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# LYAPUNOV FUNCTIONALS AND EXPONENTIAL STABILITY AND INSTABILITY IN MULTI DELAY DIFFERENTIAL EQUATIONS

MEGAN CABLE AND YOUSSEF RAFFOUL

Abstract. We use Lyapunov functionals to obtain sufficient conditions that guarantee exponential decay of solutions to zero of the multi delays differential equation

$$x'(t) = a(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t - h_i).$$

The highlight of the paper is allowing $a(t)$ to change signs. Instability criteria of the zero solution is obtained. Moreover, we will provide an example, in which we show that our theorems provide an improvement of some of the recent literature.

## 1. Introduction

In this paper we consider the scalar linear differential equation with multiple delays

$$x'(t) = a(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t - h_i) \tag{1.1}$$

where $a, b$ are continuous with $0 < h_i \leq h^*$ for $i = 1, \cdot\cdot, n$, for some positive constant $h^*$. We will use Lyapunov functionals and obtain some inequalities regarding the solutions of (1.1) from which we can deduce exponential asymptotic stability of the zero solution. Also, we will provide a criteria for the instability of the zero solution of (1.1) by means of Liapunov functional. There are many results concerning equations similar to (1.1). In this paper we will compare our results to the recent paper [9] by Wang and show that the results of this research are better in some cases. Due to the choice of the Lyapunov functionals, we will deduce some inequalities on all solutions. As a consequence, the exponential decay of all solutions to zero is concluded. The main task in achieving this is to be able to relate the solutions back to $V$. That is, to find a lower bound on $V$ in terms of $x$, where $x$ is a solution of (1.1). For more on the stability of (1.1) when the delay is constant and the sign condition on $a(t)$ is required, we refer the reader to [4]. Also, for a general reading on stability delay differential equations, we refer the reader to [1], [2], [3], [5], [6], [7] and [9]. In the case $n = 1$, equation (1.1) reduces

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to the equation that was studied in [9]. Later on, we set $n = 1$ in (1.1) and show our work improve the results of [9].
Let $\psi : [-r_0, 0] \to (-\infty, \infty)$ be a given continuous initial function with

$$||\psi|| = \max_{-r_0 \le s \le 0} |\psi(s)|.$$

It should cause no confusion to denote the norm of a continuous function $\varphi : [-r, \infty) \to (-\infty, \infty)$ with

$$||\varphi|| = \sup_{-r \le s < \infty} |\varphi(s)|.$$

The notation $x_t$ means that $x_t(\tau) = x(t+\tau), \tau \in [-h^*, 0]$ as long as $x(t+\tau)$ is defined. Thus, $x_t$ is a function mapping an interval $[-h^*, 0]$ into $\mathbb{R}$. We say $x(t) \equiv x(t, t_0, \psi)$ is a solution of (1.1) if $x(t)$ satisfies (1.1) for $t \ge t_0$ and $x_{t_0} = x(t_0 + s) = \psi(s),\ s \in [-h^*, 0]$.
In preparation for our main results, we notice that (1.1) is equivalent to

$$x'(t) = \big(a(t) + \sum_{i=1}^{n} b_i(t+h_i)\big)x(t) - \sum_{i=1}^{n} \frac{d}{dt} \int_{t-h_i}^{t} b_i(s+h_i)x(s)ds \qquad (1.2)$$

## 2. **Exponential Stability**

Now we turn our attention to the exponential decay of solutions of equation (1.1). For simplicity, we let

$$Q(t) = \sum_{i=1}^{n} b_i(t+h_i) + a(t).$$

**Lemma 1.** *Assume for $\delta > 0$,*

$$-\frac{\delta}{(\delta+1)h^*} \le \sum_{i=1}^{n} b_i(t+h_i) + a(t) \le -\delta h^* \sum_{i=1}^{n} b_i^2(t+h_i), \qquad (2.1)$$

*hold. If*

$$\begin{aligned} V(t) &= \big[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i)x(s)\, ds\big]^2 \\ &+ \delta \sum_{i=1}^{n} \int_{-h_i}^{0} \int_{t+s}^{t} b_i^2(z+h_i)x^2(z)dzds \qquad (2.2) \end{aligned}$$

*then, along the solutions of (1.1) we have*

$$V'(t) \le Q(t)V(t).$$

**Proof**. First we note that due to condition (2.1), $Q(t) < 0$ for all $t \ge 0$. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.1) and define $V(t)$ by (2.2). Then along solutions of (1.2) we have

$$
\begin{aligned}
V'(t) &= 2[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)x(s)ds]Q(t)x(t)\\
&+ \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+h_i)x^2(t)ds\\
&- \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)x^2(t+s)ds\\
&\le 2[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)x(s)\,ds]Q(t)x(t)\\
&+ \delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)x^2(t)\\
&- \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)x^2(t+s)ds\\
&\le Q(t)[x^2(t)+2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)x(s)ds]\\
&+ \delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)x^2(t)\\
&- \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)x^2(t+s)ds+Q(t)x^2(t)\\
&\le Q(t)V(t)-Q(t)\big[\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s-h_i)x(s)ds\big]^2\\
&- \delta Q(t)\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)x^2(z)dzds\\
&- \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)x^2(t+s)ds\\
&+ \big(\delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)+Q(t)\big)x^2(t). \qquad (2.3)
\end{aligned}
$$

In what to follow we perform some calculations to simplify (2.3). First, if we let $u=t+s$, then

$$\delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)x^2(t+s)ds=\delta\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(u+h_i)x^2(u)du. \quad (2.4)$$

Also, since $-Q(t) > 0$, we have by the aid of Holder's inequality that

$$-Q(t)\big[\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)x(s)ds\big]^2 \leq -Q(t)h^*\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)x^2(s)ds \quad (2.5)$$

Finally, we easily observe

$$\delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)x^2(z)dz\ ds \leq \delta h^*\int_{t-h_i}^{t}\sum_{i=1}^{n} b_i^2(s+h_i)x^2(s)ds. \quad (2.6)$$

By invoking (2.1) and substituting expressions (2.4)-(2.6) into (2.3), yield

$$\begin{aligned} V'_{(1.1)}(t) &\leq Q(t)V(t) + \big(\delta h^*\sum_{i=1}^{n} b_i^2(t+h_i) + Q(t)\big)x^2(t) \\ &+ [-\big(\delta+1\big)h^*Q(t) - \delta]\int_{t-h_i}^{t} b_i^2(s+h_i)x^2(s)ds \\ &\leq Q(t)V(t). \end{aligned} \quad (2.7)$$

**Theorem 2.1.** *Suppose condition* (2.1) *hold.*
*Then any solution $x(t) = x(t, t_0, \psi)$ of (1.1) satisfies the exponential inequalities*

$$|x(t)| \leq \sqrt{2\frac{\delta}{2+\delta}V(t_0)e^{\frac{1}{2}\int_{t_0}^{t-h^*/2}[a(s)+\sum_{i=1}^{n} b_i(s+h_i)]ds}} \quad (2.8)$$

*for $t \geq t_0 + h^*/2$ and*

$$x(t) = ||\psi||e^{\int_{t_0}^{t}\sum_{i=1}^{n} b_i(s+h_i)ds}\big[1 + \int_{t_0}^{t}|a(u)|e^{-\int_{t_0}^{t}\sum_{i=1}^{n} b_i(s+h_i)ds}du\big] \quad (2.9)$$

*for $t \in [t_0, t_0 + h^*/2]$.*

**Proof**. By changing the order of integration we have

$$\begin{aligned} \delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)x^2(z)dz\ ds &= \delta\sum_{i=1}^{n}\int_{t-h_i/2}^{t}\int_{-h_i}^{z-t} b_i^2(z+h_i)x^2(z)dsdz \\ &= \delta\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(z+h_i)x^2(z)(z-t+h_i)dz. \end{aligned}$$

Also, we observe that

$$\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(z+h_i)x^2(z)(z-t+h_i)dz$$

$$\begin{aligned}
&= \sum_{i=1}^{n}\int_{t-h_i}^{t-\frac{h_i}{2}} b_i^2(z+h_i)x^2(z)(z-t+h_i)dz \\
&+ \sum_{i=1}^{n}\int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i)x^2(z)(z-t+h_i)dz \\
&\geq \sum_{i=1}^{n}\int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i)x^2(z)(z-t+h_i)dz \\
&\geq \sum_{i=1}^{n}\frac{h_i}{2}\int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i)x^2(z)dz,
\end{aligned} \tag{2.10}$$

where we have used the fact that when $t-\frac{h_i}{2}\leq z\leq t \implies \frac{h_i}{2}\leq z-t+h_i\leq h_i$. Let $V(t)$ be given by (2.2). Then,

$$\begin{aligned}
V(t) &\geq \delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)x^2(z)dzds \\
&\geq \delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-\frac{h_i}{2}}^{t} b_i^2(s+h_i)x^2(s)ds.
\end{aligned}$$

Consequently,

$$V\big(t-\frac{h_i}{2}\big) \geq \delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-h_i}^{t-\frac{h_i}{2}} b_i^2(s+h_i)x^2(s)ds \tag{2.11}$$

Note that since $V'(t)\leq 0$ we have for $t\geq t_0+h^*/2$ that
$0\leq V(t)+V\big(t-\frac{h_i}{2}\big)\leq 2V\big(t-\frac{h_i}{2}\big)$.
We note that

$$\frac{1}{2}\Big(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)x(s)ds\Big)^2 \leq \sum_{i=1}^{n}\frac{h_i}{2}\int_{t-h_i}^{t} b_i^2(s+h_i)x^2(s)ds.$$

Using (2.10) and (2.11), we get

$$
\begin{aligned}
& V(t)+V\big(t-\frac{h_i}{2}\big) \\
= \;& \Big(x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)x(s)ds\Big)^2+\delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)x^2(z)dz\; ds+V\big(t-\frac{h_i}{2}\big) \\
\geq \;& \Big(x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)x(s)ds\Big)^2+\delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-\frac{h_i}{2}}^{t} b_i^2(s+h_i)x^2(s)ds \\
+ \;& \delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-h_i}^{t-\frac{h_i}{2}} b_i^2(s+h_i)x^2(s)ds \\
\geq \;& \Big(x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)x(s)ds\Big)^2+\delta\frac{1}{2}\big(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)x(s)ds\big)^2 \\
= \;& \frac{2}{2+\delta}x^2(t)+\big[\frac{1}{\sqrt{1+\delta/2}}x(t)+\sqrt{1+\delta/2}\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)x(s)ds\big]^2 \\
\geq \;& \frac{2}{2+\delta}x^2(t)
\end{aligned}
$$

Consequently,

$$\frac{2}{2+\delta}x^2(t)\leq V(t)+V(t-\frac{h_i}{2})\leq 2V(t-\frac{h_i}{2})$$

An integration of (2.7) from $t_0$ to $t$ yields the inequality

$$V(t)\leq V(t_0)e^{\int_{t_0}^{t}[a(s)+\sum_{i=1}^{n} b_i(s+h_i)]ds}.$$

This implies that

$$V\big(t-\frac{h_i}{2}\big)\leq V(t_0)e^{\int_{t_0}^{t-\frac{h_i}{2}}[a(s)+\sum_{i=1}^{n} b_i(s+h_i)]\; ds}.$$

Therefore,

$$|x(t)|\leq\sqrt{2\frac{2+\delta}{\delta}V(t_0)}\;\; e^{\frac{1}{2}\int_{t_0}^{t-\frac{h_i}{2}}[a(s)+\sum_{i=1}^{n} b_i(s+h_i)]ds}$$

for $t\geq t_0+h^*/2$.

For $t\in[t_0,t_0+h^*/2]$, equation (1.1) can be written as

$x'(t)=a(t)x(t)+\sum_{i=1}^{n} b_i(t)\psi(t-h_i)$. Since $\psi(t)$ is the known initial function, we can easily solve for $x(t)$ using the variations of parameters formula. That is

$$x(t)=e^{\int_{t_0}^{t}a(s)ds}\big[\psi(t_0)+\int_{t_0}^{t}\sum_{i=1}^{n} b_i(u)\psi(u-h_i)e^{-\int_{t_0}^{t}a(s)ds}du\big]$$

Thus for $t \in [t_0, t_0 + h^*/2]$, the above expression implies

$$|x(t)| \leq ||\psi|| e^{\int_{t_0}^{t} a(s)ds} \big[1 + \int_{t_0}^{t_0+h^*/2} \sum_{i=1}^{n} |b_i(u)| e^{-\int_{t_0}^{t} a(s)ds} du\big]$$

This completes the proof.
The prove of the next corollary is a direct consequence of inequality (2.8)

**Corollary 1** Suppose condition (2.1) hold and $\sum_{i=1}^{n} b_i^2(s+h_i)]ds \geq \gamma$ for some constant $\gamma > 0$, then the zero solution of (1.1) is exponentially stable.
Next we compare our results with the results of [9]. In [9], Wang used similar method and showed the constant delay equation

$$x'(t) = a(t)x(t) + b(t)x(t-h) \tag{2.12}$$

and derived similar inequalities to (2.8) and (2.9) provided that

$$-\frac{1}{2h} \leq a(t) + b(t+h) \leq -hb^2(t+h) \tag{2.13}$$

hold. If $n = 1$ we see that our equation reduces to (2.12). For the sake of comparison, take $a(t) = -2 + \sqrt{11},\ b(t) = -\sqrt{11},\ h = \frac{1}{5},\ \delta = \frac{2}{3}$, then we easily see that our condition (2.1) is satisfied, while condition (2.13) of [9] is not.

We end this paper by giving a criteria for instability via Lyapunov functional.

## 3. Criteria For Instability

In this section, we use a non-negative definite Liapunov functional and obtain criteria that can be easily applied to test for instability of the zero solution of (1.1).

**Theorem 3.1.** *Suppose there exists a positive constant $H > h^*$ such that*

$$a(t) + \sum_{i=1}^{n} b(t+h_i) - H \sum_{i=1}^{n} b^2(t+h_i) \geq 0. \tag{3.1}$$

*If*

$$\begin{aligned} V(t) &= \Big[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b(s+h_i)x(s)ds\Big]^2 \\ &\quad -H \sum_{i=1}^{n} \int_{t-h_i}^{t} b^2(s+h_i)x^2(s)ds, \end{aligned} \tag{3.2}$$

*then, along the solutions of (1.1) we have*

$$V'(t) \geq [a(t) + \sum_{i=1}^{n} b(t+h_i)]V(t).$$

**Proof**. First we observe that condition (3.1) implies that $Q(t) > 0$ for all $t \geq 0$. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.1) and define $V(t)$ by (3.2). Then along solutions of (1.1) we have

$$\begin{aligned}
V'_{(1.1)}(t) &= 2\Big[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b(s+h_i)x(s)ds\Big]\Big[x'(t) + \sum_{i=1}^{n} b(t+h_i)x(t) - \sum_{i=1}^{n} b(t)x(t-h_i)\Big] \\
&- H\Big[\sum_{i=1}^{n} b^2(t+h_i)x^2(t) - b^2(t)x^2(t-h_i)\Big] \\
&= 2\Big[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b(s+h_i)x(s)ds\Big]Q(t)x(t) \\
&- H\sum_{i=1}^{n} b^2(t+h_i)x^2(t) + H\sum_{i=1}^{n} b^2(t)x^2(t-h_i) \\
&= Q(t)\Big\{[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b(s+h_i)x(s)ds]^2 - H\sum_{i=1}^{n} \int_{t-h_i}^{t} b^2(s+h_i)x^2(s)ds\Big\} \\
&+ Q(t)\Big[H\sum_{i=1}^{n} \int_{t-h_i}^{t} b^2(s+h_i)x^2(s)ds - \Big(\sum_{i=1}^{n} \int_{t-h_i}^{t} b(s+h_i)x(s)ds\Big)^2\Big] \\
&+ \big[Q(t) - H\sum_{i=1}^{n} b^2(t+h_i)\big]x^2(t) + H\sum_{i=1}^{n} b^2(t)x^2(t-h_i) \\
&\geq Q(t)V(t) \qquad (3.3)
\end{aligned}$$

where we have used

$$\Big(\sum_{i=1}^{n} \int_{t-h_i}^{t} b(s+h_i)x^2(s)ds\Big)^2 \leq h^* \sum_{i=1}^{n} \int_{t-h_i}^{t} b^2(s+h_i)x^2(s)ds$$

and (3.1). This completes the proof.

**Lemma 2.** Suppose condition (3.1) hold. Then the zero solution of (1.1) is unstable, provided that

$$\sum_{i=1}^{n} \int^{\infty} b^2(s+h_i)\, ds = \infty.$$

**Proof**. An integration of (3.3) from $t_0$ to $t$ yields

$$V(t) \geq V(t_0)e^{\int_{t_0}^{t} \left(a(s) + \sum_{i=1}^{n} b(s+h_i)\right)\, ds}. \qquad (3.4)$$

Let $V(t)$ be given by (3.2). Then

$$\begin{aligned} V(t) &= x^2(t) + 2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b(s+h_i)x(s)ds + \Big[\sum_{i=1}^{n}\int_{t-h_i}^{t} b(s+h_i)x(s)ds\Big]^2 \\ &- H\sum_{i=1}^{n}\int_{t-h_i}^{t} b^2(s+h_i)x^2(s)ds. \end{aligned} \tag{3.5}$$

Let $\beta = H - h^*$. Then from

$$(\frac{\sqrt{h_i}}{\sqrt{\beta}}a - \frac{\sqrt{\beta}}{\sqrt{h_i}}b)^2 \geq 0,$$

we have

$$2ab \leq \frac{h_i}{\beta}a^2 + \frac{\beta}{h_i}b^2.$$

With this in mind we arrive at,

$$\begin{aligned} 2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b(s+h_i)x(s)ds &\leq 2|x(t)||\sum_{i=1}^{n}\int_{t-h_i}^{t} b(s+h_i)x(s)ds| \\ &\leq \frac{h_i}{\beta}x^2(t) + \frac{\beta}{h_i}\Big[\sum_{i=1}^{n}\int_{t-h_i}^{t} b(s+h_i)x(s)ds\Big]^2 \\ &\leq \frac{h^*}{\beta}x^2(t) + \frac{\beta}{h_i}h_i\sum_{i=1}^{n}\int_{t-h_i}^{t} b^2(s+h_i)x^2(s)ds. \end{aligned}$$

A substitution of the above inequality into (3.5) yields,

$$\begin{aligned} V(t) &\leq x^2(t) + \frac{h^*}{\beta}x^2(t) + (\beta + h^* - H)\int_{t-h_i}^{t} b^2(s+h_i)x^2(s)ds \\ &= \frac{\beta + h^*}{\beta}x^2(t) \\ &= \frac{H}{H-h^*}x^2(t) \end{aligned}$$

Using inequality (3.4), we get

$$\begin{aligned} |x(t)| &\geq \sqrt{\frac{H}{H-h^*}D}\, V^{1/2}(t) \\ &= \sqrt{\frac{H}{H-h^*}}\, V^{1/2}(t_0)e^{\int_{t_0}^{t}\left(a(s)+\sum_{i=1}^{n} b(s+h_i)\right)\, ds} \\ &\geq \sqrt{\frac{H}{H-h^*}}\, V^{1/2}(t_0)e^{\sum_{i=1}^{n}\int_{t_0}^{t} b^2(s+h_i)\big)\, ds}. \end{aligned}$$

This completes the proof.

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Department of Mathematics, University of Dayton, Dayton, OH 45469-2316 USA

*E-mail address*: youssef.raffoul@notes.udayton.edu