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ON DERIVATION OF OSKOLKOV'S EQUATIONS FOR NON-COMPRESSIBLE VISCOUS KELVIN-VOIGHT FLUID BY STOCHASTIC ANALYSIS ON THE GROUPS OF DIFFEOMORPHISMS

YURI GLIKLIKH AND MARINA ZALYGAEVA

Abstract. We continue the research started in [5] and obtain by the methods of stochastic analysis on the groups of diffeomorphisms, a derivation of Oskolkov's equations non-compressible viscous Kelvin-Voight fluid.

Introduction

In this paper we use the Lagrangian approach to hydrodynamics, suggested by V.I. Arnold [1] and then by D. Ebin and J. Marsden [2]. Their ideas and results obtained development in [3], [4] where it was shown that the description of the motion of viscous fluid includes stochastic perturbations of the flow of perfect fluid. Such description (see [3]) with the use of some additional constructions and the machinery of the stochastic analysis allowed us to obtain Burgers, Reynolds and Navier-Stokes's classical equations where the Byurgers equation arises upon transition from the Lagrangian description to Euler in tangent space at the unit of the group of diffeomorphisms, while Reynolds's equation and Navier-Stokes equation – in the tangent space at the unit of group of diffeomorphisms, preserving volume, respectively. In [5] these results are generalized to the case of fluids having viscous term described as some second order differential operator, maybe, depending on time parameter t, i.e., instead of the use of Laplace operator in Euler description we have obtained constructions with the operator who's matrix has the form $\tilde{\mathfrak{B}} = \frac{1}{2}(\tilde{\mathfrak{B}}^{ij}(t))\frac{\partial}{\partial x^i \partial x^j}$ that allows us to deal with non-Newtonian fluids. Here we continue the research started in [5] and obtain by the methods of stochastic analysis on the groups of diffeomorphisms derivation of Oskolkov's equations non-compressible viscous Kelvin-Voight fluid. we involve the machinery of the so-called Nelson's mean derivativwes (see [6] – [8] and [4]).

Everywhere in the book we use Einsteins summation convention with respect a shared upper and lower index. The symbol $\frac{\partial}{\partial x^i}$ denotes both the i-th vector of the basis in $\mathbb{R}^n$ and the corresponding partial derivative.

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1. Mean Derivatives

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, where $t \in [0,T]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and such that $\xi(t)$ is an L^1-random variable for all t. The "present" ("now") for $\xi(t)$ is the least complete σ-subalgebra $\mathcal{N}_t^\xi$ of $\mathcal{F}$ that includes preimages of the Borel set of $\mathbb{R}^n$ under the map $\xi(t) : \Omega \to \mathbb{R}^n$. We denote by E_t^ξ the conditional expectation with respect to $\mathcal{N}_t^\xi$. The least complete σ-subalgebra that includes preimages of the Borel set of $\mathbb{R}^n$ under all maps $\xi(s) : \Omega \to \mathbb{R}^n$ for $s \le t$ (resp. $s > t$) is called the "*past*" (resp. "*future*") *σ-algebra* and is denoted by $\mathcal{P}_t^\xi$ (resp. $\mathcal{F}_t^\xi$).

Below we most often deal with the diffusion processes of the form

$$\xi(t) = \xi_0 + \int_0^t a(s, \xi(s))ds + \int_0^t \mathbf{B}(s)dw(s) \tag{1.1}$$

in $\mathbb{R}^n$ and in the flat torus $\mathcal{T}^n$. In (1.1) $w(t)$ is a Wiener process, $a(t,x)$ is a vector field and $\mathbf{B}(t)$ is a linear operator in $\mathbb{R}^n$ depending on time t.

Following Nelson [6]-[8] and [4], we give the definition of forward and backward mean derivatives of a stochatic process $\xi(t)$.

Definition 1.1. (i) The *forward mean derivative* $D\xi(t)$ of the process $\xi(t)$ at t is the L^1-random variable of the form

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} \right)$$

where the limit is supposed to exist in $L^1(\Omega, \mathcal{F}, \mathrm{P})$ and $\Delta t \to +0$ means that $\Delta t \to 0$ and $\Delta t > 0$.

(ii) The *backward mean derivative* $D_*\xi(t)$ of $\xi(t)$ at t is the L^1-random variable

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t} \right)$$

where (as well as in (i)) the limit is supposed to exist in $L^1(\Omega, \mathcal{F}, \mathrm{P})$ and $\Delta t \to +0$ means the same as in (i).

Notice that, generally speaking, $D\xi(t) \neq D_*\xi(t)$ (but, if $\xi(t)$ almost surely (a.s.) has smooth sample trajectories, these derivatives evidently coincide).

From the properties of conditional expectation it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as compositions of $\xi(t)$ and Borel measurable vector fields

$$Y^0(t,x) = \lim_{\Delta t \to 0+} E_t^\xi \left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} | \xi(t) = x \right)$$

$$Y_*^0(t,x) = \lim_{\Delta t \to 0+} E_t^\xi \left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t} | \xi(t) = x \right)$$

on $\mathbb{R}^n$ (following [8] we call them the *regressions*): $D\xi(t) = Y^0(t, \xi(t))$ and $D_*\xi(t) = Y_*^0(t, \xi(t))$. Note that for a process of type (1.1) $D\xi(t) = a(t, \xi(t))$ and so $Y^0(t,x) = a(t,x)$.

Let $Z(t,x)$ be C^2-smooth vector field on $\mathbb{R}^n$. Define the mean derivatives of Z along $\xi(t)$ as follows:

Definition 1.2.

$$DZ(t,\xi(t)) = \lim_{\Delta t\to 0+} E_t^{\xi}\left(\frac{Z(t+\Delta t,\xi(t+\Delta t)) - Z(t,\xi(t))}{\Delta t}\right)$$

$$D_*Z(t,\xi(t)) = \lim_{\Delta t\to 0+} E_t^{\xi}\left(\frac{Z(t,\xi(t)) - Z(t-\Delta t,\xi(t-\Delta t))}{\Delta t}\right)$$

$DZ(t,\xi(t))$ and $D_*Z(t,\xi(t))$ have regressions, which we denote by the symbols DZ and D_*Z, respectively (see details in [8]).

Lemma 1.3. *For process (1.1) in $\mathbb{R}^n$, the following formulae take place:*

$$DZ = \frac{\partial}{\partial t}Z + (Y^0\cdot\nabla)Z + \tilde{\mathfrak{B}}(t)Z \tag{1.2}$$

$$D_*Z = \frac{\partial}{\partial t}Z + (Y^0_*\cdot\nabla)Z - \tilde{\mathfrak{B}}(t)Z \tag{1.3}$$

where $\nabla = (\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^n})$ and $\tilde{\mathfrak{B}}(t) = \frac{1}{2}\tilde{\mathfrak{B}^{ij}}(t)\frac{\partial}{\partial x^i\partial x^j}$ is the second order differential operator with the matrix $(\tilde{\mathfrak{B}}^{ij}(t)) = \mathbf{B}(t)\mathbf{B}^*(t)$

2. Groups of diffeomorphisms

Here we describe some properties of the groups of diffeomorphisms on flat n-dimensionsl torus that are necessary below (see details in [2]).

Let $\mathcal{T}^n$ be a flat n-dimensional torus and $\mathcal{D}^s(\mathcal{T}^n)$ be its group of sobolev H^s-diffeomorphisms ($s > n/2+1$). Recall that for $s > n/2+1$ the mappings from H^s are C^1-smooth.

$\mathcal{D}^s(\mathcal{T}^n)$ is a group with respect to the composition with unit $e = id$. The tangent space $T_e\mathcal{D}^s(\mathcal{T}^n)$ is the space of all H^s-vector fields on $\mathcal{T}^n$. We denote by β the subspace in $T_e\mathcal{D}^s(\mathcal{T}^n)$ consisting of the divergence-free H^s-vector fields on $\mathcal{T}^n$.

The tangent bundle $T\mathcal{D}^s(\mathcal{T}^n)$ is the set of H^s-mappings from $\mathcal{T}^n$ to $T\mathcal{T}^n$ such that under projection on $\mathcal{T}^n$ yield the mappings from $\mathcal{D}^s(\mathcal{T}^n)$.

In an arbitrary $T_f\mathcal{D}^s(\mathcal{T}^n)$ we can define the L^2-inner product by the formula

$$(X,Y) = \int_{\mathcal{T}^n}\langle X(m),Y(m)\rangle_{f(m)}\mu(dm) \tag{2.1}$$

The family of these inner products forms the so-called weak Riemannian metric on $\mathcal{D}^s(\mathcal{T}^n)$. In particular, in $T_e\mathcal{D}^s(\mathcal{T}^n)$ (3.1) takes the form

$$(X,Y)_e = \int_{\mathcal{T}^n}\langle X(m),Y(m)\rangle_m\mu(dm). \tag{2.2}$$

The right translation $R_f : \mathcal{D}^s(T^n) \to \mathcal{D}^s(T^n)$, where $R_f(\Theta) = \Theta\circ f$ for $\Theta, f \in \mathcal{D}^s(T^n)$, is a C^∞-smooth mapping. The tangent map of the right translation has the form $TR_f(X) = X\circ f$ for $X \in T\mathcal{D}^s(T^n)$.

On the other hand, the left translation $L_f : \mathcal{D}^s(T^n) \to \mathcal{D}^s(T^n)$, where $L_f(\Theta) = f\circ\Theta$ for $\Theta, f \in \mathcal{D}^s(T^n)$, is only continuous. Specify a vector $x \in \mathbb{R}^n$ and denote by $l_x : T^n \to T^n$ the mapping $l_x(m) = m + x$ modulo factorization with respect to the integral lattice. Note that the left translation Ll_x is C^∞-smooth.

Recall that $T\mathcal{T}^n = \mathcal{T}^n \times \mathbb{R}^n$. Introduce the operators

$$B : T\mathcal{T}^n \to \mathbb{R}^n$$

the projection onto the second factor in $\mathcal{T}^n \times \mathbb{R}^n$, and

$$A(m) : \mathbb{R}^n \to \mathcal{T}^n_m,$$

inverse to B linear isomorphism of $\mathbb{R}^n$ onto the tangent space to $\mathcal{T}^n$ at $m \in \mathcal{T}^n$.

Introduce

$$Q_{g(m)} = A(g(m)) \circ B,$$

where $g \in \mathcal{D}^s(\mathcal{T}^n)$, $m \in \mathcal{T}^n$. For every $Y \in T_f\mathcal{D}^s(\mathcal{T}^n)$ we get that $Q_gY = A(g(m)) \circ B(Y(m)) \in T_g\mathcal{D}^s(\mathcal{T}^n)$ for all $f \in \mathcal{D}^s(\mathcal{T}^n)$.

Lemma 2.1. *The following relations hold*

$$TR_{g^{-1}}(Q_gX) = Q_e(TR_{g^{-1}}X),$$
$$TR_g(Q_{g^{-1}}X) = Q_e(TR_gX).$$

Lemma 2.2. Q_g *is the parallel translation in* $\mathcal{D}^s(\mathcal{T}^n)$ *with respect to the Levi-Civita connection of metric (3.1)*

The proofs if Lemmas 2.1 and 2.2 can be found, e.g., in [4].

Thus, for a smooth vector field $Y(t)$ along a smooth curve $g(t)$ in $\mathcal{D}^s(\mathcal{T}^n)$, the covariant derivative at time instant t^* is defined as

$$\frac{\bar{D}}{dt}Y(t)|_{t=t^*} = \frac{d}{dt}(Q_{g(t^*)}Y(t))|_{t=t^*}.$$

Recall that (see [4]), that the geodesic is a smooth curve $g(t)$ in $\mathcal{D}^s(\mathcal{T}^n)$ such that

$$\frac{\bar{D}}{dt}\dot{g}(t) = 0. \tag{2.3}$$

For such a curve $g(t)$ let us construct the vector $v(t) \in T_e\mathcal{D}^s(\mathcal{T}^n)$ by the formula $v(t) = \dot{g}(t) \circ g^{-1}(t)$

Lemma 2.3. *If* $g(t)$ *is a geodesic, the curve* $R_fg(t)$ *is also geodesic.*

Lemma 2.4. *Let* $g(t)$ *be a geodesic and* $x \in \mathbb{R}^n$ *be a certain vector. Then* $l_xg(t)$ *is a geodesic.*

Consider the operator $\bar{A} : \mathcal{D}^s(\mathcal{T}^n) \times \mathbb{R}^n \to T\mathcal{D}^s(\mathcal{T}^n)$ such that $\bar{A}_e$ coincides with A introduced above, and for every $g \in \mathcal{D}^s(\mathcal{T}^n)$ the mapping $\bar{A}_g : \mathbb{R}^n \to T_g\mathcal{D}^s(\mathcal{T}^n)$ is obtained from $\bar{A}_e$ by right translation, i.e,, for $X \in \mathbb{R}^n$:

$$\bar{A}_g(X) = TR_g \circ A_e(X) = (A \circ g)(X).$$

Every right-invariant vector field $\bar{A}(X)$ is C^∞-smooth on $\mathcal{D}^s(\mathcal{T}^n)$ for every $X \in \mathbb{R}^n$.

For an arbitrary point $m \in T^n$ we denote by $exp_m : T_mT^n \to T^n$ the mapping that sends the vector $X \in T_mT^n$ to the point $m + X$ modulo factorization with respect to integral lattice on T^n. The family of these mappings generates the mapping $\overline{exp} : T_e\mathcal{D}^s(T^n) \to \mathcal{D}^s(T^n)$ that sends the vector $X \in T_e\mathcal{D}^s(T^n)$ to $e+X \in \mathcal{D}^s(T^n)$, where $e+X$ is the diffeomorphism of T^n of the form: $(e+X)(m) = m + X(m)$.

Consider the composition $\overline{exp} \circ \bar{A}_e : \mathbb{R}^n \to \mathcal{D}^s(\mathcal{T}^n)$. By construction, for an arbitrary $X \in \mathbb{R}^n$ we obtain that $\overline{exp} \circ \bar{A}_e(X)(m) = m + X$, i.e., the same vector X is added to every point m.

Let $w(t)$ be a Wiener processin $\mathbb{R}^n$, given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ Construct the stochastic process

$$W(t) = \overline{exp} \circ \bar{A}_e(\int_0^t \mathbf{B}(s)dw(s)) \tag{2.4}$$

in $\mathcal{D}^s(\mathcal{T}^n)$ where $\mathbf{B}(t)$ is the linear operator from formula (1.1). By construction, for $\omega \in \Omega$ the corresponding sample path $W_\omega(t)$ is the diffeomorphism of the form $W_\omega(t)(m) = m + \int_0^t \mathbf{B}(s)dw(s)_\omega$. Note that for a specified $\omega \in \Omega$ and specified $t \in \mathbb{R}$ we get that $\int_0^t \mathbf{B}(s)dw(s)_\omega$ is a constant vector in $\mathbb{R}^n$. This means that for given ω and t the action of $W_\omega(t)$ coincides with $l_{\int_0^t \mathbf{B}(s)dw(s)_\omega}$.

Let $\mathcal{D}^s_\mu(\mathcal{T}^n)$ be the group of sobolev volume-preserving H^s-diffeomiorphisms on $\mathcal{T}^n$ $(s > n/2 + 1)$, a subgroup and a Hilbert submanifold of $\mathcal{D}^s(\mathcal{T}^n)$ (see [2, 4]). As well as for $\mathcal{D}^s(\mathcal{T}^n)$, one can here introduce the right translation and the left translation. The former is C^∞-smooth and the latter is continuous.

The tangent space to $\mathcal{D}^s_\mu(\mathcal{T}^n)$ at the unit $e = id$ is denoted by $T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$. It is the space of all divergence-free H^s-vector fields on $\mathcal{T}^n$. The tangent space at $\eta \in \mathcal{D}^s_\mu(\mathcal{T}^n)$ consists of the compisitions $X \circ \eta$ where $X \in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$. Note that the tangent map of right translation has the same form as the right translation itself: $TR_gX = X \circ g$, $X \in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$.

The right-invariant vector field $\bar{X}$ on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ is generated by the unique vector $X \in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ by the formula $\bar{X}_g = TR_gX = X \circ g$. Note that $\bar{X}$ is C^k-smooth if and only if X as a vector filed on $\mathcal{T}^n$ belongs to the sobolev class H^{s+k}. In particular, $\bar{X}$ is C^∞-smooth if and only if X is C^∞-smooth.

Note that the field of operators A can be considered as a mapping $A : \mathbb{R}^n \to T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$.

On $\mathcal{D}^s_\mu(\mathcal{T}^n)$ we use the weak Riemannian metric that is the restriction of (3.1) to tangent spaces of $\mathcal{D}^s_\mu(\mathcal{T}^n)$. Consider the orthogonal projection with respect to inner product (3.8)

$$P : H^s \to T_e\mathcal{D}^s_\mu(\mathcal{T}^n).$$

From Hodge decomposition (see e.g. [2, 4]) it follows that this projection does exist and that the kernel of P is the space of all gradients. Thus, for an arbitrary $Y \in H^s$ the presentation

$$P(Y) = Y - \text{grad}p, \tag{2.5}$$

holds where p is a certain H^{s+1}-function on $\mathcal{T}^n$ (unique up to additive constants).

The covariant derivative on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ is introduced by the formula $\frac{\tilde{D}}{dt} = P\frac{\bar{D}}{dt}$.

Consider equation

$$\frac{\tilde{D}}{dt}\dot{g}(t) = \bar{F}(t, g(t), \dot{g}(t)), \tag{2.6}$$

on $\mathcal{D}^s_\mu(\mathcal{T}^n)$. If F is smooth enough, for every initial conditions $g(0) = e$ and $\dot{g}(0) = u_0 \in T_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ equation (2.6) has a solution well-posed on a certain time interval $t \in [0, T]$. This solution is a flow of ideal incompressible fluid on $\mathcal{T}^n$ under external force F. If $F = 0$, it is a geodesic of Levi-Civita connection of metric (3.8) and it describes the flow in the absence of external forces. Below, if the contrary is not said, we deal with $g(t)$ in the case $F = 0$.

3. Viscous hydrodynamics

Let $s > n/2+1$, so we will deal with diffeomorphisms from $\mathcal{D}^s(\mathcal{T}^n)$, that are $\mathcal{C}^\infty$-smooth, and $Vect^{(s)}$, consists of all $\mathcal{C}^2$-smooth vector fields. Everywhere below we will use the same Wiener process $W(t)$, constructed from specified Wiener process $w(t)$ in $\mathbb{R}^n$ by the formula (1.3).

Let $g(t)$ be a geodesic map on $\mathcal{D}^s(\mathcal{T}^n)$ with initial conditions $g(0) = e$ and $\dot{g}(0) = v_0 \in T_e\mathcal{D}^s(\mathcal{T}^n)$. Such map exists on the interval $t \in [0, T]$. Consider $v(t) = \dot{g}(t) \circ g^{-1}(t) \in T_e\mathcal{D}^s(\mathcal{T}^n)$. This infinite-dimensional vector can be represented like the vector field on $\mathcal{T}^n$, that we denote by $v(t, m)$. It's well known that $v(t)$ satisfies the Hopf equation

$$\frac{\partial}{\partial t}v + (v, \nabla)v = 0.$$

Introduce on $\mathcal{T}^n$ the vector field $V(t, m) = E(v(t, m - \mathbf{B}(t)w(t)))$ (in infinite-dimensional notation $V(t) = E(Q_eTR^{-1}_{W(t)}v(t))$, where E is ordinary mathematical expectation).

Theorem 3.1. ([5]) *Vector field $V(t, m)$ satisfies the analogue of Burgers equation*

$$\frac{\partial}{\partial t}V(T - t, m) + (V(T - t, m) \cdot \nabla)V(T - t, m) - \tilde{\mathfrak{B}}(t)V(T - t, m) = 0,$$

where $\tilde{\mathfrak{B}}(t)$ - differential operator of second order $\tilde{\mathfrak{B}}(t) = \frac{1}{2}(\tilde{\mathfrak{B}}^{ij})(t)\frac{\partial}{\partial x^i \partial x^j}$, wuth matrix $\tilde{\mathfrak{B}}^{ij}(t) = \mathbf{B}(t)\mathbf{B}^*(t)$.

Now we consider the case of viscous incompressible fluid. As well as above, we suggest that the fluid moves on flat n-dinemsional torus $\mathcal{T}^n$. Let $\mathcal{D}^s_\mu(\mathcal{T}^n)$ be the set of Sobolev H^s-diffeomorphisms of $\mathcal{T}^n$, preserving volume($s > n/2 + 1$). Let the map $g(t)$ on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ be the solution of following equation

$$\frac{D}{dt}\dot{g}(t) = F(t, g(t), \dot{g}(t)).$$

where F is external force. Consider $u(t) = \dot{g}(t) \circ g^{-1}(t) \in \mathcal{T}_e\mathcal{D}^s_\mu(\mathcal{T}^n)$ and the Wiener process $W(t)$, given by (1.3). We notice that $u(t)$ satisfies Euler equation

$$\frac{\partial}{\partial t}u + (u, \nabla)u - gradp = TR_g^{-1}F(t, g(t), \dot{g}(t)).$$

By the analogy with above constructions introduce on $\mathcal{T}^n$ the vector field

$$U(t, m) = E(u(t, m - \int_0^t B(s)dw(s))).$$

In infinite-dimensional notation $U(t) = E(Q_eTR^{-1}_{W(t)}u(t))$.

Theorem 3.2. ([5]) *The vector field $U(t,m)$ satisfies the analogue of Reynolds equation*

$$\frac{\partial}{\partial t}U + E\left[((U\cdot\nabla)U)(t, m - \int\limits_0^t B(s)dw(s))\right] - \tilde{\mathfrak{B}}(t)U - gradp = 0, \tag{3.1}$$

where $\tilde{\mathfrak{B}}(t)$ - differential operator of second order $\tilde{\mathfrak{B}}(t) = \frac{1}{2}(\tilde{\mathfrak{B}}^{ij})(t)\frac{\partial}{\partial x^i \partial x^j}$, with matrix $\tilde{\mathfrak{B}}^{ij}(t) = \mathbf{B}(t)\mathbf{B}^*(t)$

We notice that

$$E\left[((U\cdot\nabla)U)(t, m - \int\limits_0^t B(s)dw(s))\right] = (U\cdot\nabla)U + E\left[(\breve{U}_{u(t)}\cdot\nabla)\breve{U}_{u(t)}\right]$$

It can help us to transform (3.1) into analogue of standart form of Reynolds equation

$$\frac{\partial}{\partial t}U + (U\cdot\nabla)U - \tilde{\mathfrak{B}}(t)U - gradp = E\left[(\breve{U}_{u(t)}\cdot\nabla)\breve{U}_{u(t)}\right]. \tag{3.2}$$

With a slight modification we can annihilate the external force in (3.2). For this we introduce special random force by the formula

$$\mathfrak{F}_\omega(t, X_\omega) = Q_e TR_{W(t)} PE\left[(\breve{U}_{X_\omega}\cdot\nabla)\breve{U}_{X_\omega}\right],$$

where $X_\omega(m) \in \mathcal{T}_e\mathcal{D}^{s+1}_\mu(\mathcal{T}^n)$ – random divergence-free H^{s+1} vector field. Then introduce right-invariant force field $\bar{\mathfrak{F}}_\omega(t, Y_\omega) = TR_g\mathfrak{F}_\omega(t, TR_g^{-1}Y_\omega)$, where $TR_g^{-1}Y_\omega$ – divergence-free H^{s+1} vector field.

Consider on $\mathcal{D}^s_\mu(\mathcal{T}^n)$ the equation

$$\frac{\bar{D}}{dt}\dot{g}_\omega(t) = \bar{\mathfrak{F}}_\omega(t, g_\omega(t), \dot{g}_\omega(t)).$$

Suppose that for initial conditions $g_\omega(0) = e$ and $\dot{g}_\omega(0) = u_o \in \mathcal{T}_e\mathcal{D}^{s+1}_\mu(\mathcal{T}^n)$ there is exists the unique equation g_ω in H^{s+1}. Consider divergence-free H^{s+1} vector field $u_\omega(t,m)$ on $\mathcal{T}^n$, given by the ratio $u_\omega(t,m) = \dot{g}_\omega(t,m)\circ g_\omega^{-1}(t,m)$. The analogue of U, introduced in (13), is

$$\mathbb{U} = E\left(u_\omega(t, m - \int\limits_0^t B(s)dw(s)_\omega)\right). \tag{3.3}$$

Theorem 3.3. ([5]) *The vector field $\mathbb{U}$ (3.3) satisfies the analogue of Navier-Stokes equation*

$$\frac{\partial}{\partial t}\mathbb{U} + (\mathbb{U}\cdot\nabla)\mathbb{U} - \tilde{\mathfrak{B}}(t)\mathbb{U} - gradp = 0, \tag{3.4}$$

where $\tilde{\mathfrak{B}}(t)$ - differential operator of second order $\tilde{\mathfrak{B}}(t) = \frac{1}{2}(\tilde{\mathfrak{B}}^{ij})(t)\frac{\partial}{\partial x^i \partial x^j}$, with matrix $\tilde{\mathfrak{B}}^{ij}(t) = \mathbf{B}(t)\mathbf{B}^*(t)$.

Now let's turn to the main result of this paper.

Consider the equation

$$\frac{\bar{D}}{dt}\dot{g}_\omega(t) = \bar{\mathfrak{F}}_\omega(t, g_\omega(t), \dot{g}_\omega(t)). \tag{3.5}$$

on $\mathcal{D}^s_\mu(\mathcal{T}^n)$, where $\bar{\mathfrak{F}}_\omega(t, g_\omega(t), \dot{g}_\omega(t))$ is the right-invariant force vector field. For $Y_\omega \in T_g D^s_\mu(T^n)$, $g \in D^{s+1}_\mu(T^n)$ and $\omega \in \Omega$ it is determined by the formula

$$\bar{\mathfrak{F}}_\omega(t, g_\omega(t), Y_\omega) = TR_g \mathfrak{F}(t, TR_g^{-1} Y_\omega).$$

Using (3.3) we construct another random vector field

$$\mathfrak{F}_\omega(t, g_\omega(t), \dot{g}_\omega(t)) = Q_e TR_{W(t)} PE\left[(\breve{U}_{X_\omega} \cdot \nabla)\breve{U}_{X_\omega}\right] +$$

$$+ Q_e TR_{W(t)} \frac{d}{dt} \Delta E Q_e TR^{-1}_{W(t)} u_\omega(t). \tag{3.6}$$

Notice, that in (21) $Q_e TR_{W(t)} \frac{d}{dt} \Delta E Q_e TR^{-1}_{W(t)} u_\omega(t) = Q_e TR_{W(t)} \frac{d}{dt} \Delta \mathbb{U}(t)$.

The analog of above-mentioned vector $\mathbb{U}$ stays the same as in (**??**).

$$\mathbb{U} = E\left(u_\omega(t, m - \int_0^t B(s) dw(s)_\omega)\right), \tag{3.7}$$

where $u_\omega(t, g_\omega(t)) = \dot{g}_\omega(t)$ and $g_\omega(t)$ is a solution of (3.5). Note that there is a difference between (3.5) and (3.4) in in the use of different formulas of external forces (see (3.3) and (3.6)).

Apply the Ito formula for $u_\omega(t, g_\omega(t))$

$$d(u_\omega(t, m - \int_0^t B(s) dw(s)_\omega)) =$$

$$\frac{\partial}{\partial t} u_\omega(t, m - \int_0^t B(s) dw(s)_\omega) dt + \tilde{\mathfrak{B}}(t) u_\omega(t, m - \int_0^t B(s) dw(s)_\omega) dt.$$

Also it's known that $u_\omega(t, m - \int_0^t B(s) dw(s)_\omega)$ satisfies Euler equation

$$\frac{\partial}{\partial t} u_\omega(t, m - \int_0^t B(s) dw(s)_\omega)(t)) =$$

$$= -P[(u_\omega(t, m - \int_0^t B(s) dw(s)_\omega) \cdot \nabla) u_\omega(t, m - \int_0^t B(s) dw(s)_\omega)] + \mathfrak{F}_\omega(t, u_\omega(t)).$$

So

$$\frac{\partial}{\partial t}\mathbb{U}(t, m) = E(\frac{d}{dt} u_\omega(t, m - \int_0^t B(s) dw(s)_\omega)) =$$

$$= -E[(u_\omega \cdot \nabla) u_\omega] + \mathfrak{F}_\omega(t, u_\omega(t)) +$$

$$+ gradp + E Q_e TR^{-1}_{W(t)} \mathfrak{F}_\omega(t, u_\omega(t)). \tag{3.8}$$

By construction of (3.6) we get

$$\begin{aligned} & EQ_eTR^{-1}_{W(t)}\mathfrak{F}_\omega(t,u_\omega(t)) \\ = & EQ_eTR^{-1}_{W(t)}\mathfrak{F}_\omega(t,u_\omega(t))\Big[Q_eTR_{W(t)}PE\Big[(\breve{U}_{u_\omega}\cdot\nabla)\breve{U}_{u_\omega}\Big] \\ & +Q_eTR_{W(t)}\frac{d}{dt}\Delta EQ_eTR^{-1}_{W(t)}u_\omega(t)\Big] = \\ & = PE\left[(\breve{U}_{u_\omega}\cdot\nabla)\breve{U}_{u_\omega}\right]+\frac{d}{dt}\Delta E(u_\omega(t)) = \\ & = PE\left[(\breve{U}_{u_\omega}\cdot\nabla)\breve{U}_{u_\omega}\right]+\frac{d}{dt}\Delta\mathbb{U}(t,m). \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8) we obtain

Theorem 3.4. *The vector field* $\mathbb{U}$*, given by (2.4), satisfies the system of Oskolkov's equations*

$$\frac{\partial}{\partial t}\mathbb{U}-E\left[((\mathbb{U}\cdot\nabla)\mathbb{U})(t,m-\int_0^t B(s)dw(s))\right]-$$
$$-\frac{\partial}{\partial t}\Delta\mathbb{U}-\frac{\sigma^2}{2}\Delta\mathbb{U}-gradp=0,$$

modeling the dynamics of non-compressible viscous Kelvin-Voight fluid of the order 0.

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Yuri E. Gliklikh & Marina E. Zalygaeva: Voronezh State University, Universitetskaya pl. 1, Voronezh 394018, Russia, 115432, Russia

E-mail address: yeg@math.vsu.ru zalygaeva@math.vsu.ru