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A MORE GENERAL CLASS OF LINEAR OCTAGONAL FUZZY NUMBERS AND ITS APPLICATION TO FUZZY MULTI ATTRIBUTE DECISION MAKING PROBLEM

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Abstract. Over the years, Octagonal Fuzzy Numbers have shown optimal solutions in real applications over triangular and trapezoidal fuzzy numbers. In this paper, variations in the k - levels of Octagonal Fuzzy Numbers are introduced which yield a More General Class of Linear Octagonal Fuzzy Numbers (MGLOFNs) - MGLOFN of the $\mathcal{LH}$ type or MGLOFN of the $\mathcal{HL}$ type. The α - cut, measure and arithmetic operations of both the classes are studied in detail and its application in a Fuzzy Multi Attribute Decision Making Problem (FMADM) is explored with a secondary data. The problem of choosing the best bicycle helmet is discussed by modeling it as a FMADM involving MGLOFNs.

1. Introduction

In order to cope with inaccurate quantitative data in a practical way, Lotfi A. Zadeh [9],[10],[11] introduced the idea of fuzzy sets in 1965 and fuzzy numbers in 1975 and established its applications in various fields (to cite a few [1],[6]). R. E. Bellman and L. A. Zadeh [2] pioneered decision making in a fuzzy environment in 1970, paving the path for the creation of several multi-attribute decision making approaches. Fuzzy Multiple Attribute Decision Making Problem (FMADM) with different methods and applications was studied in 1992 by Chen et al. [4]. There has been a wide usage of various fuzzy numbers such as triangular fuzzy numbers, trapezoidal fuzzy numbers in almost every field, to solve problems in hand [8],[3]. Over the decade, Octagonal Fuzzy Numbers(OFNs) have shown optimum results than triangular, trapezoidal fuzzy numbers in several real life applications[1],[5].

The uniform spread in octagonal and trapezoidal fuzzy numbers caters to only specific varieties of problems. Therefore, we need a better quantifier than OFNs, since the need for mathematical modeling is more demanding. This study analyzes the non-uniform spreads of OFNs, which give rise to a new class of OFNs known as "A More General Class of Linear Octagonal Fuzzy Numbers" that perform optimally in a number of practical circumstances.

This paper is arranged as follows: Section 2 introduces a More General class of Linear Octagonal Fuzzy Number (MGLOFN) and its types namely, MGLOFN of $\mathcal{LH}$ type and MGLOFN of $\mathcal{HL}$ type. In Section 3, the problem of selecting a best

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helmet based on various factors is discussed. In Section 4, we give the computation and outcome of the problem. Section 5 gives the conclusion.

2. A More General Linear Octagonal Fuzzy Number

In this section, we formally define a more general linear octagonal fuzzy number (MGLOFN) and study its properties.

Definition 2.1. *A fuzzy number $\tilde{A}$ is said to be a more general linear octagonal fuzzy number (MGLOFN) denoted $\widetilde{A^\dagger}$ whose membership function $\mu_{\widetilde{A^\dagger}}$ is given by*

$$\mu_{\widetilde{A^\dagger}}(x) = \begin{cases} k^{(1)}(\frac{x-a_1}{a_2-a_1}) & a_1 \le x \le a_2 \\ k^{(1)} & a_2 \le x \le a_3 \\ k^{(1)} + (1-k^{(1)})(\frac{x-a_3}{a_4-a_3}) & a_3 \le x \le a_4 \\ 1 & a_4 \le x \le a_5 \\ k^{(2)} + (1-k^{(2)})(\frac{a_6-x}{a_6-a_5}) & a_5 \le x \le a_6 \\ k^{(2)} & a_6 \le x \le a_7 \\ k^{(2)}(\frac{a_8-x}{a_8-a_7}) & a_7 \le x \le a_8 \\ 0 & otherwise \end{cases}$$

where $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, k^{(1)}, k^{(2)}$ are real numbers such that $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le a_7 \le a_8$, $k^{(1)}, k^{(2)} \in [0,1]$

Remark 2.2. *The collection of all MGLOFNs is denoted by $\mathcal{F}^\dagger(\mathbb{R})$. It consists of two types.*
MGLOFN of $\mathcal{LH}$ type:
If $0 < k^{(1)} < k^{(2)} < 1$, then the MGLOFN is said to be of $\mathcal{LH}$ type (lower $k^{(1)}$ higher $k^{(2)}$) and is represented as $\tilde{A^\dagger_{\mathcal{LH}}} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k^{(1)}, k^{(2)})$. We denote such collection to be $\mathcal{F}^\dagger_{\mathcal{LH}}(\mathbb{R})$. The diagrammatic representation is shown in Figure 1
MGLOFN of $\mathcal{HL}$ type:
If $0 < k^{(2)} < k^{(1)} < 1$ $(1 > k^{(1)} > k^{(2)} > 0)$, then the MGLOFN is said to be of $\mathcal{HL}$ type (higher $k^{(1)}$ lower $k^{(2)}$) and is represented as $\tilde{A^\dagger_{\mathcal{HL}}} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k^{(1)}, k^{(2)})$. We denote such collection to be $\mathcal{F}^\dagger_{\mathcal{HL}}(\mathbb{R})$. The diagrammatic representation is shown in Figure 2

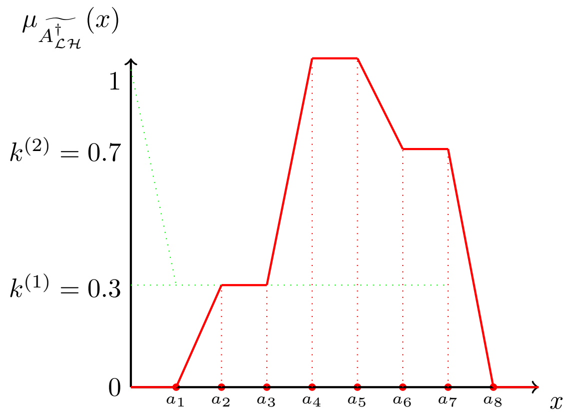

FIGURE 1. MGLOFN of the $\mathcal{LH}$ type

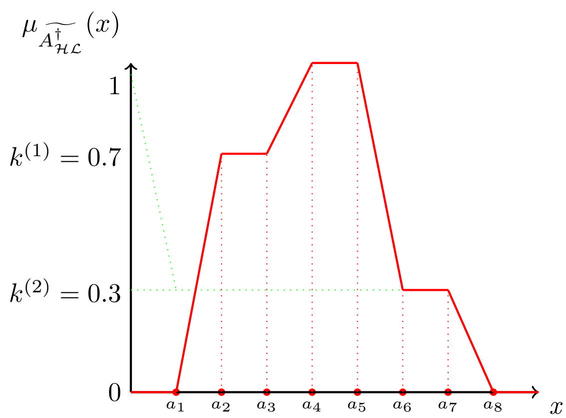

FIGURE 2. MGLOFN of the $\mathcal{HL}$ type

Remark 2.3. *1.If $k^{(1)} = k^{(2)} = k$, then the MGLOFN reduces to an octagonal fuzzy number $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k)$ defined in [3]*
2. If $k^{(1)} = 0$ and $k^{(2)} = 0$, the MGLOFN reduces to the trapezoidal fuzzy number (a_3, a_4, a_5, a_6)
3. If $k^{(1)} = 1$ and $k^{(2)} = 1$, the MGLOFN reduces to the trapezoidal fuzzy number (a_1, a_2, a_7, a_8)
4. If $k^{(1)} = 0$ and $k^{(2)} = 1$, the MGLOFN reduces to the trapezoidal fuzzy number (a_3, a_4, a_7, a_8)
5. If $k^{(1)} = 1$ and $k^{(2)} = 0$, the MGLOFN reduces to the trapezoidal fuzzy number (a_1, a_2, a_5, a_6)

The necessity of the more general class of octagonal fuzzy numbers is explained by a real life situation involving subjective assessment of temperature in a fuzzy setup.

Example:

To express 'felt temperature' in a clinical laboratory with few individuals, for medical testing purposes, they are questioned as to how they felt about their body temperature in this setting. Some people may react with extremes of cold, warmth or heat. MGLOFNs can be used to express these perceptions in a more realistic way.

Linguistic terms	**MGLOFNs**
Very Cold (VC)	(33.5,34.5,35.5,36.5,37.5,38.5,39.5,40.5;0.85,0.15)
Cold (C)	(33.5,34.5,35.5,36.5,37.5,38.5,39.5,40.5;0.65,0.35)
Normal (N)	(33.5,34.5,35.5,36.5,37.5,38.5,39.5,40.5;0.5,0.5)
Warm (W)	(33.5,34.5,35.5,36.5,37.5,38.5,39.5,40.5;0.35,0.65)
Very Warm (VW)	(33.5,34.5,35.5,36.5,37.5,38.5,39.5,40.5;0.15,0.85)

TABLE 1. Representation of Subjective Temperature

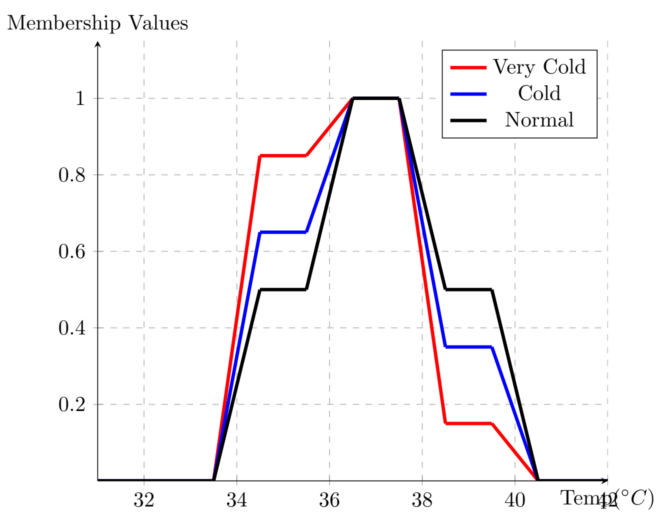

FIGURE 3. MGLOFNs of $\mathcal{HL}$ type

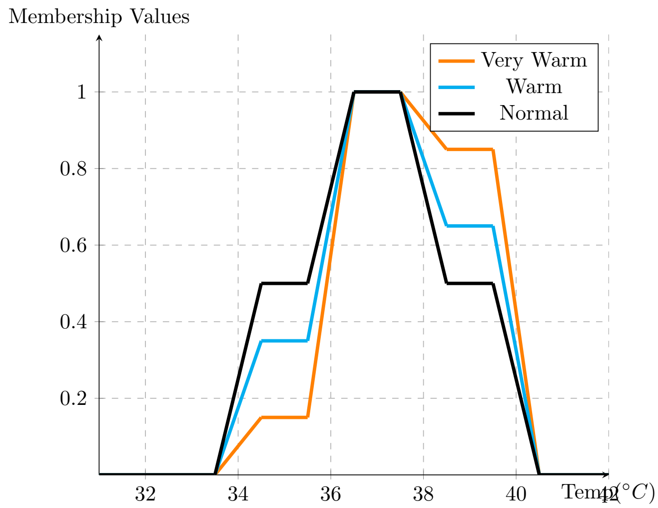

FIGURE 4. MGLOFNs of $\mathcal{LH}$ type

In the Table 1, representation of linguistic terms of subjective temperature are expressed in MGLOGNs for different values of $k^{(1)}$ and $k^{(2)}$. In Figure 3, *'very cold', 'cold'* are represented by MGLOFNs of $\mathcal{HL}$ type and in Figure 4, *'very warm', 'warm'* are represented by MGLOFNs of $\mathcal{LH}$ type. Note that the normal temperature is represented by Linear Octagonal Fuzzy Number as $k^{(1)} = k^{(2)}$.

The structural properties built on the above said classes is unique. Henceforth, there is a need to study these classes seperately and the same is developed further.

2.1. MGLOFN of the $\mathcal{LH}$ type. In this section, we introduce the α - cut of MGLOFN of the $\mathcal{LH}$ type and a measure on $\mathcal{F}^{\dagger}_{\mathcal{LH}}(\mathbb{R})$. The arithmetic operations of addition and scalar multiplication are defined using both α - cut approach and coordinate-wise approach and are compared.

Definition 2.4. *The α - cut of a MGLOFN of the $\mathcal{LH}$ type is defined as follows: If $\widetilde{A^{\dagger}_{\mathcal{LH}}} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k^{(1)}, k^{(2)})$, where $k^{(1)} < k^{(2)}$ then*

$$[\widetilde{A^{\dagger}_{\mathcal{LH}}}]_\alpha = [(\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha, (\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha] \quad = \begin{cases} [((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_1, ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_1] \; for \; \alpha \in (0, k^{(1)}] \\ [((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_2, ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_2] \; for \; \alpha \in (k^{(1)}, k^{(2)}] \\ [((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_3, ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_3] \; for \; \alpha \in (k^{(2)}, 1] \end{cases}$$

where

$$((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_1 = \frac{a_1 k^{(1)} + \alpha a_2 - \alpha a_1}{k^{(1)}} \qquad ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_1 = \frac{a_8 k^{(2)} - \alpha a_8 + \alpha a_7}{k^{(2)}}$$

$$((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_2 = \frac{-a_3 - \alpha a_4 + \alpha a_3 + k^{(1)} a_4}{k^{(1)} - 1} \qquad ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_2 = \frac{a_8 k^{(2)} - \alpha a_8 + \alpha a_7}{k^{(2)}}$$

$$((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_3 = \frac{-a_3 - \alpha a_4 + \alpha a_3 + k^{(1)} a_4}{k^{(1)} - 1} \qquad ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_3 = \frac{-a_6 - \alpha a_5 + \alpha a_6 + k^{(2)} a_5}{k^{(2)} - 1}$$

Example 2.5. *If* $\widetilde{A^{\dagger}_{\mathcal{LH}}} = (2,3,4,6,8,9,11,12;0.3,0.8)$ *then the* α *- cut of* $[\widetilde{A^{\dagger}_{\mathcal{LH}}}]_{\alpha}$*, for* $k^{(1)} = 0.3, k^{(2)} = 0.8$ *is given by*

$$[\widetilde{A^{\dagger}_{\mathcal{LH}}}]_{\alpha} = \begin{cases} [\frac{10\alpha+6}{3}, \frac{48-5\alpha}{4}] \; for \; \alpha \in (0, 0.3] \\ [\frac{20\alpha+22}{7}, \frac{48-5\alpha}{4}] \; for \; \alpha \in (0.3, 0.8] \\ [\frac{20\alpha+22}{7}, 13-5\alpha] \; for \; \alpha \in (0.8, 1] \end{cases}$$

When $\alpha = 0.1$, $[\widetilde{A^{\dagger}_{\mathcal{LH}}}]_{\alpha} = [\frac{7}{3}, \frac{95}{8}]$*; When* $\alpha = 0.5$, $[\widetilde{A^{\dagger}_{\mathcal{LH}}}]_{\alpha} = [\frac{32}{7}, \frac{91}{8}]$*;*
When $\alpha = 0.9$, $[\widetilde{A^{\dagger}_{\mathcal{LH}}}]_{\alpha} = [\frac{40}{7}, \frac{17}{2}]$.

Definition 2.6. *A measure on the MGLOFN of the* $\mathcal{LH}$ *type* $\widetilde{A^{\dagger}_{\mathcal{LH}}}$ *is defined as follows:*

$$\mathcal{M}(\widetilde{A^{\dagger}_{\mathcal{LH}}}) = \frac{1}{2}\int_0^{k^{(1)}} \left(((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_{\alpha})_1 + ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_{\alpha})_1 \right) d\alpha +$$
$$\frac{1}{2}\int_{k^{(1)}}^{k^{(2)}} \left(((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_{\alpha})_2 + ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_{\alpha})_2 \right) d\alpha +$$
$$\frac{1}{2}\int_{k^{(2)}}^{1} \left(((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_{\alpha})_3 + ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_{\alpha})_3 \right) d\alpha, \quad 0 < k^{(1)} < k^{(2)} < 1$$

Computing, we get

$$\mathcal{M}(\widetilde{A^{\dagger}_{\mathcal{LH}}}) = \frac{1}{4}\{(a_1+a_2-a_3-a_4)k^{(1)} + (a_3+a_4+a_5+a_6) - (a_5+a_6-a_7-a_8)k^{(2)}\}$$

Remark 2.7. *Any two MGLOFNs of* $\mathcal{LH}$ *type* $\widetilde{A^{\dagger}_{\mathcal{LH}}}$ *and* $\widetilde{B^{\dagger}_{\mathcal{LH}}}$ *could be compared using the following:*
1. $\widetilde{A^{\dagger}_{\mathcal{LH}}} \prec \widetilde{B^{\dagger}_{\mathcal{LH}}} \quad if \quad \mathcal{M}(\widetilde{A^{\dagger}_{\mathcal{LH}}}) < \mathcal{M}(\widetilde{B^{\dagger}_{\mathcal{LH}}})$
2. $\widetilde{A^{\dagger}_{\mathcal{LH}}} \approx \widetilde{B^{\dagger}_{\mathcal{LH}}} \quad if \quad \mathcal{M}(\widetilde{A^{\dagger}_{\mathcal{LH}}}) = \mathcal{M}(\widetilde{B^{\dagger}_{\mathcal{LH}}})$
3. $\widetilde{A^{\dagger}_{\mathcal{LH}}} \succ \widetilde{B^{\dagger}_{\mathcal{LH}}} \quad if \quad \mathcal{M}(\widetilde{A^{\dagger}_{\mathcal{LH}}}) > \mathcal{M}(\widetilde{B^{\dagger}_{\mathcal{LH}}})$

Arithmetic Operations:
We introduce the arithmetic operations on MGLOFNs of the $\mathcal{LH}$ type having same $k^{(1)}$ and $k^{(2)}$ values. Let $\widetilde{A^{\dagger}_{\mathcal{LH}}} \approx (a_1, a_2, ..., a_8; k^{(1)}, k^{(2)})$, $\widetilde{B^{\dagger}_{\mathcal{LH}}} \approx (b_1, b_2, ..., b_8; k^{(1)}, k^{(2)})$ be two MGLOFNs of the $\mathcal{LH}$ type. The arithmetic operations are defined using both α - cut approach and coordinate-wise approach.
Using extension principle, the α - cut approach for addition and scalar multiplication are given below:
Addition:
For $\widetilde{A^{\dagger}_{\mathcal{LH}}}, \widetilde{B^{\dagger}_{\mathcal{LH}}} \in \mathcal{F}^{\dagger}_{\mathcal{LH}}(\mathbb{R})$, the addition is calculated by adding their corresponding

α - cuts using interval arithmetic, we have

$$[\widetilde{A^{\dagger}_{\mathcal{LH}}}]_{\alpha} \oplus [\widetilde{B^{\dagger}_{\mathcal{LH}}}]_{\alpha} = \begin{cases} [a_1+b_1+\frac{\alpha}{k^{(1)}}(a_2-a_1+b_2-b_1), a_8+b_8-\frac{\alpha}{k^{(2)}}(a_8-a_7+b_8-b_7)] \\ \qquad for\ \alpha \in (0, k^{(1)}] \\ [a_3+b_3+\frac{\alpha-k^{(1)}}{1-k^{(1)}}(a_4-a_3+b_4-b_3), a_8+b_8-\frac{\alpha}{k^{(2)}}(a_8-a_7+b_8-b_7)] \\ \qquad for\ \alpha \in (k^{(1)}, k^{(2)}] \\ [a_3+b_3+\frac{\alpha-k^{(1)}}{1-k^{(1)}}(a_4-a_3+b_4-b_3), a_6+b_6-\frac{\alpha-k^{(2)}}{1-k^{(2)}}(a_6-a_5+b_6-b_5)] \\ \qquad for\ \alpha \in (k^{(2)}, 1] \end{cases}$$

and the membership function is given by

$$\mu_{\left(\widetilde{A^{\dagger}_{\mathcal{LH}}}\oplus\widetilde{B^{\dagger}_{\mathcal{LH}}}\right)}(x) = \begin{cases} k^{(1)}\Big(\frac{x-(a_1+b_1)}{(a_2+b_2)-(a_1+b_1)}\Big) & a_1+b_1 \le x \le a_2+b_2 \\ k^{(1)} & a_2+b_2 \le x \le a_3+b_3 \\ k^{(1)}+(1-k^{(1)})\Big(\frac{x-(a_3+b_3)}{(a_4+b_4)-(a_3+b_3)}\Big) & a_3+b_3 \le x \le a_4+b_4 \\ 1 & a_4+b_4 \le x \le a_5+b_5 \\ k^{(2)}+(1-k^{(2)})\Big(\frac{(a_6+b_6)-x}{(a_6+b_6)-(a_5+b_5)}\Big) & a_5+b_5 \le x \le a_6+b_6 \\ k^{(2)} & a_6+b_6 \le x \le a_7+b_7 \\ k^{(2)}\Big(\frac{(a_8+b_8)-x}{(a_8+b_8)-(a_7+b_7)}\Big) & a_7+b_7 \le x \le a_8+b_8 \\ 0 & \text{otherwise} \end{cases}$$

Scalar Multiplication:
Let $\widetilde{A^{\dagger}_{\mathcal{LH}}} = (a_1, a_2, ..., a_8; k^{(1)}, k^{(2)})$ be a MGLOFN of the $\mathcal{LH}$ type. The scalar multiplication of $\widetilde{A^{\dagger}_{\mathcal{LH}}}$ by a scalar, $\lambda \ge 0$ is given by

$$\lambda[\widetilde{A^{\dagger}_{\mathcal{LH}}}]_{\alpha} = \begin{cases} [\lambda a_1+\frac{\alpha}{k^{(1)}}(\lambda a_2-\lambda a_1), \quad \lambda a_8-\frac{\alpha}{k^{(2)}}(\lambda a_8-\lambda a_7)], & \alpha \in (0, k^{(1)}] \\ [\lambda a_3+\frac{\alpha-k^{(1)}}{1-k^{(1)}}(\lambda a_4-\lambda a_3), \quad \lambda a_8-\frac{\alpha}{k^{(2)}}(\lambda a_8-\lambda a_7)], & \alpha \in (k^{(1)}, k^{(2)}] \\ [\lambda a_3+\frac{\alpha-k^{(1)}}{1-k^{(1)}}(\lambda a_4-\lambda a_3), \quad \lambda a_6-\frac{\alpha-k^{(2)}}{1-k^{(2)}}(\lambda a_6-\lambda a_5)], & \alpha \in (k^{(2)}, 1] \end{cases}$$

Definition 2.8. *If $\widetilde{A^{\dagger}_{\mathcal{LH}}}$ be a MGLOFN of the $\mathcal{LH}$ type, then the scalar multiplication denoted $\lambda\widetilde{A^{\dagger}_{\mathcal{LH}}}$ is defined to have the membership function $\mu_{\lambda\widetilde{A^{\dagger}_{\mathcal{LH}}}}$*

corresponding to the α - cut given by equation (2.1)

$$\mu_{\lambda\widetilde{A^{\dagger}_{\mathcal{LH}}}}(x) = \begin{cases} \frac{k^{(1)}}{\lambda}\left(\frac{x-a_1\lambda}{a_2-a_1}\right) & \lambda a_1 \le x \le \lambda a_2 \\ k^{(1)} & \lambda a_2 \le x \le \lambda a_3 \\ \frac{(1-k^{(1)})(x-\lambda a_3)+k^{(1)}\lambda(a_4-a_3)}{\lambda(a_4-a_3)} & \lambda a_3 \le x \le \lambda a_4 \\ 1 & \lambda a_4 \le x \le \lambda a_5 \\ \frac{(1-k^{(2)})(\lambda a_6-x)+k^{(2)}\lambda(a_6-a_5)}{\lambda(a_6-a_5)} & \lambda a_5 \le x \le \lambda a_6 \\ k^{(2)} & \lambda a_6 \le x \le \lambda a_7 \\ \frac{k^{(2)}}{\lambda}\left(\frac{\lambda a_8-x}{a_8-a_7}\right) & \lambda a_7 \le x \le \lambda a_8 \\ 0 & otherwise \end{cases}$$

Definition 2.9. *Let $\widetilde{A^{\dagger}_{\mathcal{LH}}} = (a_1, a_2, ..., a_8; k^{(1)}, k^{(2)})$ and $\widetilde{B^{\dagger}_{\mathcal{LH}}} = (b_1, b_2, ..., b_8; k^{(1)}, k^{(2)})$, $0 < k^{(1)} < k^{(2)} < 1$ be two MGLOFNs of $\mathcal{LH}$ type and let $\lambda \ge 0$ be any real number, then the coordinate-wise addition and scalar multiplication are defined as follows:*

$$\widetilde{A^{\dagger}_{\mathcal{LH}}}+\widetilde{B^{\dagger}_{\mathcal{LH}}} = (a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4, a_5+b_5, a_6+b_6, a_7+b_7, a_8+b_8; k^{(1)}, k^{(2)})$$

$$\lambda\widetilde{A^{\dagger}_{\mathcal{LH}}} = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \lambda a_5, \lambda a_6, \lambda a_7, \lambda a_8; k^{(1)}, k^{(2)})$$

Theorem 2.10. *The α - cut approach and coordinate-wise approach of addition and scalar multiplication of MGLOFNs of $\mathcal{LH}$ type yield the same result.*

Proof. **Addition:** Let $\widetilde{A^{\dagger}_{\mathcal{LH}}} = (a_1, a_2, ..., a_8; k^{(1)}, k^{(2)})$ and $\widetilde{B^{\dagger}_{\mathcal{LH}}} = (b_1, b_2, ..., b_8; k^{(1)}, k^{(2)})$ be the two MGLOFNs of the $\mathcal{LH}$ type with α - cuts denoted by $[\widetilde{A^{\dagger}_{\mathcal{LH}}}]_\alpha$ and $[\widetilde{B^{\dagger}_{\mathcal{LH}}}]_\alpha$. Then,

$$\begin{aligned}
[\widetilde{A^{\dagger}_{\mathcal{LH}}}]_\alpha \oplus [\widetilde{B^{\dagger}_{\mathcal{LH}}}]_\alpha &= [(\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha, (\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha] + [(\widetilde{B^{\dagger}_{\mathcal{LH}}})^L_\alpha, (\widetilde{B^{\dagger}_{\mathcal{LH}}})^M_\alpha] \\
&= \begin{cases} \left[((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_1 + ((\widetilde{B^{\dagger}_{\mathcal{LH}}})^L_\alpha)_1, ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_1 + ((\widetilde{B^{\dagger}_{\mathcal{LH}}})^M_\alpha)_1\right] & for\ \alpha \in (0, k^{(1)}] \\ \left[((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_2 + ((\widetilde{B^{\dagger}_{\mathcal{LH}}})^L_\alpha)_2, ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_2 + ((\widetilde{B^{\dagger}_{\mathcal{LH}}})^M_\alpha)_2\right] & for\ \alpha \in (k^{(1)}, k^{(2)}] \\ \left[((\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_3 + ((\widetilde{B^{\dagger}_{\mathcal{LH}}})^L_\alpha)_3, ((\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_3 + ((\widetilde{B^{\dagger}_{\mathcal{LH}}})^M_\alpha)_3\right] & for\ \alpha \in (k^{(2)}, 1] \end{cases} \\
&= \begin{cases} [a_1 + b_1 + \frac{\alpha}{k^{(1)}}(a_2 - a_1 + b_2 - b_1), a_8 + b_8 - \frac{\alpha}{k^{(2)}}(a_8 - a_7 + b_8 - b_7)] \\ \qquad for\ \alpha \in (0, k^{(1)}] \\ [a_3 + b_3 + \frac{\alpha-k^{(1)}}{1-k^{(1)}}(a_4 - a_3 + b_4 - b_3), a_8 + b_8 - \frac{\alpha}{k^{(2)}}(a_8 - a_7 + b_8 - b_7)] \\ \qquad for\ \alpha \in (k^{(1)}, k^{(2)}] \\ [a_3 + b_3 + \frac{\alpha-k^{(1)}}{1-k^{(1)}}(a_4 - a_3 + b_4 - b_3), a_6 + b_6 - \frac{\alpha-k^{(2)}}{1-k^{(2)}}(a_6 - a_5 + b_6 - b_5)] \\ \qquad for\ \alpha \in (k^{(2)}, 1] \end{cases} \\
&= [(a_1 + b_1, a_2 + b_2, ..., a_8 + b_8; k^{(1)}, k^{(2)})]_\alpha \\
&= [\widetilde{A^{\dagger}_{\mathcal{LH}}} + \widetilde{B^{\dagger}_{\mathcal{LH}}}]_\alpha
\end{aligned}$$

Scalar Multiplication: Let $\widetilde{A^{\dagger}_{\mathcal{LH}}} = (a_1, a_2, ..., a_8; k^{(1)}, k^{(2)})$ be a MGLOFN with α - cut given by Equation (2.1) and $\lambda \geq 0$, then

$$\begin{aligned}
&\lambda[\widetilde{A^{\dagger}_{\mathcal{LH}}}]_\alpha \\
&\qquad = [\lambda(\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha, \lambda(\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha] \\
&\qquad = \begin{cases} [(\lambda(\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_1, (\lambda(\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_1] \; for \; \alpha \in (0, k^{(1)}] \\ [(\lambda(\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_2, (\lambda(\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_2] \; for \; \alpha \in (k^{(1)}, k^{(2)}] \\ [(\lambda(\widetilde{A^{\dagger}_{\mathcal{LH}}})^L_\alpha)_3, (\lambda(\widetilde{A^{\dagger}_{\mathcal{LH}}})^M_\alpha)_3] \; for \; \alpha \in (k^{(2)}, 1] \end{cases}
\end{aligned}$$

$$= \begin{cases} [\lambda a_1 + \frac{\alpha}{k^{(1)}}(\lambda a_2 - \lambda a_1), \quad \lambda a_8 - \frac{\alpha}{k^{(2)}}(\lambda a_8 - \lambda a_7)], & \alpha \in (0, k^{(1)}] \\ [\lambda a_3 + \frac{\alpha - k^{(1)}}{1 - k^{(1)}}(\lambda a_4 - \lambda a_3), \quad \lambda a_8 - \frac{\alpha}{k^{(2)}}(\lambda a_8 - \lambda a_7)], & \alpha \in (k^{(1)}, k^{(2)}] \\ [\lambda a_3 + \frac{\alpha - k^{(1)}}{1 - k^{(1)}}(\lambda a_4 - \lambda a_3), \quad \lambda a_6 - \frac{\alpha - k^{(2)}}{1 - k^{(2)}}(\lambda a_6 - \lambda a_5)], & \alpha \in (k^{(2)}, 1] \end{cases}$$

$$= [(\lambda a_1, \lambda a_2, ..., \lambda a_8; k^{(1)}, k^{(2)})]_\alpha = [\lambda \widetilde{A^{\dagger}_{\mathcal{LH}}}]_\alpha$$

□

2.2. MGLOFN of $\mathcal{HL}$ type. In this section, we introduce the α - cut and a measure on $\mathcal{F}^{\dagger}_{\mathcal{HL}}(\mathbb{R})$. The arithmetic operations of addition and scalar multiplication are defined using both α - cut approach and coordinate-wise approach are compared.

Definition 2.11. *The α - cut of a MGLOFN of the $\mathcal{HL}$ type is defined as follows: If $\widetilde{A^{\dagger}_{\mathcal{HL}}} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k^{(1)}, k^{(2)})$, where $k^{(1)} > k^{(2)}$ then*

$$[\widetilde{A^{\dagger}_{\mathcal{HL}}}]_\alpha = [(\widetilde{A^{\dagger}_{\mathcal{HL}}})^L_\alpha, (\widetilde{A^{\dagger}_{\mathcal{HL}}})^M_\alpha] = \begin{cases} [((\widetilde{A^{\dagger}_{\mathcal{HL}}})^L_\alpha)_1, ((\widetilde{A^{\dagger}_{\mathcal{HL}}})^M_\alpha)_1], \; for \; \alpha \in (0, k^{(2)}] \\ [((\widetilde{A^{\dagger}_{\mathcal{HL}}})^L_\alpha)_2, ((\widetilde{A^{\dagger}_{\mathcal{HL}}})^M_\alpha)_2], \; for \; \alpha \in (k^{(2)}, k^{(1)}] \\ [((\widetilde{A^{\dagger}_{\mathcal{HL}}})^L_\alpha)_3, ((\widetilde{A^{\dagger}_{\mathcal{HL}}})^M_\alpha)_3], \; for \; \alpha \in (k^{(1)}, 1] \end{cases}$$

where,

$$((\widetilde{A^{\dagger}_{\mathcal{HL}}})^L_\alpha)_1 = \frac{a_1 k^{(1)} + \alpha a_2 - \alpha a_1}{k^{(1)}} \qquad ((\widetilde{A^{\dagger}_{\mathcal{HL}}})^M_\alpha)_1 = \frac{a_8 k^{(2)} - \alpha a_8 + \alpha a_7}{k^{(2)}}$$

$$((\widetilde{A^{\dagger}_{\mathcal{HL}}})^L_\alpha)_2 = \frac{a_1 k^{(1)} + \alpha a_2 - \alpha a_1}{k^{(1)}} \qquad ((\widetilde{A^{\dagger}_{\mathcal{HL}}})^M_\alpha)_2 = \frac{-a_6 - \alpha a_5 + \alpha a_6 + k^{(2)} a_5}{k^{(2)} - 1}$$

$$((\widetilde{A^{\dagger}_{\mathcal{HL}}})^L_\alpha)_3 = \frac{-a_3 - \alpha a_4 + \alpha a_3 + k^{(1)} a_4}{k^{(1)} - 1} \qquad ((\widetilde{A^{\dagger}_{\mathcal{HL}}})^M_\alpha)_3 = \frac{-a_6 - \alpha a_5 + \alpha a_6 + k^{(2)} a_5}{k^{(2)} - 1}$$

Example 2.12. *If $\widetilde{A^{\dagger}_{\mathcal{HL}}} = (-9, -7, -6, -4, 2, 5, 7, 9; 0.8, 0.3)$, $k^{(1)} = 0.8$ and $k^{(2)} = 0.3$ then the α - cut of $\widetilde{A^{\dagger}_{\mathcal{HL}}}$ is given by*

$$[\widetilde{A^{\dagger}_{\mathcal{HL}}}]_\alpha = \begin{cases} [\frac{8+5\alpha}{4}, 36 - 10\alpha] \; for \; \alpha \in (0, 0.3] \\ [\frac{8+5\alpha}{4}, \frac{66-10\alpha}{7}] \; for \; \alpha \in (0.3, 0.8] \\ [10\alpha - 4, \frac{66-10\alpha}{7}] \; for \; \alpha \in (0.8, 1] \end{cases}$$

When $\alpha = 0.1$, $[\widetilde{A^{\dagger}_{\mathcal{HL}}}]_{\alpha} = [\frac{17}{8}, \frac{11}{3}]$; *When* $\alpha = 0.5$, $[\widetilde{A^{\dagger}_{\mathcal{HL}}}]_{\alpha} = [\frac{21}{8}, \frac{61}{7}]$;
When $\alpha = 0.9$, $[\widetilde{A^{\dagger}_{\mathcal{HL}}}]_{\alpha} = [\frac{10}{2}, \frac{57}{7}]$.

Remark 2.13. *The measure defined in Definition 2.6 and comparison in Remark 2.7 hold good with MGLOFN of* $\mathcal{LH}$ *type* $(0 < k^{(1)} < k^{(2)} < 1)$ *is replaced by MGLOFN of* $\mathcal{HL}$ *type* $(1 > k^{(1)} > k^{(2)} > 0)$.

Remark 2.14. *The arithmetic operations like addition and scalar multiplication of two MGLOFNs of* $\mathcal{HL}$ *type using* α *- cut approach can be defined similarly along lines of the arithmetic operations of two MGLOFNs of* $\mathcal{LH}$ *type*

Definition 2.15. *Let* $\widetilde{A^{\dagger}_{\mathcal{HL}}} = (a_1, a_2, ..., a_8; k^{(1)}, k^{(2)})$ *and* $\widetilde{B^{\dagger}_{\mathcal{HL}}} = (b_1, b_2, ..., b_8; k^{(1)}, k^{(2)})$, $0 < k^{(2)} < k^{(1)} < 1$ *be two MGLOFNs of* $\mathcal{HL}$ *type and let* $\lambda \geq 0$ *be any real number, then the coordinate-wise addition and scalar multiplication are defined as follows:*

$$\widetilde{A^{\dagger}_{\mathcal{HL}}} + \widetilde{B^{\dagger}_{\mathcal{HL}}} = (a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4, a_5+b_5, a_6+b_6, a_7+b_7, a_8+b_8; k^{(1)}, k^{(2)})$$

$$\lambda\widetilde{A^{\dagger}_{\mathcal{HL}}} = (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \lambda a_5, \lambda a_6, \lambda a_7, \lambda a_8; k^{(1)}, k^{(2)})$$

Remark 2.16. *On similar lines of Theorem 2.10, the* α *- cut approach and coordinate-wise approach of addition and scalar multiplication of MGLOFNs of the* $\mathcal{HL}$ *type yield the same result.*

3. Mathematical Formulation of a Fuzzy Multi Attribute Decision Making Problem using MGLOFNs

In this section, we use MGLOFNs to give a mathematical model to the FMADM Problem. A fuzzy analogue of Simple Additive Weighting (SAW) method [8] involving MGLOFNs is formulated to solve the same.

Consider the FMADM problem having n alternatives $A_1, A_2, ..., A_n$ and m criteria $C_1, C_2, ..., C_m$. We assign weights $w_1, w_2, ..., w_m$ to each criteria. The linguistic evaluations involved in the problem are represented using MGLOFNs. The information obtained for n alternatives corresponading to m criteria is represented as a decision matrix

$$\widetilde{DM^{\dagger}_{HL}} = \begin{pmatrix} \widetilde{x^{\dagger}_{11\mathcal{HL}}} & \widetilde{x^{\dagger}_{12\mathcal{HL}}} & \cdots & \widetilde{x^{\dagger}_{1j\mathcal{HL}}} & \cdots & \widetilde{x^{\dagger}_{1m\mathcal{HL}}} \\ \widetilde{x^{\dagger}_{21\mathcal{HL}}} & \widetilde{x^{\dagger}_{22\mathcal{HL}}} & \cdots & \widetilde{x^{\dagger}_{2j\mathcal{HL}}} & \cdots & \widetilde{x^{\dagger}_{2m\mathcal{HL}}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \widetilde{x^{\dagger}_{i1\mathcal{HL}}} & \widetilde{x^{\dagger}_{i2\mathcal{HL}}} & \cdots & \widetilde{x^{\dagger}_{ij\mathcal{HL}}} & \cdots & \widetilde{x^{\dagger}_{im\mathcal{HL}}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \widetilde{x^{\dagger}_{n1\mathcal{HL}}} & \widetilde{x^{\dagger}_{n2\mathcal{HL}}} & \cdots & \widetilde{x^{\dagger}_{nj\mathcal{HL}}} & \cdots & \widetilde{x^{\dagger}_{nm\mathcal{HL}}} \end{pmatrix}$$

where $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$

Let $W = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_j \\ \vdots \\ w_m \end{pmatrix}$ be the weights assigned to each criteria. The problem is to choose the best alternative. We propose the following procedure to identify the best alternative.

Computational Procedure:

Step1: Input the decision matrix $\widetilde{DM^{\dagger}_{\mathcal{HL}}}$.

Step2: Evaluate the weighted decision matrix $W\widetilde{DM^{\dagger}_{HL}}$ by assigning weights to each criteria.

$$\begin{pmatrix} w_1\widetilde{\underset{\sim}{x^{\dagger}_{11\mathcal{HL}}}} + w_2\widetilde{\underset{\sim}{x^{\dagger}_{12\mathcal{HL}}}} + \cdots + w_j\widetilde{\underset{\sim}{x^{\dagger}_{1j\mathcal{HL}}}} + \cdots + w_m\widetilde{\underset{\sim}{x^{\dagger}_{1m\mathcal{HL}}}} \\ w_1x^{\dagger}_{21\mathcal{HL}} + w_2x^{\dagger}_{22\mathcal{HL}} + \cdots + w_jx^{\dagger}_{2j\mathcal{HL}} + \cdots + w_mx^{\dagger}_{2m\mathcal{HL}} \\ \vdots \\ w_1\widetilde{\underset{\sim}{x^{\dagger}_{i1\mathcal{HL}}}} + w_2\widetilde{\underset{\sim}{x^{\dagger}_{i2\mathcal{HL}}}} + \cdots + w_3\widetilde{\underset{\sim}{x^{\dagger}_{ij\mathcal{HL}}}} + \cdots + w_m\widetilde{\underset{\sim}{x^{\dagger}_{nm\mathcal{HL}}}} \vdots \\ w_1x^{\dagger}_{n1\mathcal{HL}} + w_2x^{\dagger}_{n2\mathcal{HL}} + \cdots + w_jx^{\dagger}_{nj\mathcal{HL}} + \cdots + w_mx^{\dagger}_{nm\mathcal{HL}} \end{pmatrix} = \begin{pmatrix} \widetilde{\underset{\sim}{A^{\dagger}_{1\mathcal{HL}}}} \\ A^{\dagger}_{2\mathcal{HL}} \\ \vdots \\ \widetilde{A^{\dagger}_{i\mathcal{HL}}} \\ \vdots \\ \widetilde{A^{\dagger}_{n\mathcal{HL}}} \end{pmatrix}$$

Step3: Compute $M(\widetilde{A^{\dagger}_{i\mathcal{HL}}})$ for each i.

Step4: Choose A_i for which $M(\widetilde{A^{\dagger}_{i\mathcal{HL}}})$ is maximum

Remark 3.1. *Step 2 is computed using addition and scalar multiplication in Definition (2.15)*

Remark 3.2. *Step 3 is calculated using the measure in Remark (2.14)*

4. The problem of choosing the best bicycle helmet for road style: A Case Study

Cycling has been suggested by several countries as an environmentally beneficial, fuel-free and healthful mode of transportation within a five-kilometer radius. Cycling has long been a popular means of transportation in India, since it has met the mobility needs of millions of Indians by offering efficient and cost-effective transit. As wearing a helmet is required for cycle racing and public transportation, it is critical and can mean the difference between life and death in the case of an accident. The problem of choosing the best bicycle helmet to ensure safety is valuable. In this Section, we regard such a problem using a secondary data from website ([12]) and formulated it as a Fuzzy Multi Attribute Decision Making (FMADM) problem using MGLOFNs and studied.

We consider the problem of a businessman (Mr.R), who wants to become a dealer of bicycle helmets. He wants to look into factors like safety, fit, comfort, cost and risk during impacts etc.

Problem:

Mr.R goes through several websites and came across the website of Virginia Tech Bicycle Helmet Ratings, which satisfies his expectations to some extent as the ratings concentrate on reduction in concussion risk from a range of impacts, score for better protection and cost. The information from this website are discussed in the following.

The information gathered from the website needed for our study: The website provides score, star ratings and cost of various helmets for road, mountain, urban and multi-sports. The helmets used for road purpose are considered for the study.

The problem is to identify a suitable mathematical model to incorporate the information and choose the best helmet. The complete scenario is modeled as FMADM problem involving MGLOFNs in the following:

Mathematical formulation of the problem :

From the website, 53 bicycle helmets of road style are considered as the alternatives say $X = A_1, A_2, ..., A_{53}$. The three features available in the website are considered as three criteria.

C_1 : Score for better protection

C_2 : Star ratings for safety

C_3 : Cost

Assessment of C_1 :

A lower score offers better protection. Based on the scores given in the website ([12]), the protection level has been identified by normalizing the data wherein score value corresponding to each alternative is divided by the lowest score and tabulated in the table below. The protection level of the helmets are evaluated using the linguistic states such as Best Protection (BP), Better Protection(BTP), Good Protection (GOP), Very Fair Protection (VFP), Fair Protection (FP), Poor Protection(PP) and Very Poor Protection(VPP).

Linguistic terms	**MGLOFNs**
Best Protection (BP)	(0.75,0.8,0.85,0.9,0.95,1,1,1;0.7,0.3)
Better Protection (BTP)	(0.65,0.7,0.75,0.8,0.85,0.9,0.95,1;0.7,0.3)
Good Protection (GOP)	(0.55,0.6,0.65,0.7,0.75,0.8,0.85,0.9;0.7,0.3)
Very Fair Protection (VFP)	(0.45,0.5,0.55,0.6,0.65,0.7,0.75,0.8;0.7,0.3)
Fair Protection (FP)	(0.35,0.4,0.45,0.5,0.55,0.6,0.65,0.7;0.7,0.3)
Poor Protection (PP)	(0.25,0.3,0.35,0.4,0.45,0.5,0.55,0.6,0.65,0.7;0.7,0.3)
Very Poor Protection (VPP)	(0.75,0.8,0.85,0.9,0.95,1,1,1;0.7,0.3)

TABLE 2. MGLOFNs of the $\mathcal{HL}$ type representing score

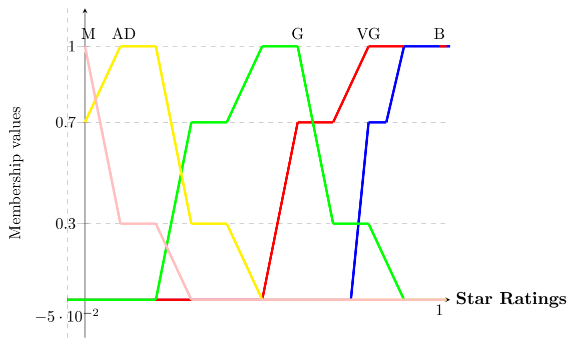

FIGURE 5. MGLOFNs representing the linguistic states

Assessment of C_2 :
Helmets with more stars provide a reduction in concussion risk from a range of impacts a cyclist might experience compared to helmets with fewer stars. The information are available as linguistic evaluations Best (B), Very Good (VG), Good (G), Adequate (AD) and Marginal (M) in the website.
Mr.R assessed each alternative linguistically and quantified using MGLOFNs as shown in the below Table 3.

Linguistic terms for Star Ratings	**Corresponding MGLOFNs**
Best (B)	(0.7,0.8,0.9,1,1,1,1,1;0.7,0.3)
Very Good (VG)	(0.5,0.6,0.7,0.8,0.9,1,1,1;0.7,0.3)
Good (G)	(0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9;0.7,0.3)
Adequate (AD)	(0,0,0,0.1,0.2,0.3,0.4,0.5;0.7,0.3)
Marginal (M)	(0,0,0,0,0,0.1,0.2,0.3;0.7,0.3)

TABLE 3. MGLOFNs of the $\mathcal{HL}$ type representing star ratings

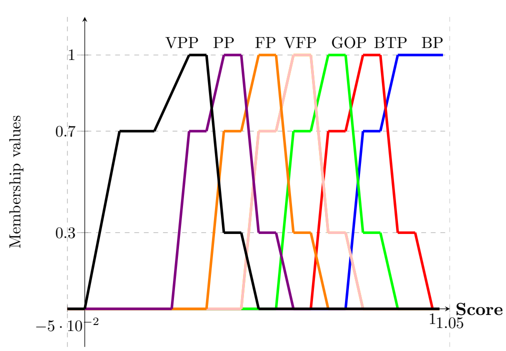

FIGURE 6. MGLOFNs representing the Star Ratings

Assessment of C_3 :
The cost given in data for different helmets is normalizied and tabulated in the Table 4.

Helmets	Score	Cost
1	1	0.7692
2	0.98	0.2083
3	0.92	0.625
4	0.91	0.2777
5	0.89	0.1831
6	0.89	0.1831
7	0.88	0.3125
8	0.85	0.2
9	0.84	0.333
10	0.83	0.25
11	0.81	0.5
12	0.81	0.5
13	0.81	0.5
14	0.81	0.2173
15	0.80	0.2
16	0.77	0.1666
17	0.76	0.5263
18	0.74	0.1960
19	0.74	0.5555
20	0.73	0.2380
21	0.73	0.1428
22	0.73	0.5
23	0.73	0.2
24	0.72	1
25	0.72	1
26	0.72	0.1538

Helmets	Score	Cost
27	0.71	0.2
28	0.70	0.333
29	0.7	0.1851
30	0.67	0.5
31	0.65	0.4761
32	0.65	0.5555
33	0.63	0.5
34	0.63	0.1818
35	0.61	0.5555
36	0.61	0.2222
37	0.60	0.3067
38	0.59	0.3846
39	0.58	0.5
40	0.58	0.8333
41	0.56	0.25
42	0.55	0.3333
43	0.55	0.5882
44	0.54	0.2777
45	0.53	0.2
46	0.52	0.2
47	0.5	0.5
48	0.49	0.1923
49	0.47	0.5263
50	0.46	0.2702
51	0.40	0.2083
52	0.37	0.5
53	0.36	0.7142

TABLE 4. Normalized value of score and cost

A MATLAB R2107b program is recorded wherein the rank in the problem of chosing the best helmet considered is acquired.

4.1. Solution to the Problem. The information corresponding to each alternative and each criteria is modeled as $\widetilde{DM^{\dagger}_{ijHL}}$ where $i = 1, 2, ..., 53$ and $j = 1, 2, 3$. Applying the procedure given in the previous section, the problem is solved using a MATHLAB program, wherein the weights used are (0.2 0.7 0.1) and MGLOFNs of $\mathcal{HL}$ type with $k^{(1)} = 0.7, k^{(2)} = 0.3$ are utilized and the outcome is tabulated below:

TABLE 5. The weighted decision matrix

i	$\widetilde{A^{\dagger}_{i\mathcal{HL}}}$	$\mathcal{M}(\widetilde{A^{\dagger}_{i\mathcal{HL}}})$	Rank
1	(0.717,0.797,0.877,0.957,0.967,0.977,0.977,0.977)	0.88925	1
2	(0.661,0.74,0.821,0.901,0.911,0.921,0.921,0.921)	0.83325	6
3	(0.703, 0.783 ,0.863, 0.943, 0.953, 0.963, 0.963, 0.963)	0.87525	2
4	(0.668, 0.748, 0.828, 0.908, 0.918, 0.928, 0.928, 0.928)	0.84025	5
5	(0.638, 0.718, 0.798, 0.878, 0.888, 0.898, 0.908, 0.918)	0.8125	14
6	(0.638, 0.718, 0.798, 0.878, 0.888, 0.898, 0.908, 0.918)	0.8125	14
7	(0.651, 0.731, 0.811, 0.891, 0.901, 0.911, 0.921, 0.931)	0.8255	9
8	(0.64, 0.72, 0.8, 0.88, 0.89, 0.9, 0.91, 0.92)	0.8145	13
9	(0.653, 0.733, 0.813, 0.893, 0.903, 0.913, 0.923, 0.933)	0.8275	8
10	(0.645, 0.725, 0.805, 0.885, 0.895, 0.905, 0.915, 0.925)	0.8195	11
11	(0.67, 0.75, 0.83, 0.91, 0.92, 0.93, 0.94, 0.95)	0.8445	4
12	(0.67, 0.75, 0.83, 0.91, 0.92, 0.93, 0.94, 0.95)	0.8445	4
13	(0.67, 0.75, 0.83, 0.91, 0.92, 0.93, 0.94, 0.95)	0.8445	4
14	(0.642, 0.722, 0.802, 0.882, 0.892, 0.902, 0.912, 0.922)	0.8165	12
15	(0.64, 0.72, 0.8, 0.88, 0.89, 0.9, 0.91, 0.92)	0.8145	13
16	(0.617, 0.697, 0.777, 0.857, 0.867, 0.877, 0.887, 0.897)	0.7915	21
17	(0.653, 0.733, 0.813, 0.893, 0.903, 0.913, 0.923, 0.933)	0.8275	8
18	(0.62, 0.7, 0.78, 0.86, 0.87, 0.88, 0.89, 0.9)	0.7945	19
19	(0.656, 0.736, 0.816, 0.896, 0.906, 0.916, 0.926, 0.936)	0.8305	7
20	(0.624, 0.704, 0.784, 0.864, 0.874, 0.884, 0.894, 0.904)	0.7985	18
21	(0.614, 0.694, 0.774, 0.854, 0.864, 0.874, 0.884, 0.894)	0.7885	23
22	(0.65, 0.73, 0.81, 0.89, 0.9, 0.91, 0.92, 0.93)	0.8245	10
23	(0.62, 0.7, 0.78, 0.86, 0.87, 0.88, 0.89, 0.9)	0.7945	19
24	(0.7, 0.78, 0.86, 0.94, 0.95, 0.96, 0.97, 0.98)	0.8745	3
25	(0.7, 0.78, 0.86, 0.94, 0.95, 0.96, 0.97, 0.98)	0.8745	3
26	(0.615, 0.695, 0.775, 0.855, 0.865, 0.875, 0.885, 0.895)	0.7895	22
27	(0.62, 0.7, 0.78, 0.86, 0.87, 0.88, 0.89, 0.9)	0.7945	19
28	(0.633, 0.713, 0.793, 0.873, 0.883, 0.893, 0.903, 0.913)	0.8075	15
29	(0.619, 0.699, 0.779, 0.859, 0.869, 0.879, 0.889, 0.899)	0.7935	20
30	(0.63, 0.71, 0.79, 0.87, 0.88, 0.89, 0.9,0.91)	0.8045	16
31	(0.628, 0.708, 0.788, 0.868, 0.878, 0.888, 0.898, 0.908)	0.8025	17
32	(0.496, 0.576, 0.656, 0.736, 0.816, 0.896, 0.906, 0.916)	0.72825	25
33	(0.49, 0.57, 0.65, 0.73, 0.81, 0.89, 0.9, 0.91)	0.72225	26
34	(0.458, 0.538, 0.618, 0.698, 0.778, 0.858, 0.868, 0.878)	0.69025	31

Table 5 (continued)

i	$\widetilde{A^{\dagger}_{i\mathcal{HL}}}$	$\mathcal{M}(\widetilde{A^{\dagger}_{i\mathcal{HL}}})$	Rank
35	(0.496, 0.576, 0.656, 0.736, 0.816, 0.896, 0.906, 0.916)	0.72825	25
36	(0.462, 0.542, 0.622, 0.702, 0.782, 0.862, 0.872, 0.882)	0.69425	30
37	(0.471, 0.551, 0.631, 0.711, 0.791, 0.871, 0.881, 0.891)	0.70325	28
38	(0.458, 0.538, 0.618, 0.698, 0.778, 0.858, 0.868, 0.878)	0.69025	31
39	(0.47, 0.55, 0.63, 0.71, 0.79, 0.87, 0.88, 0.89)	0.70225	29
40	(0.503, 0.583, 0.663, 0.743, 0.823, 0.903, 0.913, 0.923)	0.73525	24
41	(0.445, 0.525, 0.605, 0.685, 0.765, 0.845, 0.855, 0.865)	0.67725	34
42	(0.453, 0.533, 0.613, 0.693, 0.773, 0.853, 0.863, 0.873)	0.68525	32
43	(0.479, 0.559, 0.639, 0.719, 0.799, 0.879, 0.889, 0.899)	0.71125	27
44	(0.448, 0.528, 0.608, 0.688, 0.768, 0.848, 0.858, 0.868)	0.68025	33
45	(0.44, 0.52, 0.6, 0.68, 0.76, 0.84, 0.85, 0.86)	0.67225	34
46	(0.44, 0.52, 0.6, 0.68, 0.76, 0.84, 0.85, 0.86)	0.67225	34
47	(0.47, 0.55 0.63, 0.71, 0.79, 0.87, 0.88, 0.89)	0.70225	29
48	(0.419, 0.499 0.579, 0.659, 0.739, 0.819, 0.829, 0.839)	0.65125	35
49	(0.243, 0.323 0.403, 0.483, 0.563, 0.643, 0.723, 0.803)	0.491	36
50	(0.217, 0.297 0.377, 0.457, 0.537, 0.617, 0.697, 0.777)	0.465	37
51	(0.211, 0.291 0.371, 0.451, 0.531, 0.611, 0.691, 0.771)	0.459	38
52	(0.05, 0.07 0.09, 0.18, 0.26, 0.34, 0.42, 0.5)	0.21525	40
53	(0.071, 0.091 0.111, 0.201, 0.281, 0.361, 0.441, 0.521)	0.23625	39

The first helmet is the best helmet.

Remark 4.1. *Different weights can be applied to the criteria in this real-life situation depending on an individual's requirements and perspective.*

Remark 4.2. *The rank of choosing the best bicycle helmet is determined by executing the given technique using MGLOFNs of $\mathcal{LH}$ type, octagonal fuzzy numbers and trapezoidal fuzzy numbers in place of MGLOFNs of $\mathcal{HL}$. When employing trapezoidal fuzzy numbers, we note that the rank of a few helmets are close. Because of the slight variations in the helmets, a few positions have been changed. Using MGLOFNs of $\mathcal{HL}$ type and $\mathcal{LH}$ type the value of overall score for a helmet at minimum level and maximum level are obtained respectively. Thus appropriate choice of $k^{(1)}$ and $k^{(2)}$ values yeild better results.*

Remark 4.3. *The FMADM Problem involving MGLOFNs of $\mathcal{HL}$ type can be used for any decision making problem. However, the positions of the helmets vary with different weights and $k^{(1)}$ and $k^{(2)}$ values.*

5. Conclusion

In this paper, we propose a More General Class of Linear Octagonal Fuzzy Numbers which is a better quantifier than the existing fuzzy numbers in several real life applications and also study its properties. The challenge of selecting the best bicycle helmet based on various criteria is represented using MGLOGNs of the $\mathcal{HL}$ type. A new procedure for MGLOFNs is developed and solved using a MATLAB program. We see that tackling the problem using MGLOFN of the $\mathcal{HL}$ type yields more accuracy in ranking position of bicycle helmets than using octagonal and trapezoidal fuzzy numbers.

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