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EXISTENCE OF MAXIMAL AND MINIMAL SOLUTIONS FOR ATANGANA–BALEANU-CAPUTO TYPE FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. This research paper develops and extends the existence of maximal and minimal solutions for a class of nonlinear fractional integrodifferential equations involving the Atangana-Baleanu operator. By means of Arzela-Ascoli theorem with help of the upper and lower solutions method, we investigate the sufficient conditions of existence of maximal and minimal solutions for the suggested problem. An example in order to illustrate the validity of the main results.

1. Introduction

The idea of fractional calculus (FC) has been appearanced by G. W. Leibniz and G. de l'Hôpital in 1695 where an issue about the $1/2$ derivative was taken up. Numerous scientists and researchers as of late focused on fractional derivatives (FDs), principally because of their advantage in modeling fractional differential equations (FDEs) for physical and engineering processes. At this crossroads, it is significant that the mathematical models of classical differential equations do not work satisfactorily much of the cases.

Subsequently, as of late FC is utilized in many fields including mechanics, population dynamics, image processing, and different scientific areas like electrochemistry, viscoelasticity, fluid flow, and engineering [1, 2, 3, 4, 5].

The theory and applications of FC extended enormously over the nineteenth and twentieth hundreds of years, and various contributors and authors have introduced definitions for FDs and fractional integrals (FIs). The most used FDs are the Riemann-Liouville, Caputo and Hilfer types. There are other types of FDs as well, we refer to some of them, see [6, 7, 8, 9] and references therein.

Besides, the above FDs have a singularity at the beginning since the solution representations contain exponential and Mittag-Leffler function (MLF). This constraint reduces the practical applicability, additionally to utilize Caputo's FD of a function, one should evaluate its derivative, it requests regularity on differentiation. To realize the physical basis of the FC with various memory, Caputo and Fabrizio [10] presented a new definition on FD of order without a nonsingular kernel.

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This operator contains a non-singular kernel yet still conserves the most substantial peculiarity of the classical fractional operators. Utilizing this operator created better outcomes compared with the FDs singular kernel.

However, an encumbrance of this FD emerged in view of the fact that the associated integral can be written in terms of an integral of integer order. To avoid this obstacle, Atangana and Baleanu [11] introduced a new version of FDs, so-called AB operators which allows the generalized MLF as the non-singular and non-local kernel and responds to most of the properties of FDs. Many authors contributed to growing the AB-type FC, see [12, 13, 14], also, many of its uses appeared in the field of epidemiological modeling and the qualitative theory of FDEs, see Koca [15], Atangana and Gomez-Aguilar in [16], Toufik and Atangana [17], Khan et al. [18], Jarad et al. [19]. Abdo et al. [20, 21, 22, 23, 24].

Fractional integrodifferential equations (FIDEs) are generalizations of FDEs, Fredholm and Volterra integral equations. These sorts of equations emerge in numerous modeling problems in mathematical physics, like heat conduction in materials with memory, we refer here to some recent contributions on this type of problems, see [25, 26, 27, 28, 29].

On the other hand, the investigation of the existence of solutions of different types of FDEs by the using of different techniques of fixed point and upper and lower solutions methods can be found in [30, 31, 32, 33].

Motivated by above papers, and inspired by [11, 31], we investigate the sufficient conditions of existence of maximal and minimal solutions for the following AB-type FIDE

$$\begin{cases} {}^{\mathcal{ABC}}\mathcal{D}^{\mu}_{a^+}\psi(\varrho) = \mathrm{f}(\varrho, \psi(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{a^+}\psi(\varrho)), & \varrho \in [a,b], \\ \psi(a) = \psi_a, \end{cases} \tag{1.1}$$

where $0 < \mu < 1$, $0 < a < b < \infty$, ${}^{\mathcal{ABC}}\mathcal{D}^{\mu}_{a^+}$ and ${}^{\mathcal{AB}}\mathcal{I}^{\mu}_{a^+}$ are AB-type FD and FI of order μ, respectively, $\mathrm{f} : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous, and $\psi_a \in \mathbb{R}$.

We give interesting results for ABC-type FIDE. Most of our derivations are made using Schauder's fixed point theorems, and upper/lower solution method.

The paper is structured as follows: In Section 2, we give some basic results needed in the seque. In Section 3, we prove the existence of maximal and minimal solutions to the problem (1.1). An example is provided in Section 4.

2. Preliminaries

In this section, we give some essential definitions and lemmas of AB-type FC which are needed whole this paper. Assume $\mathrm{X} = \mathcal{C}([a,b], \mathbb{R})$ be a Banach space with the norm $\|\psi\| = \max\{|\psi(\varrho)|;\ \varrho \in [a,b]\};\ \psi \in \mathrm{X}$.

Definition 2.1. [11] Let $0 < \mu < 1$, and $\psi \in \mathrm{H}^1(a,b)$, $a < b$. The $\mathcal{ABC}$ FD for function ψ of order μ is given by

$${}^{\mathcal{ABC}}\mathcal{D}^{\mu}_{\mathrm{a}^+}\psi(\varrho) = \frac{\mathfrak{N}(\mu)}{1-\mu}\int_a^{\varrho} \psi'(\xi)\mathrm{E}_{\mu}\left(\frac{-\mu(\varrho-\xi)^{\mu}}{1-\mu}\right)d\xi. \tag{2.1}$$

Further, the $\mathcal{ABR}$ FD is defined by

$${}^{\mathcal{ABR}}\mathcal{D}^{\mu}_{\mathrm{a}^+}\psi(\varrho) = \frac{\mathfrak{N}(\mu)}{1-\mu}\frac{d}{d\varrho}\int_a^{\varrho} \psi(\xi)\mathrm{E}_{\mu}\left(\frac{-\mu(\varrho-\xi)^{\mu}}{1-\mu}\right)d\xi. \tag{2.2}$$

Here, $\mathfrak{N}(\mu) > 0$ is a normalization function satisfies $\mathfrak{N}(0) = \mathfrak{N}(1) = 1$ and E_μ represents the MLF.

Definition 2.2. [11] Let $0 < \mu < 1$ and ψ be function, then $\mathcal{AB}$ FI of order μ is given by

$$\begin{aligned} {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\psi(\varrho) &= \frac{1-\mu}{\mathfrak{N}(\mu)}\psi(\varrho) + \frac{\mu}{\mathfrak{N}(\mu)}\ {}^{\mathcal{RL}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\ \psi(\varrho) \\ &= \frac{1-\mu}{\mathfrak{N}(\mu)}\psi(\varrho) + \frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)}\int_a^{\varrho}\psi(\xi)(\varrho-\xi)^{\mu-1}d\xi, \end{aligned} \tag{2.3}$$

where

$${}^{\mathcal{RL}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\ \psi(\varrho) = \frac{1}{\Gamma(\mu)}\int_a^{\varkappa}\psi(\xi)(\varkappa-\xi)^{\mu-1}d\xi$$

is called the Riemman-Liouville FI [1].

Definition 2.3. [14] Let $n < \mu \leq n+1,\ n\ \in \mathbb{N}$ and ψ be a function such that $\psi^{(n)} \in \mathrm{H}^1(a,b)$. Then $\mathcal{ABC}$ FD satisfies ${}^{\mathcal{ABC}}\mathcal{D}^{\mu}_{\mathrm{a}^+}\psi(\varrho) = {}^{\mathcal{ABC}}\mathcal{D}^{\eta}_{\mathrm{a}^+}\ \psi^{(n)}(\mathrm{t})$, where $\eta = \mu - n$.

Lemma 2.4. [14] *For $n < \mu \leq n+1,\ n\ \in \mathbb{N}$,*

$${}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\ {}^{\mathcal{ABC}}\mathcal{D}^{\mu}_{\mathrm{a}^+}\ \psi(\varrho) = \psi(\varrho) + \mathrm{d}_0 + \mathrm{d}_1\,(\varrho - \mathrm{a}) + \mathrm{d}_2\,(\varrho - \mathrm{a})^2 + \cdots + \mathrm{d}_n\,(\varrho - \mathrm{a})^n\,,$$

where d_i($i = 0,1,2,...,n$) is an arbitrary constant.

Theorem 2.5. [34] *(Arzela-Ascoli's Theorem). Let X be a Banach space. A subset F in X is relatively compact iff it is uniformly bounded and equicontinuous.*

3. Main results

In this section, we prove the existence of maximal and minimal solutions for (1.1).

Lemma 3.1. [11] *Let $0 < \mu < 1$ and $\mathrm{h} : [a,b] \to \mathbb{R}$ is a continuous with $\mathrm{h(a)} = 0$. Then the linear $\mathcal{ABC}$-type FDF*

$$\begin{aligned} {}^{\mathcal{ABC}}\mathcal{D}^{\mu}_{a^+}\psi(\varrho) &= \mathrm{h}(\varrho), \quad \varrho \in [a,b], \\ \psi(a) &= \psi_a \end{aligned} \tag{3.1}$$

is equivalent to

$$\psi(\varrho) = \psi_a + \frac{1-\mu}{\mathfrak{N}(\mu)}\mathrm{h}(\varrho) + \frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)}\int_0^{\varrho}\mathrm{h}(\xi)(\varrho-\xi)^{\mu-1}d\xi. \tag{3.2}$$

As result of Lemma 3.1, we get the following Lemma:

Lemma 3.2. *Assume that $\mathrm{f} : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous with $\mathrm{f}(\mathrm{a}, \psi(\mathrm{a}), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\psi(\mathrm{a})) = 0$. Then the nonlinear $\mathcal{ABC}$-type FIDF (1.1) is equivalent to the fractional integral equation (FIE)*

$$\begin{aligned} \psi(\varrho) &= \psi_a + \frac{1-\mu}{\mathfrak{N}(\mu)}\mathrm{f}(\varrho, \psi(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\psi(\varrho)) \\ &\quad + \frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)}\int_a^{\varrho}(\varrho-\xi)^{\mu-1}\mathrm{f}(\xi, \psi(\xi), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\psi(\xi))d\xi. \end{aligned} \tag{3.3}$$

Now, we shall introduce the concept of upper and lower solutions for the FIE (3.3), which play an important role in our forthcoming analysis.

Definition 3.3. A pair of functions $(\underline{\psi},\overline{\psi}) \in \mathrm{X}\times\mathrm{X}$ is called to be upper and lower solutions of FIE (3.3), respectively, if

$$\begin{aligned}\underline{\psi}(\varrho) &\leq \psi_a + \frac{1-\mu}{\mathfrak{N}(\mu)}\mathrm{f}(\varrho,\underline{\psi}(\varrho),{}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}+}\underline{\psi}(\varrho)) \\ &\quad + \frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)}\int_a^{\varrho}(\varrho-\xi)^{\mu-1}\mathrm{f}(\xi,\underline{\psi}(\xi),{}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}+}\underline{\psi}(\xi))d\xi\end{aligned}$$

and

$$\begin{aligned}\overline{\psi}(\varrho) &\geq \psi_a + \frac{1-\mu}{\mathfrak{N}(\mu)}\mathrm{f}(\varrho,\overline{\psi}(\varrho),{}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}+}\overline{\psi}(\varrho)) \\ &\quad + \frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)}\int_a^{\varrho}(\varrho-\xi)^{\mu-1}\mathrm{f}(\xi,\overline{\psi}(\xi),{}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}+}\overline{\psi}(\xi))d\xi.\end{aligned}$$

In the sequel, we denote an admissible set of solutions for FIE (3.3) governed by a pair of upper
and lower solutions $(\underline{\psi},\overline{\psi})$ as follows

$$\Pi_{(\underline{\psi},\overline{\psi})} = \left\{\quad \psi\in\mathrm{X}:\ \underline{\psi}(\varrho)\leq\psi(\varrho)\leq\overline{\psi}(\varrho),\quad \varrho\in[a,b]\right\},$$

where ψ is a solution of (3.3).

Theorem 3.4. *Let* $\mathrm{f}\in C([a,b],\mathbb{R}\times\mathbb{R},\mathbb{R})$ *with* $\mathrm{f}(\mathrm{a},\psi(\mathrm{a}),{}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}+}\psi(\mathrm{a}))=0$*. Suppose that* $(\underline{\psi},\overline{\psi})\in\mathrm{X}\times\mathrm{X}$ *is a pair of upper and lower solutions of FIE (3.3) with* $\underline{\psi}(\varrho)\leq\overline{\psi}(\varrho)$ *for* $\varrho\in[a,b]$*. If* $(\psi,\omega)\mapsto\mathrm{f}(\varrho,\psi,\omega)$ *is nondecreasing, that is*

$$\mathrm{f}(\varrho,\psi_1,\omega_1)\leq\mathrm{f}(\varrho,\psi_2,\omega_2),\ \textit{for}\ \psi_1\leq\psi_2,\ \textit{and}\ \omega_1\leq\omega_2,$$

then there exist maximal and minimal solutions $\psi_L,\psi_U\in\Pi_{(\underline{\psi},\overline{\psi})}$ *such that for each* $\psi\in\Pi_{(\underline{\psi},\overline{\psi})}$,

$$\psi_L(\varrho)\leq\psi(\varrho)\leq\psi_U(\varrho),\quad \varrho\in[a,b].$$

Proof. initially, we structure two sequences $\{\omega_n\}$ and $\{\varpi_n\}$ as follows

$$\begin{cases}\omega_0=\underline{\psi},\\ \quad\omega_{n+1}(\varrho)=\psi_a+\frac{1-\mu}{\mathfrak{N}(\mu)}\mathrm{f}(\varrho,\omega_n(\varrho),{}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}+}\omega_n(\varrho))\\ \quad+\frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)}\int_a^{\varrho}(\varrho-\xi)^{\mu-1}\mathrm{f}(\xi,\omega_n(\xi),{}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}+}\omega_n(\xi))d\xi,\ \varrho\in[a,b],n=0,1,...\end{cases}$$

and

$$\begin{cases}\varpi_0=\overline{\psi},\\ \quad\varpi_{n+1}(\varrho)=\psi_a+\frac{1-\mu}{\mathfrak{N}(\mu)}\mathrm{f}(\varrho,\varpi_n(\varrho),{}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}+}\varpi_n(\varrho))\\ \quad+\frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)}\int_a^{\varrho}(\varrho-\xi)^{\mu-1}\mathrm{f}(\xi,\varpi_n(\xi),{}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}+}\varpi_n(\xi))d\xi,\ \varrho\in[a,b],n=0,1,...\end{cases}$$

Next, we divide the proof into several parts.

Part 1: Sequences $\{\omega_n\}$ and $\{\varpi_n\}$ satisfy

$$\begin{aligned}\underline{\psi}(\varrho) &= \omega_0(\varrho)\leq\omega_1(\varrho)\leq...\leq\omega_n(\varrho)\leq... \\ &\quad ...\leq\varpi_n(\varrho)\leq...\leq\varpi_1(\varrho)\leq\varpi_0(\varrho)=\overline{\psi}(\varrho),\end{aligned}\tag{3.4}$$

for $\varrho \in [a,b]$. Now, we will prove that sequence $\{\omega_n\}$ is nondecreasing and

$$\omega_n(\varrho) \leq \varpi_0(\varrho), \text{ for } \varrho \in [a,b], \ n = 0, 1, ...$$

As per the assumptions, we have $\underline{\psi}(\varrho) = \omega_0(\varrho) \leq \overline{\psi}(\varrho) = \varpi_0(\varrho)$, for $\varrho \in [a,b]$ and

$$\begin{aligned} \omega_1(\varrho) = \psi_a + \tfrac{1-\mu}{\mathfrak{N}(\mu)} \mathrm{f}(\varrho, \omega_0(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_0(\varrho)) \\ + \tfrac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)} \int_a^{\varrho} (\varrho - \xi)^{\mu-1} \mathrm{f}(\xi, \omega_0(\xi), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_0(\xi)) d\xi \\ \geq \omega_0(\varrho), \ \varrho \in [a,b], \end{aligned}$$

As $(\psi, \omega) \mapsto \mathrm{f}(\varrho, \psi, \omega)$ is nondecreasing, then it is clear that

$$\mathrm{f}(\varrho, \omega_0(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_0(\varrho)) \leq \mathrm{f}(\varrho, \varpi_0(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\varpi_0(\varrho)).$$

Hence

$$\begin{aligned} \omega_1(\varrho) \leq \psi_a + \tfrac{1-\mu}{\mathfrak{N}(\mu)} \mathrm{f}(\varrho, \varpi_0(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\varpi_0(\varrho)) \\ + \tfrac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)} \int_a^{\varrho} (\varrho - \xi)^{\mu-1} \mathrm{f}(\xi, \varpi_0(\xi), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\varpi_0(\xi)) d\xi \\ \leq \varpi_0(\varrho), \ \varrho \in [a,b], \end{aligned}$$

By induction, we suppose that

$$\omega_{n-1}(\varrho) \leq \omega_n(\varrho) \leq \varpi_0(\varrho), \ \varrho \in [a,b].$$

According to the definition of $\{\omega_n\}$, we have

$$\begin{aligned} \omega_n(\varrho) = \psi_a + \tfrac{1-\mu}{\mathfrak{N}(\mu)} \mathrm{f}(\varrho, \omega_{n-1}(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_{n-1}(\varrho)) \\ + \tfrac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)} \int_a^{\varrho} (\varrho - \xi)^{\mu-1} \mathrm{f}(\xi, \omega_{n-1}(\xi), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_{n-1}(\xi)) d\xi, \ \varrho \in [a,b], n = 0, 1, ... \end{aligned}$$

and

$$\begin{aligned} \omega_{n+1}(\varrho) = \psi_a + \tfrac{1-\mu}{\mathfrak{N}(\mu)} \mathrm{f}(\varrho, \omega_n(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\varrho)) \\ + \tfrac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)} \int_a^{\varrho} (\varrho - \xi)^{\mu-1} \mathrm{f}(\xi, \omega_n(\xi), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\xi)) d\xi, \ \varrho \in [a,b], n = 0, 1, ... \end{aligned}$$

From the monotonicity of f, we get

$$\omega_n(\varrho) \leq \omega_{n+1}(\varrho) \leq \varpi_0(\varrho), \ \varrho \in [a,b].$$

Now, by induction, we assume that $\omega_n(\varrho) \leq \varpi_n(\varrho)$, for $\varrho \in [a,b]$.

Similarly, we facilely conclude from the monotonicity of f that

$$\omega_{n+1}(\varrho) \leq \varpi_{n+1}(\varrho), \text{for } \varrho \in [a,b].$$

Moreover, the sequence $\{\varpi_n\}$ is nonincreasing.

Part 2: Sequences $\{\omega_n\}$ and $\{\varpi_n\}$ are relatively compact in X.

From the continuity of f and since $\underline{\varphi}, \overline{\psi} \in \mathrm{X}$ along with Part 1, we conclude that $\omega_n, \varpi_n \in \mathrm{X}$.

It follows from (3.5) that $\{\omega_n\}$ and $\{\varpi_n\}$ are uniformly bounded. Now, for any $\varrho_1, \varrho_2 \in [a,b]$ with $\varrho_1 \leq \varrho_2$, we have

$$\begin{aligned}
& |\omega_{n+1}(\varrho_2) - \omega_{n+1}(\varrho_1)| \\
= & \frac{1-\mu}{\mathfrak{N}(\mu)} \left| \mathrm{f}(\varrho_2, \omega_n(\varrho_2), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\varrho_2)) - \mathrm{f}(\varrho_1, \omega_n(\varrho_1), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\varrho_1)) \right| \\
& + \frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)} \left| \int_a^{\varrho_2} (\varrho_2 - \xi)^{\mu-1} \mathrm{f}(\xi, \omega_n(\xi), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\xi)) d\xi \right. \\
& \left. - \int_a^{\varrho_1} (\varrho_1 - \xi)^{\mu-1} \mathrm{f}(\xi, \omega_n(\xi), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\xi)) d\xi \right| \\
\leq & \frac{1-\mu}{\mathfrak{N}(\mu)} \left| \mathrm{f}(\varrho_2, \omega_n(\varrho_2), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\varrho_2)) - \mathrm{f}(\varrho_1, \omega_n(\varrho_1), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\varrho_1)) \right| \\
& + \frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)} \left| \int_a^{\varrho_1} \left[(\varrho_2 - \xi)^{\mu-1} - (\varrho_1 - \xi)^{\mu-1} \right] \mathrm{f}(\xi, \omega_n(\xi), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\xi)) d\xi \right| \\
& + \frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)} \left| \int_{\varrho_1}^{\varrho_2} (\varrho_2 - \xi)^{\mu-1} \mathrm{f}(\xi, \omega_n(\xi), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\xi)) d\xi \right| \\
\leq & \frac{1-\mu}{\mathfrak{N}(\mu)} \left| \mathrm{f}(\varrho_2, \omega_n(\varrho_2), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\varrho_2)) - \mathrm{f}(\varrho_1, \omega_n(\varrho_1), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega_n(\varrho_1)) \right| \\
& + \frac{\mathrm{M}}{\mathfrak{N}(\mu)\Gamma(\mu)} \left[(\varrho_2 - a)^{\mu} - (\varrho_1 - a)^{\mu} + 2(\varrho_2 - \varrho_1)^{\mu} \right] \\
\to & \ 0, \quad \text{as } |\varrho_1 - \varrho_2| \to 0,
\end{aligned}$$

where $\mathrm{M} > 0$. This means that $\{\omega_n\}$ is equicontinuous in X. From Theorem 2.5, we infer that $\{\omega_n\}$ is relatively compact in X. Analogically, we get $\{\varpi_n\}$ is also relatively compact in X.

Part 3: There exist minimal and maximal solutions in $\Pi_{(\underline{\psi},\overline{\psi})}$.

Part 1 and Part 2 indicate that $\{\omega_n\}$ and $\{\varpi_n\}$ are monotone and relatively compact in X. Clearly, there exist continuous functions ω and ϖ with

$$\omega_n(\varrho) \leq \omega(\varrho) \leq \varpi(\varrho) \leq \varpi_n(\varrho), \ \forall \varrho \in [a,b], \ n \in \mathbb{N}$$

such that $\{\omega_n\}$ and $\{\varpi_n\}$ converge uniformly to ω and ϖ in X, respectively. Therefore, ω and ϖ are two solutions of (3.3), i.e.,

$$\begin{aligned}
\omega(\varrho) &= \psi_a + \frac{1-\mu}{\mathfrak{N}(\mu)} \mathrm{f}(\varrho, \omega(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega(\varrho)) \\
&\quad + \frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)} \int_a^{\varrho} (\varrho - \xi)^{\mu-1} \mathrm{f}(\xi, \omega(\xi), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\omega(\xi)) d\xi, \\
\varpi(\varrho) &= \psi_a + \frac{1-\mu}{\mathfrak{N}(\mu)} \mathrm{f}(\varrho, \varpi(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\varpi(\varrho)) \\
&\quad + \frac{\mu}{\mathfrak{N}(\mu)\Gamma(\mu)} \int_a^{\varrho} (\varrho - \xi)^{\mu-1} \mathrm{f}(\xi, \varpi(\xi), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{\mathrm{a}^+}\varpi(\xi)) d\xi,
\end{aligned}$$

for all $\varrho \in [a,b]$. Even so, the fact (3.5) guarantees that

$$\underline{\psi}(\varrho) \leq \omega(\varrho) \leq \varpi(\varrho) \leq \overline{\psi}(\varrho), \ \text{for } \varrho \in [a,b].$$

Eventually, we will show that ω and ϖ are the minimal and maximal solutions in $\Pi_{(\underline{\psi},\overline{\psi})}$, respectively. For any $\psi \in \Pi_{(\underline{\psi},\overline{\psi})}$, we have

$$\underline{\psi}(\varrho) \leq \psi(\varrho) \leq \overline{\psi}(\varrho), \text{ for } \varrho \in [a,b].$$

Known that f is nondecreasing, we induct

$$\underline{\psi}(\varrho) \leq \omega_n(\varrho) \leq \psi(\varrho) \leq \varpi_n(\varrho) \leq \overline{\psi}(\varrho), \text{ for all } \varrho \in [a,b] \text{ and } n \in \mathbb{N}.$$

As $n \to \infty$, we obtain

$$\underline{\psi}(\varrho) \leq \omega(\varrho) \leq \psi(\varrho) \leq \varpi(\varrho) \leq \overline{\psi}(\varrho), \text{ for all } \varrho \in [a,b] \text{ and } n \in \mathbb{N}.$$

This means that

$$\psi_L(\varrho) = \omega(\varrho), \text{ and } \psi_U(\varrho) = \varpi(\varrho)$$

which are the minimal and maximal solutions in $\Pi_{(\underline{\psi},\overline{\psi})}$, respectively. This completes the proof. □

Corollary 3.5. *Suppose that assumptions of Theorem 3.4 are satisfied. Then ABC-type FIDF (1.1) has at least one solution in* X.

Proof. By the hypotheses and Theorem 3.4, we notice that $\Pi_{(\underline{\psi},\overline{\psi})} \neq \emptyset$, i.e., the solution family of the FIE (3.3) is nonempty in X. This combines along with Lemma 3.2 to confirm that ABC-type FIDF (1.1) has at least one solution in X. □

4. An Example

In this portion, we provide an example to enlighten our results.

Example 4.1. Consider the following $\mathcal{ABC}$-type FDF

$$\begin{aligned} {}^{\mathcal{ABC}}\mathcal{D}^{\frac{1}{2}}_{0^+}\, \psi(\varrho) &= \tfrac{8}{3\sqrt{\pi}}\varrho^{\frac{3}{2}} - \varrho^2 + {}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{2}}_{0^+}\, \psi(\varrho), \qquad 0 \leq \varrho \leq 1, \\ \psi(0) &= 0, \end{aligned} \tag{4.1}$$

where $\mu = \frac{1}{2}$, and $\mathrm{f}(\varrho, \psi, {}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{2}}_{0^+}\, \psi) = \frac{8}{3\sqrt{\pi}}\varrho^{\frac{3}{2}} - \varrho^2 + {}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{2}}_{0^+}\, \psi$. It is clear that $\mathrm{f}(0, \psi(0), {}^{\mathcal{AB}}\mathcal{I}^{\mu}_{0^+}\psi(0)) = 0$. Then, the corresponding FIE is obtained by

$$\begin{aligned} \psi(\varrho) \;\; = \;\; & \frac{4}{3\sqrt{\pi}}\varrho^{\frac{3}{2}} - \varrho^2 + {}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{2}}_{0^+}\, \psi(\varrho) \\ & + {}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{2}}_{0^+}\left[\frac{8}{3\sqrt{\pi}}\varrho^{\frac{3}{2}} - \varrho^2 + {}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{2}}_{0^+}\, \psi(\varrho)\right],\ \varrho \in [0,1]. \end{aligned} \tag{4.2}$$

In fact, we can find that $\underline{\varphi}(\varrho) = 0$ and $\overline{\psi}(\varrho) = \frac{8}{3\sqrt{\pi}}\varrho^{\frac{3}{2}} + \frac{1}{2}\varrho^2$ which are respectively the lower and upper solutions of (4.3). Moreover, $\mathrm{f}(\cdot, \psi, \omega)$ is nondecreasing. So, all assumptions of Theorem 3.4 are satisfied. so, we construct the sequences $\{\omega_n\}$ and $\{\varpi_n\}$ by

$$\begin{cases} \omega_0(\varrho) = \underline{\psi}, \\ \;\omega_{n+1}(\varrho) = \frac{1}{2}\mathrm{f}(\varrho, \omega_n(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{2}}_{0^+}\omega_n(\varrho)) \\ \qquad + {}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{2}}_{0^+}\left[\mathrm{f}(\varrho, \omega_n(\varrho), {}^{\mathcal{AB}}\mathcal{I}^{\frac{1}{2}}_{0^+}\omega_n(\varrho))\right],\ \text{for } \varrho \in [0,1], \text{ and } n = 0, 1, ... \end{cases}$$

and

$$\begin{cases} \varpi_0 = \overline{\psi}, \\ \varpi_{n+1}(\varrho) = \frac{1}{2}\mathrm{f}(\varrho, \varpi_n(\varrho), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\frac{1}{2}}\varpi_n(\varrho)) \\ \quad + {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\frac{1}{2}}\left[\mathrm{f}(\varrho, \varpi_n(\varrho), {}^{\mathcal{AB}}\mathcal{I}_{0^+}^{\frac{1}{2}}\varpi_n(\varrho))\right], \text{ for } \varrho \in [0,1], \text{ and } n = 0, 1, ... \end{cases}$$

Applying Theorem 3.4 again, we have $\omega_n \to \omega \in \mathrm{X}$ and $\varpi_n \to \varpi \in \mathrm{X}$ as $n \to \infty$. Therefore, $\psi_L(\varrho) = \omega(\varrho)$, and $\psi_U(\varrho) = \varpi(\varrho)$ which are the minimal and maximal solutions in $\Pi_{(\underline{\psi},\overline{\psi})}$, where

$$\Pi_{(\underline{\psi},\overline{\psi})} = \left\{ \quad \psi \in \mathrm{X} : \ 0 \leq \psi(\varrho) \leq \frac{8}{3\sqrt{\pi}}\varrho^{\frac{3}{2}} + \frac{1}{2}\varrho^2, \quad \varrho \in [0,1] \right\}.$$

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