Journal of Mathematical Control Science & Applications
Vol. 9 No. 1 (January-June, 2023)
ISSN: 0974-0570

NONLINEAR DYNAMICS OF A THREE SPECIES PREY-PREDATOR SYSTEM INCORPORATING FEAR EFFECT AND HARVESTING

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Abstract. Fear of predators is a well-recognized phenomenon that results from predation in a natural food-chain or food-web ecosystem. Wang et al. [34], published the first food-chain model that included the predator's fear effect on prey growth rate. The indirect impact of fear has been studied through its inclusion in several food web and food chain models in various ecological contexts. Some recent studies and field experiments show that predators affect their prey not only by direct capture; they also induce fear in prey species, which reduces their reproduction rate. The dynamics of a three-species prey-predator model are discussed, with the assumption that the logistic growth rate of prey is reduced as a result of both predators' fears, and the reproduction rate of intermediate consumers is affected by the top predator. We carry out the feasibility, existence of steady states, local stability analysis, and bifurcation analysis. Through numerical simulation, we show that the system stays chaotic at a low cost of fear, but an increase in the fear factor results in stability. We conclude that the chaotic dynamics of the system is controlled by fear effects, i.e., the whole prey-predator system is driven by the fear effect of predators.

1. INTRODUCTION

In an ecosystem, the order of feeding relationships among organisms is known as a food chain, in which energy and nutrients flow from one organism to another at a time along a direct and linear pathway. Each level in a food chain represents a trophic level. Several researchers have extensively studied food chain models for three or more species. The article [11], is a pioneering work on the food chain model with Holling type-II functional response in which rich chaotic dynamics occur. Later, several food chain models are proposed and studied in the presence of various functional responses by the authors [19, 21, 26]. A food web is a diagram that shows how food chains naturally connect to one another in an ecological community. The consumer-resource system is yet another term for the food web. It also includes a variety of food chains that connect with one another. K. McCann and A. Hastings [20], analyze the role of omnivore in food webs using a non-equilibrium perspective. These authors observe that the addition of omnivore to

2010 *Mathematics Subject Classification.* Primary: 34D20, 34C23, Secondary: 37G15, 65P20.
Key words and phrases. Prey-predator model; Fear effect; Existence and Stability; Bifurcation; Chaos.
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a simple food-chain model locally stabilizes the food web. The food web models with various ecological phenomena have been studied by several researchers [6, 9, 13, 12, 22, 23, 31, 32].

The anti-predator behavior includes all forms of action by an individual that is used to avoid being captured, wounded, or killed by another individual. Lima and Dill [18], perceived several components of predation risk by developing an abstraction of the predation strategies. Fear effect of prey is one such common behavior so it can play an important role on population dynamics. The capability of an organism to explore one or more such components is important for decision making in feeding organisms. This is also helpful to the individuals to determine when and how to (i) avoid predators, (ii) be social and (iii) to breathe air (fish). Most of the species including human beings are vulnerable with fear. Predators usually produce fear to their prey in the process of capturing. This motivate Wang et al. [34], to propose a prey-predator model to incorporate the cost of fear into the growth of prey population. Their results show the anti-predator response plays an important role on stabilizing the predator-prey system. A large number of research articles have been published recently incorporate the fear effect in predator-prey systems [2, 3, 4, 5, 15, 24, 25, 27, 28, 29].

This section aims to propose a simple food web model that incorporates both fear effect and harvesting. For this purpose, we first describe a simple food web model proposed by Tanabe and Namba [31], in which $P_1(t)$, $P_2(t)$ and $P_3(t)$ denote the densities of the basal resource, intermediate consumer, and an omnivorous top predator at time t, respectively:

$$\begin{cases} \frac{dP_1}{dt} &= P_1\,(r - a_{11}\,P_1) - a_{12}\,P_1\,P_2 - a_{13}\,P_1\,P_3, \\ \frac{dP_2}{dt} &= a_{21}\,P_1\,P_2 - a_{23}\,P_2\,P_3 - d_1\,P_2, \\ \frac{dP_3}{dt} &= a_{31}\,P_1\,P_3 + a_{32}\,P_2\,P_3 - d_2\,P_3, \end{cases} \tag{1.1}$$

with initial conditions

$$P_1(0) > 0, P_2(0) > 0, P_3(0) > 0. \tag{1.2}$$

Table 1: The Parameters used in the system (1.1) are non negative and have the following sense.

Parameter	**Description**
r	Prey's intrinsic growth rate
a_{11}	Intra-specific competition coefficient of prey
a_{12}	Consumption rate of resources by intermediate consumer
a_{13}	Consumption rate of resources by top predator
a_{21}	Contribution of resources to the growth of consumer
a_{23}	Consumption rate of intermediate consumer by top predator
a_{31}	Contribution of resources to the growth of top-predator
a_{32}	Contribution of consumer to the growth of top-predator
d_1	Consumer's mortality rate
d_2	Mortality rate of top predator

2. Proposed Model

Motivated by the literature, we extend the three-species food web model (1.1) by considering the fear of both predators on prey species and the fear of the top-predator on intermediate consumer. We also assume that both middle and top predators are harvested according to quadratic harvesting. The model is developed under the following two considerations:

(i) The researchers who have worked with three species of prey-predator models have considered that the predator affects prey only by direct killing. But in the literature survey, it is observed that the fear of predator also affects the growth rate of prey species. So, this fear mitigates the reproduction rate of the prey population. For this reason, it is relevant to include the fear term in the model to make it more effective. So, the growth rate of prey population can be modified as

$$\frac{dP_1}{dt} = \frac{P_1\,(r - a_{11}\,P_1)}{1 + k_1\,P_2 + k_2\,P_3}, \tag{2.1}$$

where, k_1 and k_2 are the fear parameters of middle-predator and top-predator on prey species, respectively with

$$\frac{P_1\,(r - a_{11}\,P_1)}{1 + k_1\,P_2 + k_2\,P_3} = \underbrace{\frac{(r + a_{11})\,P_1\left(1 - \frac{a_{11}}{r+a_{11}}P_1\right) + k_1\,a_{11}\,P_1\,P_2 + k_2\,a_{11}\,P_1\,P_3}{1 + k_1\,P_2 + k_2\,P_3}}_{\text{combined reproduction due to fear of both predators}} - \underbrace{a_{11}P_1}_{\substack{\text{natural}\\ \text{death rate}}}.$$

The authors [4, 10, 28, 35] have given a detailed description of the modified growth rate (2.1) of the prey population, which indicates that such a modeling process is also valid. The growth rate of consumer gets reduced due to fear of top-predator on it by a fraction $\frac{1}{1+k_3\,P_3}$ with k_3 as the fear parameter.

(ii) Many researchers [1, 8, 14, 16, 30] have studied ecological models using various types of harvesting. In particular, quadratic harvesting becomes relevant when the size of the population to be exploited becomes large [8, 14]. Although a number of prey-predator models for three species with different types of predation rates have been studied by incorporating the fear effect. As far as our knowledge goes, no researcher has studied the combined effect of fear and quadratic harvesting.

In the presence of the fear effect and the quadratic harvesting, the model (1.1) is modified as follows:

$$\begin{cases} \frac{dP_1}{dt} &= \frac{P_1(r-a_{11}\,P_1)}{1+k_1\,P_2+k_2\,P_3} - a_{12}\,P_1\,P_2 - a_{13}\,P_1\,P_3 \equiv P_1\,\psi^{[1]}(P_1,P_2,P_3), \\ \frac{dP_2}{dt} &= \frac{a_{21}\,P_1\,P_2}{1+k_3\,P_3} - a_{23}\,P_2\,P_3 - d_1\,P_2 - h_1\,P_2^2 \equiv P_2\,\psi^{[2]}(P_1,P_2,P_3), \\ \frac{dP_3}{dt} &= a_{31}\,P_1\,P_3 + a_{32}\,P_2\,P_3 - d_2\,P_3 - h_2\,P_3^2 \equiv P_3\,\psi^{[3]}(P_1,P_2,P_3), \end{cases} \tag{2.2}$$

with initial condition

$$P_1(0) > 0, P_2(0) > 0, P_3(0) > 0. \tag{2.3}$$

Here, h_1 and h_2 are the rates of harvesting of both predators, respectively. The remaining parameters have same ecological meaning as for model (1.1). The proposed system (2.2) is defined in the region $\mathbb{R}^3_+$. For simplification of forthcoming sections, we define

$\psi^{[1]}(P_1,P_2,P_3) = \dfrac{(r - a_{11}\,P_1)}{1+k_1P_2+k_2P_3} - a_{12}\,P_2 - a_{13}\,P_3$, $\psi^{[2]}(P_1,P_2,P_3) = \dfrac{a_{21}\,P_1}{1+k_3P_3}$ $- a_{23}\,P_3 - d_1 - h_1\,P_2$ and $\psi^{[3]}(P_1,P_2,P_3) = a_{31}\,P_1 + a_{32}\,P_2 - d_2 - h_2\,P_3$.

2.1. Feasibility of solutions. In this section, we provide the result for positivity and establish the condition for boundedness of solutions of the system (2.2), which ensures the meaningfulness of the proposed system (2.2) from an ecological point of view.

Theorem 2.1. *(a) The solutions* $(P_1(t),P_2(t),P_3(t))$ *of the system* (2.2) *with initial condition* (2.3) *remain positive for all* $t \geq 0$.
(b) The solutions of the system (2.2) *with initial condition (2.3) are uniformly bounded in the region*

$$\Omega = \Bigg\{(P_1,P_2,P_3) \in \mathbb{R}^3_+ : 0 < \chi(t) = P_1 + \frac{a_{12}}{a_{21}}P_2 + \frac{a_{13}}{a_{31}}P_3 < \frac{(r+M)^2}{4\,a_{11}M} + \phi\,,$$
$$\forall\,\phi > 0\Bigg\},\ \textit{if}\ 2\,a_{12}\,a_{13}\,a_{21}\,a_{31}\,(a_{32}\,a_{23} + 2\,h_1\,h_2) > a_{13}^2\,a_{21}^2\,a_{32}^2 + a_{12}^2\,a_{23}^2\,a_{31}^2.$$

Proof. The proof of this theorem given in Appendix-A (4) □

2.2. Existence and stability analysis of steady states. In this section, we have investigated steady states of the system (2.2) and obtained suitable conditions for their existence. We have also analyzed the local stability of all steady states.

2.2.1. ***Existence of various steady states.*** The steady states of the system (2.2) are given as follows:
(I) $S_0 = (0,0,0)$ is a trivial steady state of the system (2.2), which always exists.
(II) The axial steady state $S_1 = \left(\frac{r}{a_{11}},0,0\right)$ of the system (2.2) also always exists. This steady state is free from both predators.
(III) The P_1P_2-planar steady states are $\tilde{S}_1 = \left(\tilde{P}_1^{[1]},\tilde{P}_2^{[1]},0\right)$ and $\tilde{S}_2 = \left(\tilde{P}_1^{[2]},\tilde{P}_2^{[2]},0\right)$. Both are free from top-predator. The P_2-components of these steady states are given by the roots of the following quadratic equation

$$k_1\,a_{12}\,a_{21}\,P_2^2 + (h_1\,a_{11} + a_{12}\,a_{21})\,P_2 - r\,a_{21} + a_{11}\,d_1 = 0. \tag{2.4}$$

The two roots of equation (2.4), are given as

$$\tilde{P}_2^{[1]} = \frac{-Q_1 + \sqrt{Q_2}}{2\,k_1\,a_{12}\,a_{21}} \text{ and } \tilde{P}_2^{[2]} = \frac{-Q_1 - \sqrt{Q_2}}{2\,k_1\,a_{12}\,a_{21}},$$

where, $Q_1 = (h_1\,a_{11} + a_{12}\,a_{21})$ and $Q_2 = (h_1\,a_{11} + a_{12}\,a_{21})^2 - 4\,k_1\,a_{12}\,a_{21}\,(-r\,a_{21} + a_{11}\,d_1)$. The only possibility of positive root is $\tilde{P}_2 = \tilde{P}_2^{[1]}$ if $a_{11}\,d_1 < r\,a_{21}$.

Hence, $\tilde{S}_1 = \left(\frac{d_1 + h_1\,\tilde{P}_2^{[1]}}{a_{21}}, \tilde{P}_2^{[1]}, 0\right)$ is only a feasible $P_1\,P_2$-planar steady state of the system (2.2).

(IV) The P_3-components of the $P_1\,P_3$-planar steady states $\hat{S}_1 = \left(\hat{P}_1^{[1]}, 0, \hat{P}_3^{[1]}\right)$ and $\hat{S}_2 = \left(\hat{P}_1^{[2]}, 0, \hat{P}_3^{[2]}\right)$ are given by the roots of the following quadratic equation

$$k_2\,a_{13}\,a_{31}\,P_3^2 + Q_3 P_3 + Q_4 = 0, \tag{2.5}$$

where, $Q_3 = h_2\,a_{11} + a_{13}\,a_{31}$ and $Q_4 = -r\,a_{31} + a_{11}\,d_2$. The two roots of this quadratic equation are given by

$$\hat{P}_3^{[1]} = \frac{-Q_3 + \sqrt{Q}}{2\,k_2\,a_{13}\,a_{31}} \text{ and } \hat{P}_3^{[2]} = \frac{-Q_3 - \sqrt{Q}}{2\,k_2\,a_{13}\,a_{31}}, \text{ where, } Q = Q_3^2 - 4\,k_2\,a_{13}\,a_{31}\,Q_4.$$

Hence, the equation (2.5) has a positive root $\hat{P}_3 = \hat{P}_3^{[1]}$ if $a_{11}\,d_2 < r\,a_{31}$ and the system (2.2) has a unique feasible $P_1\,P_3$-planar steady state $\hat{S}_1 = \left(\frac{d_2 + h_2\,\hat{P}_3^{[1]}}{a_{31}}, 0, \hat{P}_3^{[1]}\right)$.

(V) To discuss the co-existing steady states $S_* = (P_1^*, P_2^*, P_3^*)$ of the system (2.2), we focus on the following bi-quadratic equation in P_2-component:

$$B_1 P_3^4 + B_2 P_3^3 + B_3 P_3^2 + B_4 P_3 + B_5 = 0, \tag{2.6}$$

where, B_1, B_2, B_3, B_4 and B_5 are given in Appendix-B (4).

Note that when exactly one of the coefficients B_1, B_2, B_3 and B_4 is negative with $B_5 > 0$ then from Descartes's rule of sign the equation (2.6) has a unique positive root for $P_3 = P_3^*$. The P_1 and P_2-components of $S_* = (P_1^*, P_2^*, P_3^*)$ are positive values of the expressions:

$$\begin{aligned} P_1^* =& \frac{a_{23}\,a_{32}\,P_3^* + a_{32}\,a_{23}\,k_3\,\left(P_3^*\right)^2 + d_1\,a_{32} + a_{32}\,d_1\,k_3\,P_3^* + h_1\,d_2 + d_2\,k_3\,h_1\,P_3^*}{a_{21}\,a_{32} + a_{31}\,h_1 + a_{31}\,k_3\,h_1\,P_3^*} \\ &+ \frac{h_1\,h_2\,P_3^* + h_2\,k_3\,h_1\,\left(P_3^*\right)^2}{a_{21}\,a_{32} + a_{31}\,h_1 + a_{31}\,k_3\,h_1\,P_3^*} \text{ and} \\ P_2^* =& \frac{a_{21}\,d_2 + a_{21}\,h_2\,P_3^* - a_{23}a_{31}\,P_3^* - a_{23}\,k_3\,a_{31}\,\left(P_3^*\right)^2 - d_1\,a_{31} - d_1 k_3 a_{31} P_{3*}}{a_{21}a_{32} + a_{31}\,h_1 + a_{31}\,k_3\,h_1\,P_3^*}. \end{aligned}$$

2.2.2. ***Local stability analysis of various steady states***. In this section, our aim is to analyze the local behavior of the system (2.2) in the neighborhood of steady states we have listed earlier. The Jacobian matrix of the system (2.2) evaluated at an arbitrary steady state $S = (P_1, P_2, P_3)$ is given as follows:

$$J(S) = \begin{pmatrix} R_{11} & -\frac{k_1\,P_1(r - a_{11}\,P_1)}{(1 + k_1\,P_2 + k_2\,P_3)^2} - a_{12}\,P_1 & -\frac{k_2\,P_1(r - a_{11}\,P_1)}{(1 + k_1\,P_2 + k_2\,P_3)^2} - a_{13}\,P_1 \\ \frac{a_{21}\,P_2}{1 + k_3\,P_3} & R_{22} & -\frac{k_3\,a_{21}\,P_1\,P_2}{(1 + k_3\,P_3)^2} - a_{23}\,P_2 \\ a_{31}\,P_3 & a_{32}\,P_3 & R_{33} \end{pmatrix},$$

where, $R_{11} = \frac{r - 2\,a_{11}\,P_1}{1 + k_1\,P_2 + k_2\,P_3} - a_{12}\,P_2 - a_{13}\,P_3$, $R_{22} = \frac{a_{21}\,P_1}{1 + k_3\,P_3} - a_{23}\,P_3 - d_1 - 2\,h_1\,P_2$ and $R_{33} = a_{31}\,P_1 + a_{32}\,P_2 - d_2 - 2\,h_2\,P_3$.

The Jacobian matrix $J(S)$ at trivial steady state S_0 becomes

$$J(S_0) = \begin{pmatrix} r & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}.$$

The characteristic values of the matrix $J(S_0)$ are $\lambda_{01} = r > 0$, $\lambda_{02} = -d_1 < 0$ and $\lambda_{03} = -d_2 < 0$. Thus the steady state S_0 is always a saddle point.

Next, we analyze the stability of the steady state S_1 by calculating the following Jacobian matrix

$$J(S_1) = \begin{pmatrix} -r & -\frac{r\,a_{12}}{a_{11}} & -\frac{r\,a_{13}}{a_{11}} \\ 0 & \frac{r\,a_{21}}{a_{11}} - d_1 & 0 \\ 0 & 0 & \frac{r a_{31}}{a_{11}} - d_2 \end{pmatrix}.$$

The matrix $J(S_1)$ has the characteristic values as $\lambda_{11} = -r$, $\lambda_{12} = \frac{r\,a_{21}}{a_{11}} - d_1$ and $\lambda_{13} = \frac{r\,a_{31}}{a_{11}} - d_2$. Thus, we have the following result for the stability of steady state S_1.

Theorem 2.2. *The steady state S_1 is stable if the conditions $r\,a_{21} < a_{11}\,d_1$ and $r\,a_{31} < a_{11}\,d_2$ are satisfied together otherwise it becomes unstable.*

Further, we concentrate on studying the behavior of the solutions of the system (2.2) near steady state $\tilde{S}_1$. The Jacobian matrix at steady state $\tilde{S}_1$ is

$$J(\tilde{S}_1) = \begin{pmatrix} -\frac{a_{11}\,\tilde{P}_1^{[1]}}{1+k_1\,\tilde{P}_2^{[1]}} & \tilde{R}_{12} & -\frac{k_2\,\tilde{P}_1^{[1]}\left(r - a_{11}\,\tilde{P}_1^{[1]}\right)}{\left(1+k_1\,\tilde{P}_2^{[1]}\right)^2} - a_{13}\,\tilde{P}_1^{[1]} \\ a_{21}\,\tilde{P}_2^{[1]} & -h_1\,\tilde{P}_2^{[1]} & -k_3\,a_{21}\,\tilde{P}_1^{[1]}\,\tilde{P}_2^{[1]} - a_{23}\,\tilde{P}_2^{[1]} \\ 0 & 0 & a_{31}\,\tilde{P}_1^{[1]} + a_{32}\,\tilde{P}_2^{[1]} - d_2 \end{pmatrix},$$

where, $\tilde{R}_{12} = -\frac{k_1\,\tilde{P}_1^{[1]}\left(r - a_{11}\,\tilde{P}_1^{[1]}\right)}{\left(1+k_1\,\tilde{P}_2^{[1]}\right)^2} - a_{12}\,\tilde{P}_1^{[1]}$.

The characteristic polynomial of matrix $J(\tilde{S}_1)$ is given by

$$\tilde{Q}(\lambda) = \left(\lambda - \tilde{\lambda}_1\right)\left(\lambda^2 + \frac{\tilde{A}_1}{1 + k_1\,\tilde{P}_2^{[1]}}\lambda + \frac{\tilde{A}_2\tilde{P}_1^{[1]}\,\tilde{P}_2^{[1]}}{1 + k_1\,\tilde{P}_2^{[1]}}\right), \tag{2.7}$$

where, $\tilde{\lambda}_1 = a_{31}\,\tilde{P}_1^{[1]} + a_{32}\,\tilde{P}_2^{[1]} - d_2$, $\tilde{A}_1 = a_{11}\,\tilde{P}_1^{[1]} + h_1\tilde{P}_2^{[1]} + h_1\,k_1\,\left(\tilde{P}_2^{[1]}\right)^2$ and $\tilde{A}_2 = a_{12}\,a_{21} + a_{11}\,h_1 + 2\,k_1\,a_{12}\,a_{21}\,\tilde{P}_2^{[1]}$.

Thus, we have the following result for the local stability of steady state $\tilde{S}_1$.

Theorem 2.3. *If $a_{31}\,\tilde{P}_1^{[1]} + a_{32}\,\tilde{P}_2^{[1]} < d_2$, then steady state $\tilde{S}_1$ is stable otherwise it is unstable.*

Now, we shall determine the stability of $\hat{S}_1$. The Jacobian matrix $J(S)$ evaluated at $\hat{S}_1$ is given as follows:

$$J(\hat{S}_1) = \begin{pmatrix} \frac{-a_{11}\,\hat{P}_1^{[1]}}{1+k_2\,\hat{P}_3^{[1]}} & \hat{R}_{12} & -\frac{k_2\,\hat{P}_1^{[1]}\left(r-a_{11}\,\hat{P}_1^{[1]}\right)}{\left(1+k_2\,\hat{P}_3^{[1]}\right)^2} - a_{13}\,\hat{P}_1^{[1]} \\ 0 & \frac{a_{21}\,\hat{P}_1^{[1]}}{1+k_3\,\hat{P}_3^{[1]}} - a_{23}\,\hat{P}_3^{[1]} - d_1 & 0 \\ a_{31}\,\hat{P}_3^{[1]} & a_{32}\,\hat{P}_3^{[1]} & -h_2\,\hat{P}_3^{[1]} \end{pmatrix},$$

where, $\hat{R}_{12} = -\frac{k_1\,\hat{P}_1^{[1]}\left(r-a_{11}\,\hat{P}_1^{[1]}\right)}{\left(1+k_2\,\hat{P}_3^{[1]}\right)^2} - a_{12}\,\hat{P}_1^{[1]}$.

The characteristic polynomial of matrix $J(\hat{S}_1)$ is given as following

$$\hat{Q}(\lambda) = \left(\lambda - \hat{\lambda}_1\right)\left(\lambda^2 + \frac{\hat{A}_1}{1+k_2\,\hat{P}_3^{[1]}}\lambda + \frac{\hat{A}_2\hat{P}_1^{[1]}\,\hat{P}_3^{[1]}}{1+k_2\,\hat{P}_3^{[1]}}\right), \tag{2.8}$$

where, $\hat{\lambda}_1 = \frac{a_{21}\,\hat{P}_1^{[1]}}{1+k_3\,\hat{P}_3^{[1]}} - a_{23}\,\hat{P}_3^{[1]} - d_1$, $\hat{A}_1 = a_{11}\,\hat{P}_1^{[1]} + h_2\hat{P}_3^{[1]} + h_2\,k_2\,\left(\hat{P}_3^{[1]}\right)^2$ and $\hat{A}_2 = a_{13}\,a_{31} + a_{11}\,h_2 + 2\,k_2\,a_{13}\,a_{31}\,\hat{P}_3^{[1]}$

Thus, we have the following result for the local stability of steady state $\hat{S}_1$.

Theorem 2.4. *If $\frac{a_{21}\,\hat{P}_1}{1+k_3\,\hat{P}_3^{[1]}} < a_{23}\,\hat{P}_3 + d_1$, then stationary point $\hat{S}_1$ is stable otherwise it is unstable.*

Finally, we check the local stability of co-existing steady state $S_* = (P_1^*, P_2^*, P_3^*)$. The Jacobian matrix $J(S)$ of the system (2.2) evaluated at S_* is given as following:

$$J(S_*) = \begin{pmatrix} -L_1 & -L_2 & -L_3 \\ L_4 & -h_1\,P_2^* & -L_5 \\ a_{31}, P_3^* & a_{32}\,P_3^* & -h_2\,P_3^* \end{pmatrix},$$

where, $L_1 = \frac{a_{11}\,P_1^*}{1+k_1\,P_2^*+k_2 P_3^*}$, $L_2 = \frac{P_1^*(r-a_{11}\,P_1^*)k_1}{\left(1+k_1 P_2^*+k_2\,P_3^*\right)^2} + a_{12}\,P_1^*$, $L_3 = \frac{P_1^*(r-a_{11}\,P_1^*)k_2}{\left(1+k_1\,P_2^*+k_2\,P_3^*\right)^2} + a_{13}\,P_1^*$, $L_4 = \frac{a_{21}\,P_2^*}{1+k_3\,P_3^*}$ and $L_5 = \frac{k_3\,a_{21}\,P_1^*\,P_2^*}{\left(1+k_3\,P_3^*\right)^2} + a_{23}\,P_2^*$.

The characteristic equation of the matrix is given by the following equation:

$$\lambda^3 + \xi_1\,\lambda^2 + \xi_2\,\lambda + \xi_3 = 0, \tag{2.9}$$

where, $\xi_1 = L_1 + h_1\,P_2^* + h_2\,P_3^*$, $\xi_2 = h_1\,h_2\,P_2^*\,P_3^* + a_{32}\,L_5\,P_3^* + h_2\,L_1\,P_3^* + a_{31}\,L_3\,P_3^* + h_1\,L_1\,P_2^* + L_2\,L_4$ and $\xi_3 = h_1\,h_2\,L_1\,P_2^*\,P_3^* + a_{32}\,L_1\,L_5\,P_3^* + a_{32}\,L_3\,L_4\,P_3^* + h_2\,L_2\,L_4\,P_3 - a_{31}\,L_2\,L_5\,P_3^* + h_1\,a_{31}\,L_3\,P_2^*\,P_3^*$.

Theorem 2.5. *The co-existing steady state $S_* = (P_1^*, P_2^*, P_3^*)$ is locally asymptotically stable if $\xi_1 > 0$, $\xi_3 > 0$ and $\xi_1\,\xi_2 - \xi_3 > 0$.*

We know that a simple Hopf-bifurcation occurs when the following conditions are satisfied [[7], [17]]:

(a) The matrix $J(S_*)|_{k_1=k_1{}^{[H]}}$ has a pair of purely imaginary characteristic values and remaining one characteristic value has the negative real part. For this purpose,

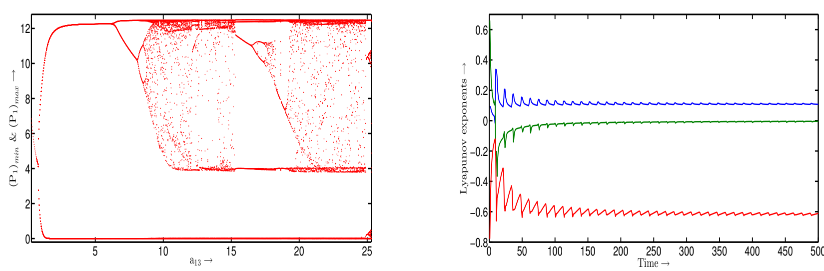
we require the following: $\xi_1 > 0,\ \xi_3 > 0$, and $\xi_1\,\xi_2 = \xi_3$ at $k_1 = {k_1}^{[H]}$.
(b) The transversality condition $\frac{d}{dk_1}\,(Re(\lambda))\,|_{k_1={k_1}^{[H]}} \neq 0$.

Example 2.6. Since, the analytical finding of Liu's criterion of existence of periodic solution through Hopf bifurcation for the system (2.2) is much complicated. So, we verify these conditions numerically. For this purpose we consider a set of parameters: $k_2 = 0.2, k_3 = 0.5, r = 5, a_{11} = 0.45, a_{12} = 1, a_{13} = 20, a_{21} = 1, a_{23} = 1, d_1 = 1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_1 = 0.1$ and $h_2 = 0.27$ and we choose fear parameter k_1 as bifurcation parameter. The system has unique Hopf bifurcation threshold $k_1^{[H]} = 1.439583$. For this choice of parameters' values the system (2.2) has only one co-existing steady state S_* as $k_1 \in (0, 2.876403)$. The co-existing steady state S_* is stable for $k_1 \in (1.439583, 2.876403)$ and unstable for $k_1 < {k_1}^{[H]} = 1.439583$. Here, we get $\xi_1 = 0.3202869752 > 0,\ \xi_3 = 2.610513025 > 0$, and $\xi_1\,\xi_2 = \xi_3 = 0.8361133721$. The characteristic equation corresponding to the matrix $J(S_*)$ at ${k_1}^{[H]} = 1.439583$ given as

$$\lambda^3 + 0.3202869752\lambda^2 + 2.610513025\lambda + 0.8361133721 = 0. \qquad (2.10)$$

The roots of this equation are -0.3202869942, $1.615708214I$ and $-1.615708214I$ which are the characteristic values of the matrix $J(S_*)$.

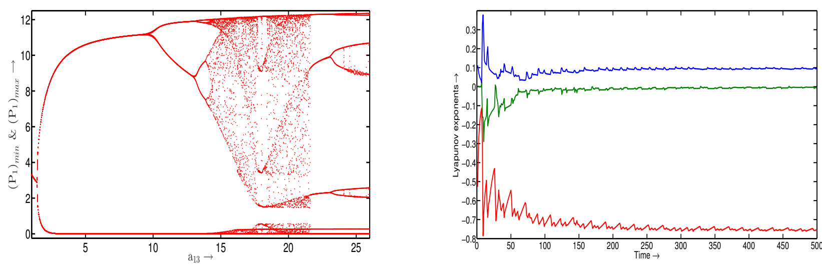
2.3. Dynamics of the system beyond Hopf-bifurcation. In this section, we discuss the different complex dynamical behaviors of the system (2.2) numerically for the impact of fears (k_1 and k_2) of both predators on prey and the impact of fear (k_3) of the top-predator on the middle-predator. Also, we study the impact of harvesting (h_1 and h_2) of both predators on the proposed system (2.2).



(a) (b)

FIGURE 1. Changes in prey species (P_1) w.r.t. parameter a_{13} is shown in (a). The maximum Lyapunov exponents are drawn in (b) for $a_{13} = 22$ with remaining parameters' values given in Example 2.7.

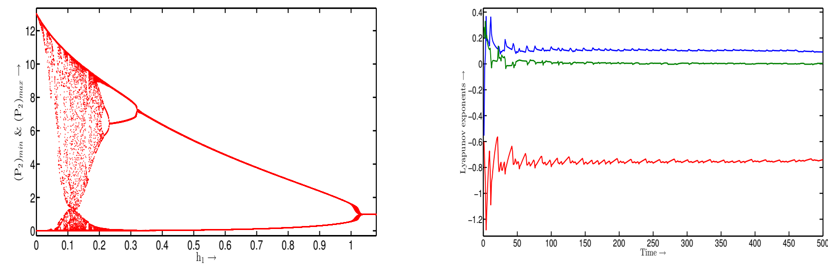
Example 2.7. First of all, we verify the results obtained in the article [[31]] by taking a_{13} as bifurcation parameter and remaining parameters' value as: $k_1 = 0, k_2 = 0, k_3 = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0$. For the chosen set of parameters, the system (2.2) has a unique Hopf-bifurcation threshold ${a_{13}}^{[H]} = 0.79634$, and the corresponding bifurcation diagrams is shown in Figure 1(a), which agrees with the diagram given in [31]. From these figures, it is clear that increasing the value of the bifurcation parameter a_{13} results in chaotic behavior. To confirm the chaotic dynamics of the system (2.2), the maximum Lyapunov exponents are also drawn in Figure 1(b) for $a_{13} = 22$.



(a) (b)

FIGURE 2. (a) represents the changes in prey species (P_1), w.r.t. prey consumption rate a_{13}. The maximum Lyapunov exponents are plotted in (b) for $a_{13} = 2$.

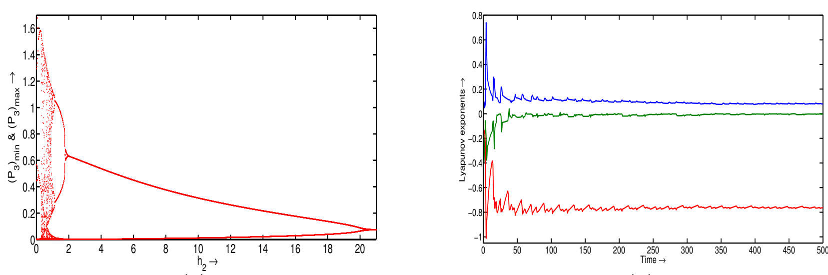
Example 2.8. Now, we check the effect of a_{13} on the dynamics of the system (2.2) in the presence of harvesting h_1 and h_2. To study the effect of a_{13}, we choose a set of parameters' values as: $k_1 = 0, k_2 = 0, k_3 = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.27$. If we choose a_{13} as the bifurcation parameter, then we get a unique Hopf-bifurcation threshold at ${a_{13}}^{[H]} = 1.39875$. The corresponding bifurcation diagrams is shown in Figure 2(a). These three figures confirm that increasing the values of the bifurcation parameter a_{13} causes the system (2.2) to remain chaotic for a narrower range of values of a_{13}. It gives us a hint for control of chaos from the proposed system (2.2). The maximum Lyapunov exponents are plotted in Figure 2(b) for $a_{13} = 22$. To observe the effects of harvesting on the dynamics of system (2.2) in absence of fear effect, we fix a set of parameters' value as: $k_1 = 0, k_2 = 0, k_3 = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 15, a_{21} = 1, a_{23} = 1, d_1 = 1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.027$ and choose h_1 as bifurcation parameter. For these parameters' value the system (2.2) has $h_1^{[H]} = 1.04111$ as a Hopf bifurcation threshold and also a unique co-existing steady state. Figure 3(a) show the chaotic dynamics of the proposed system (2.2), which has been clearly confirmed by the one positive maximum Lyapunov exponent drawn in the Figure 3(b).



(a) (b)

FIGURE 3. (a) shows variations in middle-predator (P_2), w.r.t. predator harvesting h_1. (b) represents the maximum Lyapunov exponents for $h_1 = 0.1$.

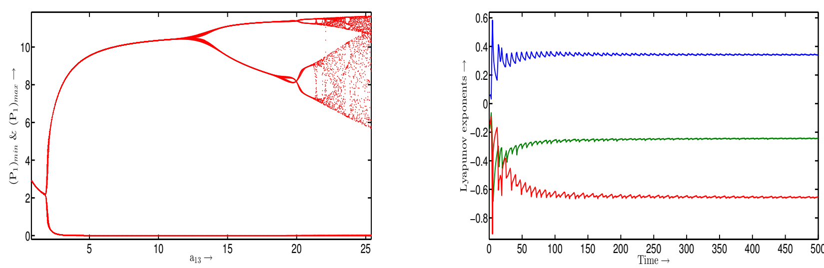
Similarly, if we take h_2 as bifurcation parameter and set: $k_1 = 0, k_2 = 0, k_3 = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 25, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2$, then $h_2^{[H]} = 20.1050$ is a unique Hopf-bifurcation threshold of the system (2.2). The corresponding chaotic bifurcation diagrams is shown in Figure 4(a) for middle-predator (P_2). This diagram shows that chaos can be controlled by harvesting an appropriate amount of middle-predator (P_2) and top-predator (P_3) populations. To confirm the existence of chaotic dynamics, we also plotted the maximum Lyapunov exponents in Figure 4(b) and this figure shows that one of the maximum Lyapunov exponents is positive, which is sufficient for the existence of chaos in the system (2.2).



(a) (b)

FIGURE 4. (a) depicts the changes in top-predator species (P_3), w.r.t. predator harvesting h_2. (b) shows the maximum Lyapunov exponents for $h_2 = 0.5$.

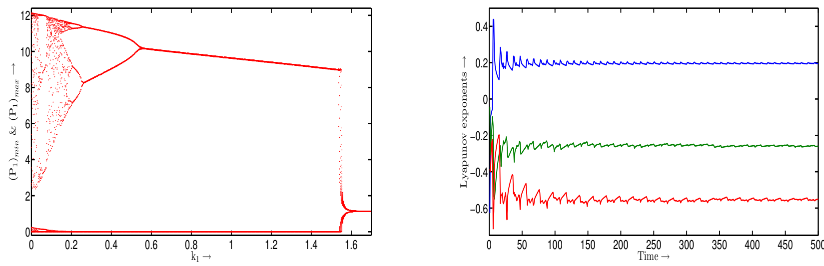
Example 2.9. To know the effect of a_{13} on the dynamics of system (2.2) in the absence of fear k_2 and k_3, and in the presence of k_1, h_1 and h_2, we choose parameters' values as: $k_1 = 0.25, k_2 = 0, k_3 = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.27$, by varying the values of a_{13}, we check the effect of consumption rate of prey by top-predator on the system dynamics. For the chosen set of parameters, the system (2.2) has a unique Hopf-bifurcation threshold ${a_{13}}^{[H]} = 1.7851$ and a unique co-existing steady state $S_* = (2.175087488, 1.265593839, 1.048528104)$. We notice that the increase in a_{13} makes system dynamics chaotic through period doubling routes, which can be seen from Figure 5(a). Also, to verify the chaotic dynamics of the proposed system (2.2), the maximum Lyapunov exponents are plotted in Figure 5(b).



(a) (b)

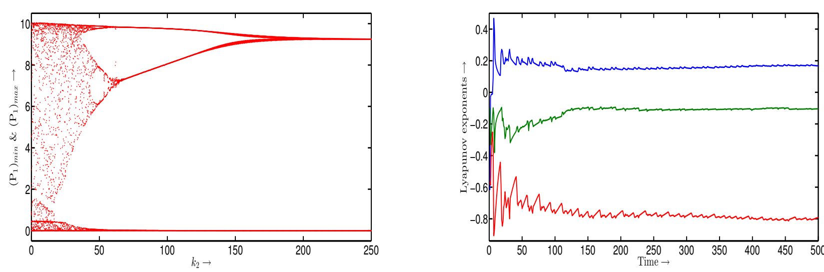
FIGURE 5. (a) describes changes in prey species (P_1), w.r.t. consumption rate of prey by middle-predator a_{13}, in presence of both fear (k_1) and harvesting (h_1 and h_2) in the system (2.2). (b) shows the maximum Lyapunov exponents for $a_{13} = 20$.

Next, we study the impact of fear on the dynamics of the proposed system (2.2). First, we investigate the impact of fear (k_1) of the middle-predator on prey in the absence of fear of the top-predator and the middle-predator, i.e. $k_2 = 0$ and $k_3 = 0$ by choosing a set of parameters with the following values: $k_2 = 0, k_3 = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 20, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.27$. For these set of parameters values, the system (2.2) has a unique Hopf-bifurcation threshold $k_1^{[H]} = 1.59582$ and also a unique co-existing steady state $S_* = (1.137461913, 1.093834981, 0.02807841525)$ if we take $k_1 = 0.08$. The bifurcation diagram is plotted in Figure 6(a), which shows that increasing the impact of fear of middle-predator (k_1) stabilizes the system (2.2) from chaotic instability. The system remains chaotic because of the low cost of fear in prey growth. It shows that prey species become more aware and save themselves from their predators. The maximum Lyapunov exponents are also plotted in Figure 6(b) from which it is clear that one maximum Lyapunov exponents (blue curve) is always positive.



(a) (b)

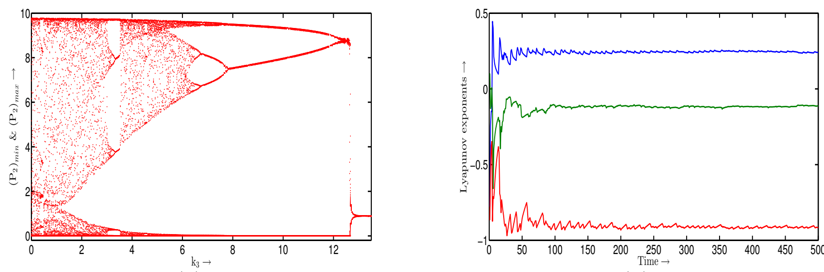
FIGURE 6. (a) shows variations in prey species (P_1), w.r.t. k_1. (b) depicts the maximum Lyapunov exponents for $k_1 = 0.08$.



(a) (b)

FIGURE 7. (a) describes the changes in the prey species (P_1) w.r.t. parameter k_2. (b) represents the maximum Lyapunov exponents for $k_2 = 2$.

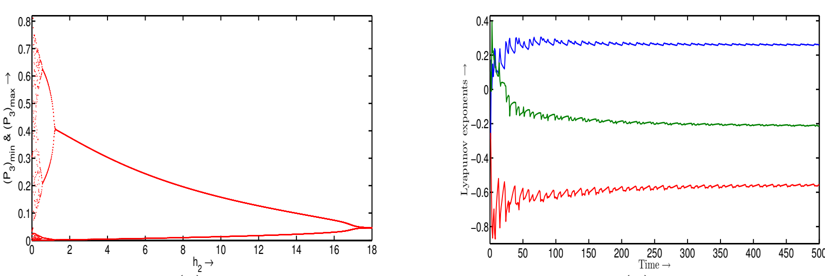
We investigate fear of the top-predator on prey (k_2) if fear of the middle-predator on prey and fear of the top-predator on middle-predator are absent in the proposed system (2.2) by selecting a set of parameters' values: $k_2 = 0, k_3 = 0, r = 5, a_{11} = 0.49, a_{12} = 1, a_{13} = 25, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.27$. A unique co-existing steady state $S_* = (1.223559715, 1.105181268, 0.1019897752)$ exists for the chosen set of parameters' values with $k_2 = 2$. Figure 7(a) shows that increasing the fear of top-predator (k_2) on prey causes the system (2.2) becomes periodically unstable through chaotic instability. The maximum Lyapunov exponents are also drawn in Figure 7(b).



(a) (b)

FIGURE 8. (a) describes the changes in the middle-predator (P_2) w.r.t. parameter k_3. (b) shows the maximum Lyapunov exponents for $k_3 = 1$.

Now, we observe the effect of (k_3) in absence of the other fear in the proposed system (2.2). We fix parameters' values: $k_1 = 0, k_2 = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 30, a_{21} = 1, a_{23} = 1, d_1 = 1.3, h_1 = 0.12, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.27$. For these chosen set of parameters values, the system (2.2) has a unique Hopf-bifurcation threshold $k_3^{[H]} = 12.72310$ and a unique co-existing steady state $S_* = (3.277516693, 0.8973729144, 0.09305401362)$ for given Hopf-bifurcation threshold. Figures 8(a) show that increasing the fear of top-predator (k_3) on middle-predator causes the system (2.2) becomes periodically unstable through chaotic instability. The maximum Lyapunov exponents are drawn in Figure 8(b).

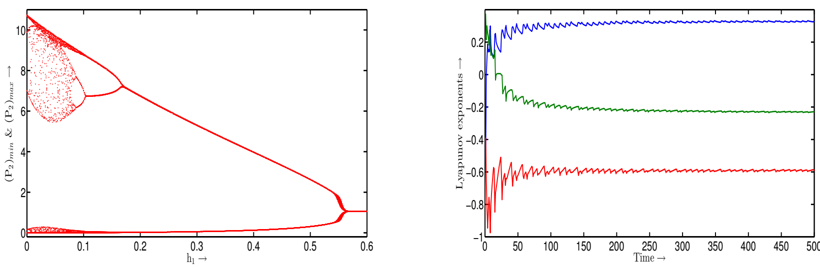


(a) (b)

FIGURE 9. (a) depicts the changes in the top-predator w.r.t. harvesting h_2. (b) represents the maximum Lyapunov exponents for $h_2 = 0.05$.

We are also curious about the impact of harvesting h_1 and h_2 on the dynamics of the system (2.2) in the presence of the fear effect (k_1). For this purpose, we first study the effect of harvesting h_1 on the system dynamics. We fixed a set of parameters' values as: $k_1 = 0.25, k_2 = 0, k_3 = 0, r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 20, a_{21} = 1, a_{23} = 1, d_1 = 1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.27$.

The corresponding bifurcation diagram is shown in Figure 10(a) for the P_2 species w.r.t. middle-predator harvesting h_1 as bifurcation parameter. The Figure 10(b) shows that one of the maximum Lyapunov exponents is positive (blue curve) which verifies existence of chaotic dynamics of the proposed system (2.2).

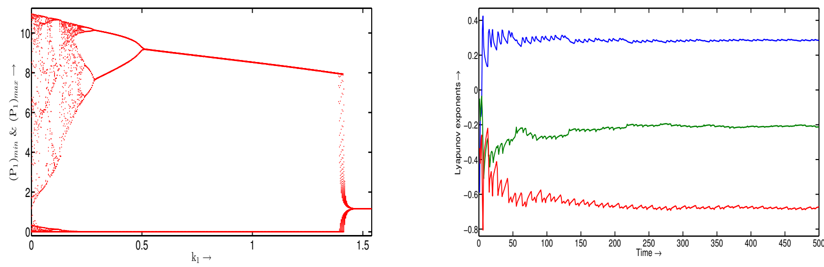


(a) (b)

FIGURE 10. (a) shows the changes in the middle-predator w.r.t. middle-predator harvesting h_1. (b) depicts the maximum Lyapunov exponents for $h_1 = 0.05$.

To check the behavior of proposed system with respect to harvesting h_2 of top-predator, we choose a set of parameters' values as: $k_1 = 0.25$, $k_2 = 0$, $k_3 = 0$, $r = 5$, $a_{11} = 0.4$, $a_{12} = 1$, $a_{13} = 25$, $a_{21} = 1$, $a_{23} = 1$, $d_1 = 1$, $h_1 = 0.1$, $a_{31} = 0.1$, $a_{32} = 1$, $d_2 = 1.2$. The corresponding bifurcation diagram is shown in Figure 9(a), from which it is clear that the system becomes stable for increasing values of h_2. Also, for the given set of parameters' values with $h_2 = 0.05$ one of the maximum Lyapunov exponents (blue curve) in Figure 9(b) is positive which confirm the appearance of chaos in proposed ecological system.

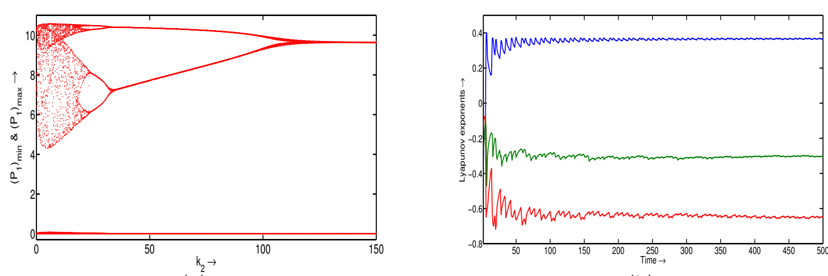
Example 2.10. In this example, we study the impact of fears k_1, k_2 and k_3 on the system (2.2). First, to investigate the effect of k_1 on the proposed system in presence of other factors, i.e., harvesting (h_1 and h_2) and fear effects (k_2 and k_3), we set parameters' values as: $k_2 = 0.2, k_3 = 0.5, r = 5, a_{11} = 0.45, a_{12} = 1, a_{13} = 20, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.027$. The bifurcation diagram presented in Figure 11(a) with respect to parameter k_1 show the appearance of chaos in the proposed system which describe the changes in prey (P_1) species. These diagrams show that the system is chaotic for small values of k_1 and chaotic dynamics become disappear for large values of k_1. To verify the appearance of chaos in the proposed system, we draw the maximum Lyapunov exponents which is depicted in Figure 11(b). From this figure, it is clear that one of the maximum Lyapunov exponents (blue curve) is positive



(a) (b)

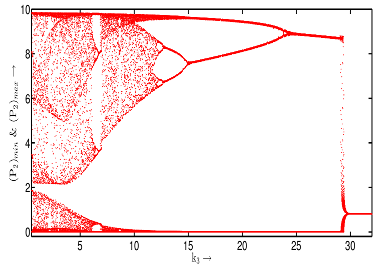
FIGURE 11. (a) is the bifurcation diagram, which shows the changes in prey species (P_1) w.r.t. middle-predator fear k_1. The maximum Lyapunov exponents are plotted in (b) for $k_1 = 0.15$.

Now, we study the dynamics of the proposed system (2.2) to observe the effect of k_2 in presence of other factors (h_1, h_2, k_1 and k_3) by choosing a set of parameters' values as: $k_1 = 0.2, k_3 = 0.5, r = 5, a_{11} = 0.45, a_{12} = 1, a_{13} = 20, a_{21} = 1, a_{23} = 1, d_1 = 1, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.027$. The bifurcation diagram with respect to fear of top-predator on prey species k_2 for prey (P_1) population is presented in Figure 12(a). From these diagrams it can be seen that chaotic dynamics of the system disappear for large values of k_2. It means fear k_2 of top-predator on prey species control the chaos of the system.

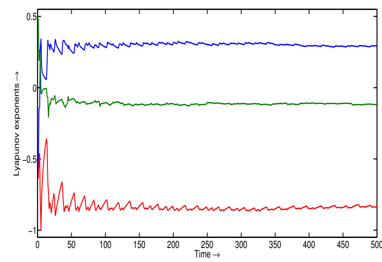


(a) (b)

FIGURE 12. (a) is the bifurcation diagram, which shows the changes in prey (P_1) species w.r.t. middle-predator fear k_2. (b) shows the maximum Lyapunov exponents for $k_2 = 0.15$.



(a)



(b)

FIGURE 13. (a) is the bifurcation diagrams, which shows the changes in middle-predator (P_2) species w.r.t. middle-predator fear k_3. (b) represent the maximum Lyapunov exponents for $k_2 = 1.5$.

The maximum Lyapunov exponents are plotted in Figure 12(b) for $k_2 = 0.15$ with remaining parameters' values same as above.

Also, we study the dynamics of the proposed system for k_3 in presence of other factors (h_1, h_2, k_1 and k_2) by choosing a set of parameters' values as: $k_1 = 0.015, k_2 = 10, r = 5, a_{11} = 0.4, a_{12} = 1, a_{13} = 25, a_{21} = 1, a_{23} = 1, d_1 = 1.3, h_1 = 0.1, a_{31} = 0.1, a_{32} = 1, d_2 = 1.2, h_2 = 0.027$. The bifurcation diagrams with respect to parameter k_3 for middle-predator is presented in Figure 13(a). To observe the appearance of chaotic behavior of the proposed system, we draw the maximum Lyapunov exponents in Figure 13(b) for $k_2 = 1.5$. From this figure, it is seen that one of the maximum Lyapunov exponents is positive.

3. Conclusion

In this article, we analyzed a prey-predator system for three species by incorporating fear effects of predators on their prey with predators harvesting. Analytically, we have derived conditions for the positivity and boundedness of solutions in support of ecological feasibility of the proposed system by using suitable differential inequalities. We have shown analytically the existence of ecological steady states and observed that the proposed system has multiple co-existing steady states. Further, the local stability analysis of each steady states of the system is performed with the help of characteristic values of the Jacobian matrix. The global stability using suitable Lyapunov functions of various steady states which are locally stable can also be performed in future studies. Since, the model system dynamics transits from the limit cycle to a stable focus, resulting in Hopf bifurcation. Therefore, we have also done Hopf bifurcation in our analysis. We have numerically shown that the system enters into Hopf-bifurcation with consumption rate of prey by intermediate consumer (a_{13}), the fear parameters (k_1, k_2 and k_3), and harvesting in predators (h_1 and h_2) as bifurcation parameters.

We observe that in absence of parameters k_1, k_2, k_3, h_1 and h_2, the proposed system which is same as a food web model proposed by [31] agrees with the results discussed in this article. To study the dynamics of the proposed system due to impact of fears of predators on their preys and harvesting of both predators, we have plotted the bifurcation diagrams with respect to each of these parameters. From the bifurcation diagrams, we observe that increase either of the parameters k_1, k_2, k_3, h_1 and h_2 can control chaos which is observed by [31] with respect to a_{13}. In case of multiple steady states more dynamics can be performed in form of saddle-node bifurcation and bifurcation of periodic solutions in future.

Acknowledgment: The first author (Dinesh K. Yadav) is thankful to UGC and BHU for the financial support under the Non-NET UGC fellowship (No.: R/Dev./Sch.(UGC Research Fellow)/2020-21/28397). The work of second author (Ravi P. Gupta) is supported by UGC-BSR Research Start-Up-Grant-2016-17 (No.R/Dev/M-14-33/0219).

AUTHOR DECLARATIONS

Conflict of Interest:
The authors confirm that there are no known conflicts of interest associated with this manuscript.

Data Availability:
The data that support the findings of this study are available within the article.

4. Appendix

Appendix-A

Proof. (a) The system (2.2) along with initial condition (2.3) gives

$$P_1(t) = P_1(0)\exp\left(\int_0^t \psi^{[1]}(P_1(\tau), P_2(\tau), P_3(\tau))d\tau\right) > 0,$$

$$P_2(t) = P_2(0)\exp\left(\int_0^t \psi^{[2]}(P_1(\tau), P_2(\tau), P_3(\tau))d\tau\right) > 0$$

and
$$P_3(t) = P_3(0)\exp\left(\int_0^t \psi^{[3]}(P_1(\tau), P_2(\tau), P_3(\tau))d\tau\right) > 0.$$

This result ensures that solutions of the system (2.2) originating from the region $\mathbb{R}^3_+$ remain inside it for all $t \geq 0$. □

Proof. (b) Let us define a function $\chi(t) = P_1 + \frac{a_{12}}{a_{21}}P_2 + \frac{a_{13}}{a_{31}}P_3$, then the derivative of $\chi(t)$ with respect to time t given as

$$\frac{d\chi(t)}{dt} = \frac{dP_1}{dt} + \frac{a_{12}}{a_{21}}\frac{dP_2}{dt} + \frac{a_{13}}{a_{31}}\frac{dP_3}{dt},$$

which can be written as

$$\begin{aligned}\frac{d\chi(t)}{dt} =& \frac{P_1\,(r - a_{11}P_1)}{1 + k_1\,P_2 + k_2\,P_3} - \frac{a_{12}\,k_3\,P_1\,P_2\,P_3}{1 + k_3\,P_3} - \frac{a_{12}\,d_2P_2}{a_{21}} - \frac{a_{13}\,d_2P_3}{a_{31}} \\ & - \left\{\frac{a_{12}\,h_1P_2^2}{a_{21}} - \left(\frac{a_{13}\,a_{32}}{a_{31}} - \frac{a_{23}\,a_{12}}{a_{21}}\right)P_2\,P_3 + \frac{a_{13}\,h_2P_2^2}{a_{31}}\right\}.\end{aligned}$$

For a positive constant M, we have

$$\frac{d\chi(t)}{dt} + M\chi(t) = \frac{P_1\,(r - a_{11}P_1)}{1 + k_1\,P_2 + k_2\,P_3} + MP_1 + \frac{a_{12}\,(M - d_1)\,P_2}{a_{21}} + \frac{a_{13}\,(M - d_2)\,P_3}{a_{31}}$$
$$- \frac{a_{12}\,k_3\,P_1\,P_2\,P_3}{1 + k_3\,P_3} - \left\{\frac{a_{12}h_1{P_2}^2}{a_{21}} - \left(\frac{a_{13}a_{32}}{a_{31}} - \frac{a_{23}a_{12}}{a_{21}}\right)P_2P_3 + \frac{a_{13}h_2P_3^2}{a_{31}}\right\}.$$

The expression

$$\frac{a_{12}h_1}{a_{21}}\,P_2^2 - \left(\frac{a_{13}a_{32}}{a_{31}} - \frac{a_{23}a_{12}}{a_{21}}\right)P_2P_3 + \frac{a_{13}h_2}{a_{31}}\,P_3^2$$

is a quadratic form and will be positive definite if it satisfies the condition:

$$2\,a_{12}\,a_{13}\,a_{21}\,a_{31}\,(a_{32}\,a_{23} + 2\,h_1\,h_2) > a_{13}^2a_{21}^2a_{32}^2 + a_{12}^2\,a_{23}^2a_{31}^2.$$

If we choose $M = \min(d_1, d_2)$, then under the above condition, we can write

$$\frac{d\chi(t)}{dt} + M\chi(t) \le -a_{11}\left(P_1 - \frac{r + M}{2\,a_{11}}\right)^2 + \frac{(r + M)^2}{4\,a_{11}}.$$

This gives

$$\frac{d\chi(t)}{dt} + M\chi(t) \le \frac{(r + M)^2}{4\,a_{11}}.$$

Now, using the theory of differential inequality [33], we obtain

$$0 < \chi(t) \le \frac{(r + M)^2}{4\,a_{11}M}\left(1 - e^{-Mt}\right) + \chi(0)\,e^{-Mt}.$$

Therefore,

$$0 < \lim_{t\to\infty} \chi(t) \le \frac{(r + M)^2}{4\,a_{11}M}.$$

Hence, all solutions of the system (2.2) with initial condition (2.3) which starts from $\mathbb{R}^3_+$ are bounded in the region Ω. □

Appendix-B

$B_1 = k_2a_{13}h_1^2k_3^2a_{31}^2 + k_1a_{12}k_3^2a_{23}^2a_{31}^2 - h_1k_2a_{12}a_{23}k_3^2a_{31}^2 - h_1k_1a_{13}a_{23}k_3^2a_{31}^2,$

$B_2 = 2\,a_{12}k_1a_{31}^2a_{23}k_3^2d_1 - a_{13}k_1a_{31}^2k_3^2h_1d_1 - 2\,a_{13}k_1a_{31}^2h_1a_{23}k_3 - a_{12}k_2a_{31}^2k_3^2h_1d_1 - 2\,a_{12}k_2a_{31}^2h_1a_{23}k_3 - a_{12}a_{31}^2k_3^2h_1a_{23} + a_{11}a_{32}a_{31}k_3^2h_1a_{23} + a_{13}k_1a_{31}k_3h_1a_{21}h_2 + 2\,a_{13}k_2a_{31}a_{21}a_{32}k_3h_1 + 2\,a_{13}k_2a_{31}^2h_1^2k_3 + a_{11}h_2a_{31}k_3^2h_1^2 + a_{13}a_{31}^2k_3^2h_1^2 + 2\,a_{12}k_1a_{31}^2a_{23}^2k_3 - a_{12}k_2a_{31}a_{21}a_{32}a_{23}k_3 - 2\,a_{12}k_1a_{31}a_{21}h_2a_{23}k_3 + a_{12}k_2a_{31}k_3h_1a_{21}h_2 - a_{13}k_1a_{31}a_{21}a_{32}a_{23}k_3,$

$B_3 = a_{11}a_{32}k_3h_1a_{21}h_2 - 2\,a_{12}k_1a_{31}a_{21}h_2a_{23} + 2\,a_{11}a_{32}a_{31}h_1a_{23}k_3 - a_{12}a_{31}a_{21}a_{32}a_{23}k_3 + a_{12}a_{31}k_3h_1a_{21}h_2 - a_{12}k_2a_{31}a_{21}a_{32}a_{23} + a_{12}k_2a_{31}h_1a_{21}h_2 - a_{13}k_1a_{31}a_{21}a_{32}a_{23} + a_{13}k_1a_{31}h_1a_{21}h_2 + 2\,a_{13}a_{31}a_{21}a_{32}k_3h_1 + 2\,a_{13}k_2a_{31}a_{21}a_{32}h_1 + a_{12}k_1a_{31}^2a_{23}^2 - ra_{31}^2k_3^2h_1^2 + 2\,a_{13}a_{31}^2h_1^2k_3 + a_{13}k_2a_{31}^2h_1^2 + a_{13}k_2a_{21}^2a_{32}^2 + a_{12}k_1a_{21}^2h_2^2 + a_{11}a_{32}^2a_{21}a_{23}k_3 + 2\,a_{11}h_2a_{31}h_1^2k_3 - 2\,a_{12}a_{31}^2h_1a_{23}k_3 - a_{12}k_2a_{31}^2h_1a_{23} - a_{13}k_1a_{31}^2h_1a_{23} + a_{13}k_1a_{21}^2a_{32}h_2 + a_{12}k_2a_{21}^2a_{32}h_2 + a_{12}k_1a_{31}^2d_1^2k_3^2 - a_{12}a_{31}^2k_3^2h_1d_1 + a_{11}d_2a_{31}k_3^2h_1^2 - 2\,a_{12}k_1a_{31}a_{21}d_2a_{23}k_3 - 2\,a_{12}k_1a_{31}a_{21}h_2d_1k_3 - a_{12}k_2a_{31}a_{21}a_{32}d_1k_3 + a_{12}k_2a_{31}k_3h_1a_{21}d_2 - a_{13}k_1a_{31}a_{21}a_{32}d_1k_3 + a_{13}k_1a_{31}k_3h_1a_{21}d_2 + 4\,a_{12}k_1a_{31}^2a_{23}d_1k_3 + a_{11}a_{32}a_{31}k_3^2h_1d_1 - 2\,a_{12}k_2a_{31}^2h_1d_1k_3$

$-2\,a_{13}k_1a_{31}^2h_1d_1k_3,$

$B_4 = a_{13}a_{21}^2a_{32}^2+a_{13}a_{31}^2h_1^2-2\,ra_{31}a_{21}a_{32}k_3h_1-a_{12}a_{31}^2h_1a_{23}-2\,ra_{31}^2h_1^2k_3+a_{11}h_2a_{31}h_1^2+a_{11}a_{32}^2a_{21}a_{23}+a_{12}a_{21}^2a_{32}h_2+2\,a_{13}a_{31}a_{21}a_{32}h_1+a_{11}a_{32}a_{31}h_1a_{23}-a_{12}a_{31}a_{21}a_{32}a_{23}+a_{12}a_{31}h_1a_{21}h_2+a_{11}a_{32}h_1a_{21}h_2+2\,a_{12}k_1a_{31}^2d_1^2k_3+2\,a_{12}k_1a_{31}^2a_{23}d_1+a_{11}a_{32}^2a_{21}d_1k_3+a_{12}k_2a_{21}^2a_{32}d_2+a_{13}k_1a_{21}^2a_{32}d_2+2\,a_{12}k_1a_{21}^2d_2h_2-2\,a_{12}a_{31}^2h_1d_1k_3-a_{12}k_2a_{31}^2h_1d_1-a_{13}k_1a_{31}^2h_1d_1+2\,a_{11}d_2a_{31}h_1^2k_3-2\,a_{12}k_1a_{31}a_{21}d_2d_1k_3-2\,a_{12}k_1a_{31}a_{21}d_2a_{23}-2\,a_{12}k_1a_{31}a_{21}h_2d_1+2\,a_{11}a_{32}a_{31}h_1d_1k_3-a_{12}a_{31}a_{21}a_{32}d_1k_3+a_{12}a_{31}k_3h_1a_{21}d_2-a_{12}k_2a_{31}a_{21}a_{32}d_1+a_{12}k_2a_{31}h_1a_{21}d_2-a_{13}k_1a_{31}a_{21}a_{32}d_1+a_{13}k_1a_{31}h_1a_{21}d_2+a_{11}a_{32}k_3h_1a_{21}d_2$

and $B_5 = a_{11}a_{32}h_1a_{21}d_2-2\,a_{12}k_1a_{31}a_{21}d_2d_1+a_{12}k_1a_{21}^2d_2^2+a_{12}k_1a_{31}^2d_1^2-a_{12}a_{31}^2h_1d_1-ra_{21}^2a_{32}^2+a_{11}d_2a_{31}h_1^2-ra_{31}^2h_1^2+a_{11}a_{32}^2a_{21}d_1+a_{11}a_{32}a_{31}h_1d_1+a_{12}a_{21}^2a_{32}d_2-a_{12}a_{31}a_{21}a_{32}d_1+a_{12}a_{31}h_1a_{21}d_2-2\,ra_{31}a_{21}a_{32}h_1$**.**

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