(t,\omega)\right).$$

Introduce the space $\mathbf{C^lL_2}$, $l \in \{0\} \cup \mathbb{N}$, of random processes from $\mathbf{CL_2}$, whose paths are differentiable (almost sure) by Nelson–Gliklikh on $\mathcal{J}$ up to the order l inclusively, define the norm in the space by the following formula:

$$\|\eta\|^2_{\mathbf{C^lL_2}} = \sup_{\mathcal{J}}\left(\sum_{k=0}^{l} D\overset{\circ}{\eta}^{l}(t,\omega)\right).$$

By definition, we understand the Nelson–Gliklikh derivative of the order zero $\overset{\circ}{\eta}^{0}$ as the original stochastic process, by the space $\mathbf{C^lL_2}$, $l \in \{0\} \cup \mathbb{N}$ we understand *the space of* $\mathbf{K}$*–"noises"*.

Let us consider a real separable Hilbert space $\mathfrak{U}$ ($\mathfrak{F}$) with orthonormal basis $\{\varphi_k\}$ ($\{\psi_k\}$). Introduce a monotonic sequence $K = \{\lambda_k\} \subset \mathbb{R}$ such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < \infty$. Denote by $\mathbf{U_K L_2}$ ($\mathbf{F_K L_2}$) the Hilbert space, which is a completion of the linear span of $\mathbf{K}$-*random variables*

$$\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k, \quad \xi_k \in \mathbf{L_2} \quad \left(\zeta = \sum_{k=1}^{\infty} \mu_k \zeta_k \psi_k \quad \zeta_k \in \mathbf{L_2}\right)$$

by the norm

$$\|\xi\|_{\mathbf{U}}^2 = \sum_{k=1}^{\infty} \lambda_k^2 D\xi_k, \quad \left(\|\zeta\|_{\mathbf{F}}^2 = \sum_{k=1}^{\infty} \mu_k^2 D\zeta_k\right).$$

Note that for existence of a $\mathbf{K}$-*random variable* $\xi \in \mathbf{U_K L_2}$ ($\zeta \in \mathbf{F_K L_2}$) it is enough to consider a sequence of random variables $\{\xi_k\} \subset \mathbf{L_2}$ ($\{\zeta_k\} \subset \mathbf{L_2}$) having uniformly bounded dispersions $D\xi_k \le Const$ $(D\zeta_k \le Const)$, $k \in \mathbb{N}$.

Construct the *space of differentiable* $\mathbf{K}$*–"noises"*. Consider the inteval $(\epsilon, \tau) \subset \mathbb{R}$. A mapping $\eta\colon (\epsilon, \tau) \to \mathbf{U_K L_2}$ given by the formula

$$\eta(t) = \sum_{k=1}^{\infty} \lambda_k \xi_k(t) \varphi_k,$$

where the sequence $\{\xi_k\} \subset \mathbf{CL_2}$, is called a $\mathfrak{U}$*-valued continuous stochastic* $\mathbf{K}$*-process*, if the series on the right converges uniformly on any compact in $\mathcal{J}$ by the norm $\|\cdot\|_{\mathbf{U}}$ and paths of the process $\eta = \eta(t)$ are almost sure continuous.

A continuous stochastic $\mathbf{K}$-process

$$\overset{\circ}{\eta}(t) = \sum_{k=1}^{\infty} \lambda_k \overset{\circ}{\xi}_k(t) \varphi_k, \tag{1.1}$$

is called *continously differentiable by Nelson–Gliklikh* on $\mathcal{J}$, if the series converges uniformly on any compact in $\mathcal{J}$ by the norm $\|\cdot\|_{\mathbf{U}}$ and paths of the process $\overset{\circ}{\eta} = \overset{\circ}{\eta}(t)$ are almost sure continuous.

Denote by $\mathbf{C^l}(\mathcal{J}, \mathbf{U_K L_2})$, $l \in \{0\} \cup \mathbb{N}$ the space of *differentiable* $\mathbf{K}$*–"noises"*, whose paths almost sure differentiable by Nelson–Gliklikh on $\mathcal{J}$ up to the order l inclusively, with the following norm:

$$\|\eta\|^2_{\mathbf{C^l}(\mathcal{J}, \mathbf{U_K L_2})} = \sup_{\mathcal{J}} \left(\sum_{k=0}^{\infty} \lambda_k^2 \sum_{j=1}^{l} D \overset{\circ}{\eta}{}^{j} \right).$$

An example of *continously differentiable by Nelson–Gliklikh* up to the order l inclusively $\mathbf{K}$-*process* is Wiener $\mathbf{K}$-*process* (see, for example, [6])

$$W_{\mathbf{K}}(t) = \sum_{k=1}^{\infty} \lambda_k \beta_k(t) \varphi_k$$

where $\{\beta_k\} \subset \mathbf{C^l L_2}$ is a sequence of Brownian motions on $\mathbb{R}_+$.

Similarly, the space of $\mathbf{C^l}(\mathcal{J}, \mathbf{F_K L_2})$, i.e. *differentiable* $\mathbf{K}$*–"noises"* on $\mathbf{F_K L_2}$, are constructed.

2. Stochastic Sobolev type equation

Let us consider a real separable Hilbert space $\mathfrak{U}$ ($\mathfrak{F}$) with orthonormal basis $\{\varphi_k\}$ ($\{\psi_k\}$).

Lemma 2.1. *Let the sequences $\{\lambda_k\}$ and $\{\mu_k\}$ be associated with the inequality $\mu_k^2 D\zeta_k \le \lambda_k^2 D\xi_k$. Then the operator $A \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ iff $A \in \mathcal{L}(\mathbf{U_K L_2};\mathbf{F_K L_2})$.*

Let us give an idea of the proof. Statement is obviously, since the inequality hold according to comparison test for infinite series with non-negative (real-valued) terms

$$\|A\xi\|_{\mathbf{F}}^2 \le M \sum_{k=1}^{\infty} \mu_k^2 D\zeta_k \|\psi_k\|_{\mathfrak{U}}^2 \le M \|\xi\|_{\mathbf{U}}^2 .$$

Therefore, in terms of the theory of relative σ-bounded operators (see, e.g., [7]) holds the following

Lemma 2.2. *Let the sequences $\{\lambda_k\}$ and $\{\mu_k\}$ be associated with the inequality $\mu_k^2 D\zeta_k \le \lambda_k^2 D\xi_k$. The operator $M \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ is σ-bounded with respect to the operator $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ iff $M \in \mathcal{L}(\mathbf{U_K L_2};\mathbf{F_K L_2})$ is σ-bounded with respect to the operator $L \in \mathcal{L}(\mathbf{U_K L_2};\mathbf{F_K L_2})$. Moreover, the L-spectrum of the operator M is the same in both cases.*

Using Lemma 2.2 we can consider the theory of relative σ-bounded operators in the space random $\mathbf{K}$-variables. Consider the auxiliary problem with the initial Cauchy condition

$$\eta(0) = \xi_0 \tag{2.1}$$

for the abstract equation

$$L \overset{\circ}{\eta} (\omega, t) = M\eta(\omega, t) + Nf, \tag{2.2}$$

where $L, M, N \in \mathcal{L}(\mathbf{U_K L_2};\mathbf{F_K L_2})$, $\eta \in \mathbf{C^{l+1}}(\mathcal{J}, \mathbf{F_K L_2})$ is a desired random $\mathbf{K}$-process, $f \in \mathbf{C^{l+1}}(\mathcal{J}, \mathbf{F_K L_2})$ is a "white noise".

A random $\mathbf{K}$-process $\eta \in \mathbf{C^{l+1}}(\mathcal{J}, \mathbf{F_K L_2})$ is called *a solution to equation* (2.2), if almost sure all its paths satisfy equation (2.2) for all $t \in \mathcal{J}$. The solution $\eta = \eta(t)$ to equation (2.2) is called *solution to problem* (2.1), (2.2), if it satisfies condition (2.1).

Theorem 2.3. *Let the operators $L, M \in \mathcal{L}(\mathbf{U_K L_2};\mathbf{F_K L_2})$, where the operator M is (L,σ)– bounded. Then for any random $\mathbf{K}$-process $f \in \mathbf{C^{l+1}}(\mathcal{J}, \mathbf{F_K L_2})$ such that $Nf \in \mathbf{C^{l+1}}(\mathcal{J}, \mathbf{F_K L_2})$ and any $\mathfrak{U}$-valued random variable $\xi_0 \in \mathbf{L_2}$, independent with Nf at a fixed $t \in [0,\tau]$, there exists the unique solution $\eta = \eta(t)$ to problem* (2.1), (2.2)*, which has the following form:*

$$\eta(t) = -M_0^{-1} N f^0 + U^t \xi_0 + \int_0^t U^{t-s} L_1^{-1} N f^1 ds, \text{ where } f^0 = (\mathbb{I} - Q) f, f^1 = Qf. \tag{2.3}$$

Proof. The proof of the theorem is similar to the deterministic case. Ideas and methods of the theory σ-bounded operators can be found, for example, in [7]. Note only that the Q operator should be understood as the projector

$$Q = \frac{1}{2\pi i} \int_\gamma L_\mu^L(M) d\mu \in \mathcal{L}(\mathfrak{F}),$$

where $\gamma \subset \mathbb{C}$ is a contour bounding the region containing the L-spectrum of the operator M, $L^L_\mu(M) = L(\mu L - M)^{-1}$ is the left L-resolvent of the operator M; f^0, f^1 belong to $\ker Q = \mathbf{F^0_K L_2}, \Im Q = \mathbf{F^1_K L_2}$, respectively. $\square$

3. The Cauchy–Wentzell problem in the stochastic Barenblatt–Zheltov–Kochina model

Construct a solution to problem (0.4) – (0.6) in the space of **K**-"noise" by means of reduction to the problem (2.1), (2.2).

Denote by $\{\lambda_k : \ k \in \mathbb{N}\}$ the sequence of the Laplace operator's eigenvalues with Wentzell boundary conditions, which are numbered in non-increasing order taking into account the multiplicity, and correspond to the sequence of orthonormal eigenfunctions $\{\varphi_k : \ k \in \mathbb{N}\}$. Introduce a $\mathfrak{U}$-valued random **K**-processes. Take the sequence K as the set of the Green operator's eigenvalues $\{\lambda_k : \lambda_k = \nu_k^{-1}\}$ and determine a $\mathfrak{U}$-valued random **K**-Wiener process in the form

$$W_{\mathbf{K}}(t) = \sum_{k=1}^{\infty} \nu_k \beta_k(t) \varphi_k. \tag{3.1}$$

The formula (3.1) is defined correctly due to the following asymptotics (see, e.g., [8]):

$$\lambda_n \sim -\left(\pi n + \left(\frac{-\alpha_0 + \beta_0}{\pi n} \right) + O\left(\frac{1}{n^3}\right) \right)^2.$$

According to the operator A (see, e.g. [2]), determine the operators $L = \lambda - A$, $M = \alpha A$ as elements of the space $\mathcal{L}(\mathbf{U_K L_2}; \mathbf{F_K L_2})$ by Lemma 2.2, and define the inhomogeneity function to be a derivative of the one-dimensional Wiener process

$$f = \overset{\circ}{W}_{\mathbf{K}}(t) \in \mathbf{C^{l+1}}(\mathcal{J}, \mathbf{F_K L_2}).$$

Due to the fact that the last term in the formula (2.3) has an integral singularity at zero, we transform it as follows:

$$\begin{gathered} \int_\epsilon^t U^{t-s} L_1^{-1} QN \overset{\circ}{W}_{\mathbf{K}}(s) ds = L_1^{-1} QN W_K(t) - U^{t-\epsilon} L_1^{-1} QN W_K(\epsilon) + \\ + \int_\epsilon^t U^{t-s} S L_1^{-1} QN W_{\mathbf{K}}(s) ds, \quad \text{where} \quad S = L_1^{-1} M_1. \end{gathered} \tag{3.2}$$

Integration by parts makes sense for any $\epsilon \in (0, t), t \in \mathbb{R}_+$, due to the definition of the Nelson–Gliklikh derivative. Take the limit $\epsilon \to 0$ in (3.2) and obtain

$$\int_0^t U^{t-s} L_1^{-1} QN \overset{\circ}{W}_{\mathbf{K}}(s) ds = L_1^{-1} QN W_K(t) + \int_0^t U^{t-s} S L_1^{-1} QN W_{\mathbf{K}}(s) ds.$$

Since for all $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R} \backslash \{0\}$ the operator M is (L, σ)-bounded (see, e.g., [2]), according to Theorem 2.3 the following theorem holds

Theorem 3.1. *For any $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R} \backslash \{0\}$, $N \in \mathcal{L}(\mathbf{U_K L_2}; \mathbf{F_K L_2})$ and $\xi_0 \in \mathbf{L_2}$, are independent of $W_K(t)$ there exists the unique solution $\eta = \eta(t)$ to problem*

(0.4) – (0.6)*, which has the following form:*

$$\eta(t) = -M_0^{-1}(\mathbb{I}-Q)N \overset{\circ}{W}_{\mathbf{K}}(t) + U^t\xi_0 + L_1^{-1}QNW_K(t) + \int_0^t U^{t-s}SL_1^{-1}QNW_{\mathbf{K}}(s)ds. \tag{3.3}$$

Construct the projector $Q \in \mathcal{L}(\mathfrak{F})$

$$Q = \sum_{\lambda \neq \lambda_k} < \cdot, \varphi_k >_{\mathfrak{F}} \varphi_k.$$

Then,

$$M_0^{-1}(\mathbb{I}-Q)N \overset{\circ}{W}_{\mathbf{K}}(t) = \begin{cases} 0, \text{if } \ \lambda \notin \sigma(A); \\ \frac{1}{\alpha\lambda} \sum\limits_{\lambda=\lambda_k} \frac{1}{2t} \sum\limits_{j=1}^{\infty} \frac{<\beta_j(t),\varphi_k>_{\mathfrak{F}} N\varphi_k}{(\lambda-\lambda_k)\lambda_j}, \text{if } \lambda \in \sigma(A). \end{cases}$$

$$U^t\xi_0 = \begin{cases} \sum\limits_{k=1}^{\infty} e^{\frac{\alpha\lambda_k}{\lambda-\lambda_k}t} < \xi_0, \varphi_k >_{\mathfrak{F}} \varphi_k, \text{if } \ \lambda \notin \sigma(A); \\ \sum\limits_{k=1,\lambda\neq\lambda_k}^{\infty} e^{\frac{\alpha\lambda_k}{\lambda-\lambda_k}t} < \xi_0, \varphi_k >_{\mathfrak{F}} \varphi_k, \text{if } \lambda \in \sigma(A). \end{cases}$$

$$L_1^{-1}QNW_K(t) = \begin{cases} \sum\limits_{k=1}^{\infty} \sum\limits_{j=1}^{\infty} \frac{<\beta_j(t),\varphi_k>_{\mathfrak{F}} N\varphi_k}{(\lambda-\lambda_k)\lambda_j}, \text{if } \ \lambda \notin \sigma(A); \\ \sum\limits_{k=1,\lambda\neq\lambda_k}^{\infty} \sum\limits_{j=1}^{\infty} \frac{<\beta_j(t),\varphi_k>_{\mathfrak{F}} N\varphi_k}{(\lambda-\lambda_k)\lambda_j}, \text{if } \ \lambda \in \sigma(A). \end{cases}$$

$$\int_0^t U^{t-s}SL_1^{-1}QNW_{\mathbf{K}}(s)ds = \begin{cases} \sum\limits_{k=1}^{\infty} \sum\limits_{j=1}^{\infty} \int\limits_0^t \frac{e^{\frac{\alpha\lambda_k}{\lambda-\lambda_k}(t-s)} \alpha\lambda_k <\beta_j(s),\varphi_k>_{\mathfrak{F}}}{(\lambda-\lambda_k)^2\lambda_j} \, dsN\varphi_k, \lambda \notin \sigma(A); \\ \sum\limits_{k=1,\lambda\neq\lambda_k}^{\infty} \sum\limits_{j=1}^{\infty} \int\limits_0^t \frac{e^{\frac{\alpha\lambda_k}{\lambda-\lambda_k}(t-s)} \alpha\lambda_k <\beta_j(s),\varphi_k>_{\mathfrak{F}}}{(\lambda-\lambda_k)^2\lambda_j} \, dsN\varphi_k, \lambda \in \sigma(A). \end{cases}$$

In conclusion, note that if $\lambda \in \sigma(A)$, then a random value of ξ_0 belongs to the phase space

$$\mathfrak{P}_f = \left\{ u \in domA : \alpha\lambda < u, \varphi_k >_{\mathfrak{F}} = -\sum_{j=1}^{\infty} \frac{< \beta_j(t), \varphi_k >_{\mathfrak{F}} \varphi_j}{\lambda_j}, \lambda_k = \lambda \right\}.$$

And since ξ_0 are independent of $W_K(t)$, then $cov(\xi_0, \beta_k(t)) = 0$, where $\beta_k(t)$ are (sample) paths of Wiener process that have the following form:

$$\beta_k(t) = \sum_{j=1}^{\infty} \xi_j \sin\frac{\pi}{2}(2j+1)t, \quad k = 1, 2, \cdots$$

where ξ_j are uncorrelated Gaussian random variables such that $E\xi_j = 0$, $D\xi_j = \left[\frac{\pi}{2}(2j+1)\right]^{-2}$. Further, for the sake of simplicity of calculations, we put the operator $N = \mathbb{I}$.

4. The algorithm of numerical solution for Barenblatt–Zheltova–Kochina model

Based on the theoretical results, a program for the numerical solution of problem (0.4) – (0.6) was developed and implemented in Maple 2015.This program allows to find an approximate solution to problem (0.4) – (0.6) under arbitrary initial and boundary conditions, the values of λ, α and "white noise" $W_K(t)$, and displays a graph of the approximate solution. We describe the algorithm in more detail.

It is necesary to find an approximate solution using the modify Galerkin method, since the Barenblatt–Zheltova–Kochina model may be degenerate. Let us construct Galerkin approximations solutions to the Cauchy–Wentzell problem in the following form:

$$\widetilde{u}(x,t) = u_N(x,t) = \sum_{k=1}^{N} u_k(t)\varphi_k(x), \tag{4.1}$$

where $\{\varphi_k : k \in \mathbb{N}\}$ are eigenfunctions of the one-dimensional operator A , which correspond to its eigenvalues, orthonormal by the norm $< \cdot, \cdot >_{\mathfrak{F}}$, which are numbered in non-increasing order taking into account the multiplicity.

Using the operators and functions of Maple 2015, we set the initial condition γ_0, the coefficients of ¡¡white noise¿¿ β_k and the Wentzell boundary conditions.

Substitute approximate solution (4.1) to equation (0.4) and take the scalar product of equation (0.4) and eigenfunctions $\varphi_k(x)$ with respect to $< \cdot, \cdot >_{\mathfrak{F}}$. We obtain the following system:

$$\begin{cases} (\lambda - \lambda_1)u_1'(t) = \alpha u_1(t) + f_1(t), \\ (\lambda - \lambda_2)u_2'(t) = \alpha u_2(t) + f_2(t), \\ \dots \\ (\lambda - \lambda_N)u_N'(t) = \alpha u_N(t) + f_N(t). \end{cases} \tag{4.2}$$

Depending on the parameters λ , we have algebraic or first-order differential equations in the system (4.2). Let us consider these conditions in more details.

(i) $\lambda \notin \sigma(A)$. Due to this fact, the mathematical model is non-degenerate, and all the equations in the resulting system are ordinary differential equations of the first order. For the solvability of this system with respect to $u_k(t)$, we take the scalar product of the initial conditions (0.6) and the eigenfunctions $\varphi_k(x)$ with respect to $< \cdot, \cdot >_{\mathfrak{F}}$. Then, we solve the system (4.2) with appropriate initial condition and find the coefficients $u_k(t)$ in the approximate solution $\widetilde{u}(x,t)$.

(ii) $\lambda \in \sigma(A)$. Without loss of generality suppose that $\lambda = \lambda_{m_1} = \dots = \lambda_{m_r}$, where r is the multiplicity of the root. Then, some of equations are algebraic, and some equations are ordinary differential equations of the first order. Let us consider separately systems composed of algebraic equations and differential equations of

the first order. Note that the solution to the original problem exists, according to Theorem 3.1, if the initial random variable $\xi_0(x)$ belongs to the phase space

$$\mathfrak{P}_f = \left\{ u \in domA : \alpha\lambda < u, \varphi_k >_{\mathfrak{F}} = -\sum_{j=1}^{\infty} \frac{< \beta_j(t), \varphi_k >_{\mathfrak{F}} \varphi_j}{\lambda_j}, \lambda_k = \lambda \right\}.$$

Find a solution for the obtained differential (differential and algebraic) systems with the help of built-in operators in Maple 2015 and write the numerical solution to problem (0.4) – (0.6). The block diagram of the stochastic Barenblatt–Zheltova–Kochina model is shown in Fig. 1 .

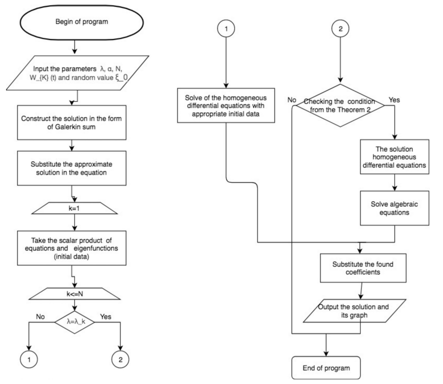


FIGURE 1. Block diagram of the algorithm.

5. Example of solution for the stochastic Cauchy–Wentzell problem

Example. Let us consider the Cauchy–Wentzell problem for the equation

$$(\lambda - A) \overset{\circ}{\eta} (\omega, t) = \alpha A\eta(\omega, t) + \overset{\circ}{W}_{\mathbf{K}} (t), \quad (\omega, t) \in [0, 1] \times (0, \tau), \quad \text{where} \tag{5.1}$$

$$\lambda = 0,\ \alpha = 0,\ \overset{\circ}{W}_{\mathbf{K}} (t) = \frac{1}{2t\lambda_1}\left(\xi_0 \sin\frac{\pi}{2}t + \xi_1 \sin\frac{\pi}{2}3t\right);\ cov(\xi_0, \xi_1) = 0;$$
$$\xi_0, \xi_1 \sim N(0, 1), \eta(0) = \gamma_0 \sim N(0, 1),$$
$$\eta_{xx}(0, t) + \eta_x(0, t) - 3\eta(0, t) = 0,$$
$$\eta_{xx}(1, t) - \eta_x(1, t) + 6\eta(1, t) = 0.$$

Let $N = 6$, then the approximate solution have the following form:

$$\widetilde{u}(x, t) = u_6(x, t) = \sum_{k=1}^{6} u_k(t)\varphi_k(x). \tag{5.2}$$

Solve the Sturm-Liouville problem and find the basis functions $\varphi_k(x)$ in decomposition (5.2). Using the method of moving chords for the transcendental equations of the corresponding form

$$ctgx = x \cdot \frac{1 + \frac{3}{x^2} - \frac{6}{x^2} - \frac{18}{x^4} - \frac{1}{x^2}}{\frac{6}{x^2} - 2 - \frac{3}{x^2}}, x = \sqrt{-\lambda_n}, \lambda_n < 0$$

$$\frac{\lambda + \sqrt{\lambda} - 3}{\lambda - \sqrt{\lambda} - 3} = \frac{e^{2\sqrt{\lambda}}(\lambda - \sqrt{\lambda} + 6)}{\lambda + \sqrt{\lambda} + 6)}, \lambda > 0$$

find and write the eigenfunctions of the one-dimensional Laplace operator.

We have the eigenvalues

$$\begin{aligned}
\lambda_1 &= -x_1^2 = -35.14514947, \\
\lambda_2 &= -x_2^2 = -84.71034130, \\
\lambda_3 &= -x_3^2 = -153.8532547, \\
\lambda_4 &= -x_4^2 = -242.7027758, \\
\lambda_5 &= -x_5^2 = -351.2803151, \\
\lambda_6 &= 5.39027.
\end{aligned}$$

Let us find $\varphi_k(x)$ and construct an orthonormal basis. Set the initial condition and "white noise" using the functions that specify random values with normal distribution. Substitute approximate solution (5.2) in equation (5.1) and take the scalar product of equation (5.1) and the eigenfunctions $\varphi_k(x)$ with respect to $< \cdot, \cdot >_{\mathfrak{F}}$. For example, write the following system for fixed ω:

$$\begin{cases}
35.1451u_1'(t) - 0.0001sin(1.5707t) - 0.0034sin(4.7123t) - 0.0409 = 0, \\
84.7103u_2'(t) - 0.0099sin(1.5707t) - 0.5739sin(4.7123t) + 0.0259 = 0, \\
153.8532u_3'(t) + 0.0021sin(1.5707t) + 0.1240sin(4.7123t) - 0.0278 = 0, \\
242.7027u_4'(t) - 0.0052sin(1.5707t) - 0.3010sin(4.7123t) + 0.0113 = 0, \\
351.2803u_5'(t) + 0.0021sin(1.5707t) + 0.1215sin(4.7123t) - 0.0214 = 0, \\
-5.3902u_6'(t) - 0.0217sin(1.5707t) - 1.2510sin(4.7123t) + 0.1966 = 0.
\end{cases} \tag{5.3}$$

Due to the fact that $\lambda \notin \sigma(A)$, the mathematical model is non-degenerate, and, according to the algorithm, all the equations in the resulting system are ordinary differential equations of the first order. Let us solve the system (5.3) with the initial conditions

$$\begin{aligned}
u_1(0) &= 0.045617, \\
u_2(0) &= 0.155882, \\
u_3(0) &= -0.021318, \\
u_4(0) &= 0.077029, \\
u_5(0) &= -0.028692, \\
u_6(0) &= 0.178588.
\end{aligned}$$

and find the Galerkin coefficients

$u_1(t) = -0.00000109cos(1.5707t) - 0.000021cos(4.7123t) + 0.001165t + 0.04564,$

$u_2(t) = -0.0000749cos(1.5707t) - 0.001437cos(4.7123t) - 0.000306t + 0.157395,$

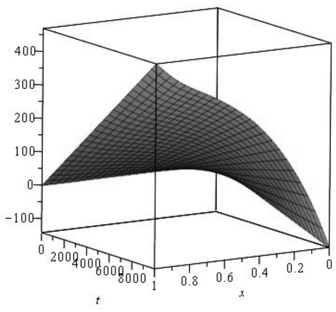
$u_3(t) = 0.00000892cos(1.5707t) + 0.000171cos(4.712t) + 0.000181t - 0.021498,$

$u_4(t) = -0.0000137cos(1.5707t) - 0.000263cos(4.7123t) - 0.00004658t + 0.0773,$

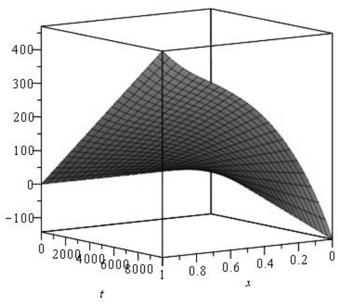
$u_5(t) = 0.00000383cos(1.5707t) + 0.0000734cos(4.7123t) + 0.0000611t - 0.0288,$

$u_6(t) = 0.002569cos(1.5707t) + 0.049251cos(4.7123t) + 0.036479t + 0.126768.$

Substituting the Galerkin coefficients in the representation, we obtain an approximate solution to the original problem. The graph of the solution in the form paths of stochastic process $\eta(t)$ is shown in Fig. 2 (a-b).



a)



b)

FIGURE 2. Paths for the solution of the problem in Example.

Conclusion

We constructed an algorithm and implementation for the numerical solution to Cauchy-Wentzel problem for the stochastic Barenblatt-Zheltova-Kochina model on the interval [0.1]. For this purpose, we used a new approach to the study of the stochastic model with "white noise", which we understand as the Nelson–Gliklikh derivative of one-dimensional Wiener process, the full description of which is given, for example, in [3, 4, 5, 6, 17].

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