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Improved nearness research III

Dieter Leseberg

Institut für Mathematik

Freie Universität Berlin

Germany

Abstract

In [9] respectively [10] generalized nearness structures were considered under the fundamental aspects of unification and extension, respectively. This interesting research was handled within the realm of the new created concept "Bounded Topology" [8], dealing besides classical structures (e.g. nearness, convergence, etc. ...) with those defined on bounded sets like set-convergences [13], supertopologies [4] or supernear structures [7], respectively. Here, we will now consider the so-called superscreen spaces, whose are in one-to-one correspondence to certain strict topological extensions, involving generalized LEADER proximities [6] and supertopologies as well in a natural way. At last we present generalized LODATO proximity spaces, here defined as preLODATO spaces including the LODATO spaces as considered in [10] leading us to a further interesting analogon in the realm of supernear spaces. In the "saturated" case all last mentioned spaces essentially coincide(up to isomorphism). As the reader will observe, that this concept is not of utmost generality, but then he is referred to [9].

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1 Basic concepts

As usual $\underline{P}X$ denotes the power set of a set X, and we use $\mathcal{B}^X \subset \underline{P}X$ to denote a collection of <u>bounded</u> subsets of X, also known as $\underline{B}$-sets [13], e.g. $\mathcal{B}^X$ has the following properties:

(b$_1$) $\varnothing \in \mathcal{B}^X$;

(b$_2$) $B_2 \subset B_1 \in \mathcal{B}^X$ imply $B_2 \in \mathcal{B}^X$;

(b$_3$) $x \in X$ implies $\{x\} \in \mathcal{B}^X$.

Then for $\underline{B}$-sets $\mathcal{B}^X, \mathcal{B}^Y$ a function $f : X \longrightarrow Y$ is called <u>bounded</u> iff f satisfies (b), e.g.

(b) $\{f[B] : B \in \mathcal{B}^X\} \subset \mathcal{B}^Y$.

Definition 1.1. For a set X, we call a tripel $(X, \mathcal{B}^X, N)$ consisting of X, $\underline{B}$-set $\mathcal{B}^X$ and a near-operator $N : \mathcal{B}^X \longrightarrow \underline{P}(\underline{P}(\underline{P}X))$ a <u>supernearness space</u> (shortly <u>supernear space</u>) iff the following axioms are satisfied, e.g.

(sn$_1$) $B \in \mathcal{B}^X$ and $\rho_2 << \rho_1 \in N(B)$ imply $\rho_2 \in N(B)$, where $\rho_2 << \rho_1$ iff $\forall F_2 \in \rho_2 \exists F_1 \in \rho_1\ F_2 \supset F_1$;

(sn$_2$) $B \in \mathcal{B}^X$ implies $\mathcal{B}^X \notin N(B) \neq \varnothing$;

(sn$_3$) $\rho \in N(\varnothing)$ implies $\rho = \varnothing$;

(sn$_4$) $x \in X$ implies $\{\{x\}\} \in N(\{x\})$;

(sn$_5$) $B_1 \subset B_2 \in \mathcal{B}^X$ imply $N(B_1) \subset N(B_2)$;

(sn$_6$) $B \in \mathcal{B}^X$ and $\rho_1 \vee \rho_2 \in N(B)$ imply $\rho_1 \in N(B)$ or $\rho_2 \in N(B)$, where $\rho_1 \vee \rho_2 := \{F_1 \cup F_2 : F_1 \in \rho_1, F_2 \in \rho_2\}$;

(sn$_7$) $B \in \mathcal{B}^X, \rho \subset \underline{P}X$ and $\{cl_N(F) : F \in \rho\} \in N(B)$ imply $\rho \in N(B)$, where $cl_N(F) := \{x \in X : \{F\} \in N(\{x\})\}$.

If $\rho \in N(B)$ for some $B \in \mathcal{B}^X$, then we call ρ a $\underline{B}$-near collection in N. For supernear spaces $(X, \mathcal{B}^X, N), (Y, \mathcal{B}^Y, M)$ a bounded function $f : X \longrightarrow Y$ is called sn-map iff it satisfies (sn), e.g.

(sn) $B \in \mathcal{B}^X$ and $\rho \in N(B)$ imply $\{f[F] : F \in \rho\} =: f\rho \in N(f[B])$.

We denote by SN the corresponding category.

Examples 1.2. (i) For a nearness space (X, ξ) let $\mathcal{B}^X$ be $\underline{B}$-set. Then we consider the tripel $(X, \mathcal{B}^X, N_\xi)$, where

$N_\xi(\varnothing) := \{\varnothing\}$ and

$N_\xi(\varnothing) := \{\rho \subset \underline{P}X : \{B\} \cup \rho \in \xi\}$, otherwise.

(ii) For a topological space (X, t) given by closure operator t let $\mathcal{B}^X$ be $\underline{B}$-set. Then we consider the tripel $(X, \mathcal{B}^X, N_t)$, where for each $B \in \mathcal{B}^X$ $N_t(B) := \{\rho \subset \underline{P}X : B \in \sec\{t(F) : F \in \rho\}\}$.

(iii) For a preLEADER space $(X, \mathcal{B}^X, \delta)$ with $\delta \subset \mathcal{B}^X \times \underline{P}X$ we consider the tripel $(X, \mathcal{B}^X, N_\delta)$, where for each $B \in \mathcal{B}^X$ $N_\delta(B) := \{\rho \subset \underline{P}X : \rho \subset \delta(B)\}$ with $\delta(B) := \{A \subset X : B\delta A\}$; hereby, $\delta \subset \mathcal{B}^X \times \underline{P}X$ satisfies the following conditions:

(bp$_1$) $\varnothing\overline{\delta}A$ and $B\overline{\delta}\varnothing$ (e.g. $\varnothing$ is not in relation to A, and analogously this is also holding for B;

(bp$_2$) $B\delta(A_1 \cup A_2)$ iff $B\delta A_1$ or $B\delta A_2$;

(bp$_3$) $x \in X$ implies $\{x\}\delta\{x\}$;

(bp$_4$) $B_1 \subset B_2 \in \mathcal{B}^X$ and $B_1\delta A$ imply $B_2\delta A$;

(bp$_5$) $B \in \mathcal{B}^X$ and $B\delta A$ with $A \subset cl_\delta(C)$ imply $B\delta C$, where $cl_\delta(C) := \{x \in X : \{x\}\delta C\}$.

For preLEADER spaces $(X, \mathcal{B}^X, \delta), (Y, \mathcal{B}^Y, \gamma)$ a bounded function $f : X \longrightarrow Y$ is called p-map iff f satisfies (p), e.g.

(p) $B \in \mathcal{B}^X, A \in X$ and $B\delta A$ imply $f[B]\gamma f[A]$. By pLESP we denote the corresponding category.

Definitions 1.3. TEXT denotes the category, whose objects are triples $E := (e, \mathcal{B}^X, Y)$ - called topological extensions - where $X := (X, cl_X), Y := (Y, cl_Y)$ are topological spaces (given by closure operators) with $\underline{B}$-set $\mathcal{B}^X$, and $e : X \longrightarrow Y$ is a function satisfying the following conditions:

(tx$_1$) $A \in \underline{P}X$ implies $cl_X(A) = e^{-1}[cl_Y(e[A])]$;

(tx$_2$) $cl_Y(e[X]) = Y$, which means the image of X under e is dense in Y. Morphisms in TEXT have the form $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$, where $f : X \longrightarrow X', g : Y \longrightarrow Y'$ are continuous maps such that f is bounded, and the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[e']{} & Y' \end{array} .$$

If $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ and $(f', g') : (e', \mathcal{B}^{X'}, Y') \longrightarrow (e'', \mathcal{B}^{X''}, Y'')$, are TEXT-morphisms, then they can be composed according to the rule:

$$(f', g') \circ (f, g) := (f' \circ f, g' \circ g) : (e, \mathcal{B}^X, Y) \longrightarrow (e'', \mathcal{B}^{X''}, Y''),$$

where "o" denotes the composition of maps.

Remark 1.4. Observe, that axiom (tx$_1$) in this definition is automatically satisfied if $e : X \longrightarrow Y$ is a topological embedding. Moreover, we only admit an ordinary $\underline{B}$-set $\mathcal{B}^X$ on X which need not be necessary coincide with the power $\underline{P}X$. In addition we mention that such an extension is called strict iff it statisfies (tx$_3$), e.g.

(tx$_3$) $\{cl_Y(e[A]) : A \subset X\}$ forms a base for the closed subsets of Y [1].

By STREXT we denote the corresponding full subcategory of TEXT.

(iv) For a topological extension $E := (e, \mathcal{B}^X, Y)$ we consider the tripel $(X, \mathcal{B}^X, N_e)$, where for each $B \in \mathcal{B}^X$ $N_e(B) := \{\rho \subset \underline{P}X : e[B] \in \sec\{cl_Y(e[F]) : F \in \rho\}\}$, where in general $\sec\mathcal{M}$ is defined by setting:

$$\sec\mathcal{M} := \{T \subset X : \forall M \in \mathcal{M}\ T \cap M \neq \varnothing\}.$$

2 Some important isomorphisms

With respect to above examples, first let us focus our attention to some special classes of supernear spaces.

Definition 2.1. A supernear space $(X, \mathcal{B}^X, N)$ is called saturated iff $\mathcal{B}^X$ is, e.g.

(s) $X \in \mathcal{B}^X$.

Remark 2.2. Note, that in above case $\mathcal{B}^X$ coincide with the power $\underline{P}X$. As shown in [9] the category NEAR, of nearness spaces and nearness preserving maps as well as TOP, the category of topological spaces and continuous maps, can be both embedded in SN^S, the full subcategory of SN, whose objects are the saturated supernear spaces.

Definition 2.3. A supernear space $(X, \mathcal{B}^X, N)$ then is called <u>supergrill space</u> if N satisfies (gri), e.g.

(gri) $B \in \mathcal{B}^X$ and $\rho \in N(B)$ imply there exists $\gamma \in \mathrm{GRL}(X)\rho \subset \gamma \in N(B)$, where $\mathrm{GRL}(X) := \{\gamma \subset \underline{P}X : \gamma$ is grill $\}$ and $\gamma \subset \underline{P}X$ is called <u>grill</u> (Choquet [3]) iff

(gri$_1$) $\varnothing \notin \gamma$;

(gri$_2$) $G_1 \cup G_2 \in \gamma$ iff $G_1 \in \gamma$ or $G_2 \in \gamma$.

We denote by G-SN the corresponding full subcategory of SN.

Remark 2.4. With respect to example (ii) we note that $(X, \mathcal{B}^X, N_t)$ is supergrill space.

Corollary 2.5. *The category of subtopological nearness spaces and related maps is isomorphic to a full subcategory of G-SN.*

Proof. Compare with example (i). □

Remark 2.6. According to example (iii) we further note that $(X, \mathcal{B}^X, N_\delta)$ is supergrill space, too.

Definition 2.7. A supergrill space $(X, \mathcal{B}^X, N)$ then is called <u>conic</u> iff N satisfies (c), e.g.

(c) $B \in \mathcal{B}^X$ implies $\cup\{\rho \subset \underline{P}X : \rho \in N(B)\} =: \cup N(B) \in N(B)$.

We denote by CG-SN the corresponding full subcategory of G-SN.

Theorem 2.8. *The category pLESP is isomorphic to the category CG-SN.*

Proof. Compare with example (iii) in connexion with [10]. □

Remark 2.9. Hence, LEADER proximity spaces [6] can be now considered as special conic supergrill spaces. Moreover each supertopological space $(X, \mathcal{B}^X, \Theta)$ is leading us to the specific conic supergrill space $(X, \mathcal{B}^X, N_\Theta)$ by setting for each $B \in \mathcal{B}^X$: $N_\Theta(B) := \{\rho \subset \underline{P}X : \rho \subset \sec\Theta(B)\}$. In this context $\Theta : \mathcal{B}^X \longrightarrow \mathrm{FIL}(X) := \{\mathcal{F} \subset \underline{P}X : \mathcal{F}$ is filter$\}$ is a <u>neighbourhood function</u>, satisfying following conditions, e.g.

(stop$_1$) $\Theta(\varnothing) = \underline{P}X$;

(stop$_2$) $B \in \mathcal{B}^X$ and $U \in \Theta(B)$ imply $U \supset B$;

(stop$_3$) $B \in \mathcal{B}^X$ and $U \in \Theta(B)$ imply there exists a set $V \in \Theta(B)$ such that always $U \in \Theta(B')\forall B' \in \mathcal{B}^X B' \subset V$.

Conversely, we define for such a conic supergrill space $(Y, \mathcal{B}^Y, M)$ the following neighbourhood function by setting: $\Theta_M(B) := \{V \subset X : V \in \sec \cup M(B)\}$. Hence, STOP can be considered as a subcategory of CG-SN, too!

Remark 2.10. Here, we point out, that a supergrill space $(X, \mathcal{B}^X, N)$ is conic if and only if for each $B \in \mathcal{B}^X \cup N(B) \in \mathrm{GRL}(X) \cap N(B)$.

Proposition 2.11. *Let (Y, t) be a topological space given by closure operator t and $\mathcal{B}^X$ $\underline{B}$-set with $X \subset Y$. We put $B\delta^t A$ iff $B \cap t(A) \neq \varnothing$ for each $B \in \mathcal{B}^X$ and $A \subset X$. Then $(X, \mathcal{B}^X, \delta^t)$ is preLEADER space*

Remark 2.12. Now, it seems to be of interest to characterize preLEADER spaces, whichever are induced by a topological space Y as above, so that a bounded set B is near to an arbitrary one iff B intersects its closure in Y. But in the following we will solve this problem under more general conditions!

3 Topological extensions and related superscreen spaces

Taking into account example (iv), we will now consider the problem for finding a one-to-one correspondence between topological extensions and related supernear spaces.

Definition 3.1. Let be given a supernear space $(X, \mathcal{B}^X, N)$. For $B \in \mathcal{B}^X, \mathcal{C} \in \mathrm{GRL}(X)$ is called B-screen in N iff it satisfies

(scr$_1$) $B \in \mathcal{C} \in N(B)$;

(scr$_2$) $A \in \mathcal{C}$ and $A \subset cl_N(F)$ imply $F \in \mathcal{C}$;

(scr$_3$) $B \in \sec\{cl_N(C) : C \in \mathcal{C}\}$.

Remark 3.2. We point out that for each $B \in \mathcal{B}^X$ with $x \in B$ $x_N := \{A \subset X : x \in cl_N(A)\}$ is B-screen in N, moreover x_N is maximal in $N(\{x\}) \setminus \{\varnothing\}$, ordered by inclusion .

Definition 3.3. A conic supernear space $(X, \mathcal{B}^X, N)$ then is called superscreen space iff it satisfies (scr), e.g.

(scr) $B \in \mathcal{B}^X \backslash \{\varnothing\}$ implies UN(B) is B-screen in N.

(see also 2.10).

Remark 3.4. According to remark 2.4 we claim that $(X, \mathcal{B}^X, N_t)$ even is superscreen space.

Lemma 3.5. *For a topological extension* $(e, \mathcal{B}^X, Y), (X, \mathcal{B}^X, N_e)$ *(see example (iv)) is a superscreen space such that* $cl_{N_e} = cl_X$.

Proof. First, we will show the equality of the closure operators. So, let be $A \in \underline{P}X$ and $x \in cl_X(A)$. Then, by (tx_1) $e(x) \in cl_Y(e[A])$, hence $\{A\} \in N_e(\{x\})$, and $x \in cl_{N_e}(A)$ follows. Conversely, let $x \in cl_{N_e}(A)$.

Then $\{A\} \in N_e(\{x\})$. Consequently $\{e(x)\} \cap cl_Y(e[A]) \neq \varnothing$; hence there exists $y \in cl_Y(e[A])$ and $y = e(x)$. As a consequence of (tx_1) we get $x \in e^{-1}[cl_Y(e[A])] \subset cl_X(A)$, which was to be proven. Secondly, it is easy to check the axioms (sn_1) to (sn_6).

to (sn_7): Let be $\{cl_{N_e}(F) : F \in \rho\} \in N_e(B)$ for $\rho \in \underline{P}X, B \in \mathcal{B}^X$ and without restriction $B \neq \varnothing$. Then $e[B] \in \sec\{cl_Y(e[A]) : A \in \{cl_{N_e}(F) : F \in \rho\}\}$. For $F \in \rho$ we get $e[B] \cap cl_Y(e[cl_{N_e}(F)]) \neq \varnothing$. Taking into account that $cl_{N_e}(F) = cl_X(F)$ and in addition (tx_1) is valid, we get $e[B] \cap cl_Y(e[F]) \neq \varnothing$, hence $\rho \in N_e(B)$ results. $(X, \mathcal{B}^X, N_e)$ is conic, because without restriction $B \in \mathcal{B}^X \backslash \{\varnothing\}$ and $F \in \cup N_e(B)$ imply there exists $\rho \in N_e(B)$ with $F \in \rho$, hence $e[B] \cap cl_Y(e[F]) \neq \varnothing$ by hypothesis, and consequently $\cup N_e(B) \in N_e(B)$ results. $(X, \mathcal{B}^X, N_e)$ is superscreen space. Let be $B \in \mathcal{B}^X \backslash \{\varnothing\}$, we put:

$$\mathcal{C}_B := \{T \subset X : e[B] \cap cl_Y(e[T]) \neq \varnothing\}.$$

Obviously, $\mathcal{C}_B \in \text{GRL}(X)$. Moreover we will verify that $\mathcal{C}_B$ is B-screen in N.

to (scr_1): $B \in \mathcal{C}_B$, since $e[B] \cap cl_Y(e[B]) \neq \varnothing$; $\mathcal{C}_B \in N_e(B)$, because $T \in \mathcal{C}_B$ implies $e[B] \cap cl_Y(e[T]) \neq \varnothing$ by definition.

to (scr_2): $T \in \mathcal{C}_B$ and $T \subset cl_{N_e}(F)$ imply $e[B] \cap cl_Y(e[T]) \neq \varnothing$, hence $e[B] \cap cl_Y(e[cl_{N_e}(F)]) \neq \varnothing$. But $cl_{N_e}(F) = cl_X(F)$, and by applying (tx_1) we get $e[B] \cap cl_Y(e[F]) \neq \varnothing$, which shows $F \in \mathcal{C}_B$.

to (scr$_3$): $T \in \mathcal{C}_B$ implies $e[B] \cap cl_Y(e[B]) \neq \varnothing$, hence $x \in e^{-1}[cl_Y(e[T])]$ results for some $x \in B$, which shows $x \in cl_X(T)$ according to (tx$_1$). But then $x \in cl_{N_e}(T)$ follows, and consequently $B \cap cl_{N_e}(T)$ is valid. So, at last we conclude $\mathcal{C}_B = \cup N(B)$.

□

Definition 3.6. We denote by SCR-SN the full subcategory of CG-SN, whose objects are the superscreen spaces.

Theorem 3.7. *Let F : TEXT$\longrightarrow$SCR-SN be defined by*

(a) For a TEXT-object $(e, \mathcal{B}^X, Y)$ we put: $F(e, \mathcal{B}^X, Y) := (X, \mathcal{B}^X, N_e)$; for a TEXT-morphism $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ we put: $F(f, g) := f$. Then F : TEXT$\longrightarrow$SCR-SN is a functor.

Proof. With respect to 3.5 we already know that $F(e, \mathcal{B}^X, Y)$ is an object of SCR-SN. Let (f, g) : $(e, \mathcal{B}^X, Y) \longrightarrow (e', B^{X'}, Y')$ be a TEXT-morphism such that $F(e, \mathcal{B}^X, Y) = (X, \mathcal{B}^X, N_e)$ and $F(e', \mathcal{B}^{X'}, Y') = (X', \mathcal{B}^{X'}, N_{e'})$. It has to be shown that $f : (X, \mathcal{B}^X, N_e) \longrightarrow (X', \mathcal{B}^{X'}, N_{e'})$ preserves B-near collections for each $B \in \mathcal{B}^X$. Without loss of generality let be $B \in \mathcal{B}^X \backslash \{\varnothing\}$ and $\rho \in N_e(B)$, hence $e[B] \in \sec\{cl_Y(e[F]) : F \in \rho\}$. Our goal is to verify that $f\rho \in N_{e'}(f[B])$. So let be $A \in f\rho$, hence $A = f[F]$ for some $F \in \rho$, and consequently $e[B] \cap cl_Y(e[F]) \neq \varnothing$ results by hypothesis. Choose $y \in cl_Y(e[F])$ with $y \in e[B]$, hence $y = e(x)$ for some $x \in B$. Then $g(y) = g(e(x)) = e'(f(x)) \in e'(f[B])$. On the other hand we have $g(y) \in g[cl_Y(e[F])] \subset cl_{Y'}(g[e[F]]) = cl_{Y'}(e'[f[F]]) = cl_{Y'}(e'[A])$ according to 1.3, which shows that $e'[f[B]] \in \sec\{cl_Y(e'[A]) : A \in f\rho\}$ implying at last $f\rho \in N_{e'}(f[B])$. Then the reminder is clear! □

4 Superscreen spaces and strict topological extensions

In the previous paragraph we have found a functor from TEXT to SCR-SN. Now, we are going to introduce a related one from SCR-SN to STREXT.

Lemma 4.1. *For a supernear space $(X, \mathcal{B}^X, N)$ we put: $X^S := \{\mathcal{C} \subset \underline{P}X : \mathcal{C}$ is B-screen in N for some $B \in \mathcal{B}^X\}$, and for each $A^S \subset X^S$ we set: $cl_{X^S}(A^S) := \{\mathcal{C} \in X^S : \triangle A^S \subset \mathcal{C}\}$, where $\triangle A^S := \{F \subset X : \forall \mathcal{C} \in A^S \; F \in \mathcal{C}\}$, so that by convention $\triangle A^S = \underline{P}X$ if $A^2 = \varnothing$. Then cl_{X^S} is topological closure operator on X^S.*

Proof. But for this verification the reader is referred to [10]. □

Theorem 4.2. *For superscreen spaces* $(X, \mathcal{B}^X, N), (Y, \mathcal{B}^Y, M)$ *let* $f : X \longrightarrow Y$ *be a sn-map. Define a function* $f^S : X^S \longrightarrow Y^S$ *by setting for each* $\mathcal{C} \in X^S : f^S(\mathcal{C}) := \{D \subset Y : f^{-1}[cl_M(D)] \in \mathcal{C}\}$*. Then the following statements are valid:*

(i) $f^S : (X^S, cl_{X^S}) \longrightarrow (Y^S, cl_{Y^S})$ *is a continuous map;*

(ii) The equality $f^S \circ e_X = e_Y \circ f$ *holds, where* $e_X : X \longrightarrow X^S$ *denotes that function which assigns the* $\{x\}$*-screen* x_N *to each* $x \in X$ *(see also remark 3.2.).*

Proof. First, let be $\mathcal{C} \in X^S$, we must show that $f^S(\mathcal{C}) \in Y^S \cdot f^S(C) \in \text{GRL}(Y)$, since $\mathcal{C} \in \text{GRL}(X)$ and f^{-1} respectively cl_M are compatible with finite union. By hypothesis $\mathcal{C} \in N(B)$ for some $B \in \mathcal{B}^X$, hence $f\mathcal{C} \in M(f[B])$, because f is sn-map. Now, we will show that $\{cl_M(D) : D \in f^S(\mathcal{C})\} << f\mathcal{C}$.

$cl_M(D)$ for some $D \in f^S(\mathcal{C})$ implies $f^{-1}[cl_M(D)] \in \mathcal{C}$, hence $cl_M(D) \supset f[f^{-1}[cl_M(D)]] \in f\mathcal{C}$. According to ($\text{sn}_7$) $f^S(\mathcal{C}) \in M(f[B])$ follows. $f[B] \in f^S(\mathcal{C})$, since $f^{-1}[cl_M(f[B])] \supset f^{-1}[f[cl_N(B)]] \supset B \in \mathcal{C}$ by hypothesis. Now, let be $D \in f^S(\mathcal{C})$ and $D \subset cl_M(F)$; we have to verify $F \in f^S(\mathcal{C})$. By hypothesis $f^{-1}[cl_M(D)] \in \mathcal{C}$. $f^{-1}[cl_M(D)] \subset cl_N(f^{-1}[cl_M(F)])$, because $x \in f^{-1}[cl_M(D)]$ implies $f(x) \in cl_M(D)$; but $cl_M(D) \subset cl_M(cl_M(D)) \subset cl_M(F)$, hence $f(x) \in cl_M(F)$. Consequently, $x \in f^{-1}[cl_M(F)] \subset cl_N(f^{-1}[cl_M(F)])$ results. Since $\mathcal{C}$ especially satisfies (scr_2) $f^{-1}[cl_M(F)] \in \mathcal{C}$ is valid, which shows $F \in f^S(\mathcal{C})$. At last let be $D \in f^S(\mathcal{C})$; we have to verify $f[B] \cap cl_M(D) \neq \varnothing$. By hypothesis $f^{-1}[cl_M(D)] \in \mathcal{C}$, hence $B \cap cl_N(f^{-1}[cl_M(D)]) \neq \varnothing$. Consequently, $\varnothing \neq f[B \cap cl_N(f^{-1}[cl_M(D)])] \subset f[B] \cap f[cl_N(f^{-1}[cl_M(D)])] \subset f[B] \cap cl_M[f(f^{-1}[cl_M(D)])] \subset f[B] \cap cl_M(cl_M(D)) \subset cl_M(D)$ results, since f is sn-map.

to (i): Let be $A^S \subset X^S, \mathcal{C} \in cl_{X^S}(A^S)$ and suppose $f^S(\mathcal{C}) \notin cl_{Y^S}(f^S[A^S])$Then $\triangle f^S[A^S] \not\subset f^S(\mathcal{C})$, hence $D \notin f^S(\mathcal{C})$ for some $D \in \triangle f^S[A^S]$, which means $f^{-1}[cl_M(D) \notin \mathcal{C}$.

But $\triangle A^S \subset \mathcal{C}$ implies $f^{-1}[cl_M(D)] \notin \mathcal{D}$ for some $\mathcal{D} \in A^S$. Therefore $D \notin f^S(\mathcal{D})$, which leads us to a contradiction, since $D \in \triangle f^S[A^S]$.

to (ii): Let x be an element of X. We will prove that the equality $f^S(e_X(x)) = e_Y(f(x))$ is valid. To this end let be $T \in e_Y(f(x))$, hence $f(x) \in cl_M(T)$, and consequently $x \in f^{-1}[cl_M(T)]$ follows which shows $f^{-1}[cl_M(T)] \in x_N = e_X(x)$. Thus $T \in f^S(e_X(x))$, which proves the inclusion $e_Y(f(x)) \subset f^S(e_X(x))$. Consequently, since $e_Y(f(x))$ is maximal in $M(\{f(x)\}) \setminus \{\varnothing\}$ (see remark 3.2 and note also that $\{cl_M(D) : D \in f^S(e_X(x))\} << fx_N \in M(\{f(x)\})$, since by supposition f is sn-map) we obtain the desired equality.

□

Theorem 4.3. *Let G : SCR-SN$\longrightarrow$STREXT be defined as follows:*

(a) For any superscreen space $(X, \mathcal{B}^X, N)$ we put: $G(X, \mathcal{B}^X, N) := (e_X, \mathcal{B}^X, X^S)$ with $X := (X, cl_N)$ and $X^S := (X^S, cl_{X^S})$;

(b) for any sn-map $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$ we put $G(f) := (f, f^S)$.

Then G : SCR-SN$\longrightarrow$STREXT is a functor.

Proof. With respect to (sn_7) cl_N is topological closure operator, and by 4.1. this also holds for cl_{X^S}. Therefore we get topological spaces with $\underline{B}$-set $\mathcal{B}^X$, and $e_X : X \longrightarrow X^S$ is a map according to 4.2.(ii). Now, we have to verify that $(e_X, \mathcal{B}^X, X^S)$ satisfies the axioms (tx_1) to (tx_3). But, as above, the reader is referred to [10].

At present it is interesting to see, how the composite functor $F \circ G$ is working on the category SCR-SN. □

Theorem 4.4. *Let G : SCR-SN$\longrightarrow$STREXT and F : TEXT$\longrightarrow$SCR-SN be the functors given in theorems 3.7 and 4.3.. For each object $(X, \mathcal{B}^X, N)$ of SCR-SN let $t_(X, \mathcal{B}^X, N)$ denote the identity map $t_(X, \mathcal{B}^X, N) := id_X : F(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$. Then $t : \mathcal{F} \circ G \longrightarrow 1_{SCR\text{-}SN}$ is natural equivalence from $F \circ G$ to the identity functor 1_{SCR-SN}, i.e. $id_X : F(G(X, \mathcal{B}^X, N)) \longrightarrow (X, \mathcal{B}^X, N)$ is in $\underline{\text{both}}$ directions a sn-map for each object $(X, \mathcal{B}^X, N)$, and the following diagram commutes for each sn-map $f : (X, \mathcal{B}^X, N) \longrightarrow (Y, \mathcal{B}^Y, M)$:*

$$\begin{array}{ccc} F(G(X, \mathcal{B}^X, N)) & \xrightarrow{id_X} & (X, \mathcal{B}^X, N) \\ {\scriptstyle F(G(f))}\Big\downarrow & & {\scriptstyle f}\Big\downarrow \\ F(G(Y, \mathcal{B}^Y, M)) & \xrightarrow{id_Y} & (Y, \mathcal{B}^Y, M). \end{array} \tag{1}$$

Proof. The commutativity of the diagram is obvious, because $F(G(f)) = f$. It remains to prove that in each case $.F(G(X, \mathcal{B}^X, N)) \underset{id_X}{\longrightarrow} (X, \mathcal{B}^X, N) \underset{id_X}{\longrightarrow} F(G(X, \mathcal{B}^X, N))$ is sn-map for any object $(X, \mathcal{B}^X, N) \in$ SCR-SN. To fix the notation, let N_1 be such that $F(G(X, \mathcal{B}^X, N)) = F(e_X, \mathcal{B}^X, X^S) = (X, \mathcal{B}^X, N_1)$. First, we show that for each $B \in \mathcal{B}^X \backslash \{\varnothing\}, \rho \in N_1(B)$ implies $\rho \in N(B)$. To this end assume $\rho \in N_1(B)$, then $e_X[B] \in \sec\{cl_{X^S}(e_X[F]) : F \in \rho\}$. It suffices to show ρ is subset of $\cup N(B)$, because by hypothesis $(X, \mathcal{B}^X, N)$ is conic. $F \in \rho$ implies there exists $y \in e_X[B] \cap cl_{X^S}(e_X[F])$, hence $y = e_X(x) = x_N \in cl_{X^S}(e_X[F])$ for some $x \in B$. Consequently, $\triangle e_X[F] \subset X_N$ results, and $F \in x_N$ follows. But $x_N \in N(\{x\})$, according to (sn$_7$), implying $x_N \in N(B)$ by (sn$_5$) showing at last that $F \in \cup N(B)$.

Conversely, let be $B \in \mathcal{B}^X \setminus \{\varnothing\}$ and $\rho \in N(B)$. We have to verify $e_X[B] \in \sec\{cl_{X^S}(e_X[F]) : F \in \rho\}$. $F \in \rho$ implies $F \in \cup N(B)$, hence $B \cap cl_N(F) \neq \varnothing$ according to 3.3.. Consequently there exists $x \in B$ with $x \in cl_N(F)$. Then $x_N = e_X(x) \in e_X[B]$. It remains prove $\triangle e_X[F] \subset e_X(x) = x_N$.

But $A \in \triangle e_X[F]$ implies $F \subset cl_N(A)$, hence $cl_N(F) \subset cl_N(A)$ follows, which shows $x \in cl_N(A)$ is valid, and $A \in x_N$ results. □

Comment 4.5. As a résumé we point out that those conic supergrill spaces $(X, \mathcal{B}^X, N)$ which can be extended to a certain topological one have a neat internal description; the condition is that for non-empty B-near collections in N $\cup N(B)$ is B-screen in N! Hence, sn-maps are all extendible!

Corollary 4.6. *If $(X, \mathcal{B}^X, N)$ is separated, that means N satisfies (sep), e.g.*

(sep) $x, z \in X$ and $\{\{z\}\} \in N(\{x\})$ imply $x = z$, then $e_X : X \longrightarrow X^S$ is injective. Conversely, for a T_1-extension $(e, \mathcal{B}^X, Y)$, where e is a topological embedding and Y T_1-space, then $(X, \mathcal{B}^X, N_e)$ is separated; because $x, z \in X$ and $\{\{z\}\} \in N_e(\{x\})$ imply $e[\{x\}] \in sec\{cl_Y(e[F]) : F \in \{z\}\}$, hence $\{e(x)\} \cap cl_Y(\{e(z)\}) \neq \varnothing$, which means $e(x) \in cl_Y(\{e(z)\})$. Since Y is T_1-space $e(x) = e(z)$ follows, and $x = z$ results by hypothesis.

5 preLODATO spaces and some other related categories

b-supertopologies were studied by Doîtchinov [4] in order to generate compactly determined extension of given space. LODATO proximity spaces are serving for an analogous purpose under more weaker conditions. In [9] was proven that there exists an one-to-one correspondence between certain "topological" extensions and related paraclan spaces in generalizing the theorem of Bentley [2]. Moreover, the mentioned relationship also includes a corresponding theorem for LODATO spaces which leading us to the famous theorem of LODATO [11] by accordant specialising!

Definition 5.1. For a set X, we call a tripel $(X, \mathcal{B}^X, \delta)$ consisting of X, B-set $\mathcal{B}^X$ and $\delta \subset \mathcal{B}^X \times X$ a preLODATO space iff the following conditions are satisfied.

(bp$_1$) $\varnothing\overline{\delta}A$ and $B\overline{\delta}\varnothing$ (e.g. $\varnothing$ is not in relation to A, and analogously this is also holding for B);

(bp$_2$) $B\delta(A_1 \cup A_2)$ iff $B\delta A_1$ or $B\delta A_2$;

(bp$_3$) $x \in X$ implies $\{x\}\delta\{x\}$;

(bp$_4$) $B_1 \subset B_2 \in \mathcal{B}^X$ and $B_1\delta A$ imply $B_2\delta A$;

(bp$_5$) $B \in \mathcal{B}^X$ and $B\delta A$ with $A \subset cl_\delta(C)$ imply $B\delta C$, where $cl_\delta(C) := \{x \in X : \{x\}\delta C\}$;

(bp$_6$) $B_1 \cup B_2 \in \mathcal{B}^X$ and $(B_1 \cup B_2)\delta A$ imply $B_1\delta A$ or $B_2\delta A$;

(bp$_7$) $B, A \subset X,\ cl_\delta(B) \in \mathcal{B}^X$ and $cl_\delta(B)\delta A$ imply $B\delta A$;

(bp$_8$) $B_1, B_2 \in \mathcal{B}^X$ and $B_1\delta B_2$ imply $B_2\delta B_1$.

Remark 5.2. According to 1.2(iii) we note that any preLODATO space is a preLEADER space. By pLOSP we denote the corresponding full subcategory of pLESP.

Additionally we note that any LODATO space [10] is a preLODATO space.

Example 5.3. Let be given a b-supertopological space $(X, \mathcal{B}^X, \Theta)$ in the sense of Doîtchinov, where $\mathcal{B}^X$ is $\underline{B}$-set on a set X and $\Theta : \mathcal{B}^X \longrightarrow \text{FIL}(X) := \{\mathcal{F} \subset \underline{P}X : \mathcal{F} \text{ is filter}\}$ function satisfying the following conditions, e.g.

(bSTOP$_1$) $\Theta(\varnothing) = \underline{P}X$;

(bSTOP$_2$) $B \in \mathcal{B}^X$ and $U \in \Theta(B)$ imply $U \supset B$;

(bSTOP$_3$) $B \in \mathcal{B}^X$ and $U \in \Theta(B)$ imply there exists a set $V \in \Theta(B)$ such that always $U \in \Theta(B')\forall B' \in \mathcal{B}^X\ B' \subset V$;

(bSTOP$_4$) $B_1 \cup B_2 \in \mathcal{B}^X$ implies $\Theta(B_1 \cup B_2) = \Theta(B_1) \cap \Theta(B_2)$;

(bSTOP$_5$) $B_1, B_2 \in \mathcal{B}^X$ and $B_1 \in \text{sec}\Theta(B_2)$ imply $B_2 \in \text{sec}\Theta(B_1)$.

Then we consider the tripel $(X, \mathcal{B}^X, \delta_\Theta)$ by setting $B\delta_\Theta A$ iff $A \in \text{sec}\Theta(B)$for each $B \in \mathcal{B}^X, A \subset X$. Hence, we point out that $(X, \mathcal{B}^X, \delta_\Theta)$ is a preLODATO space, too. The above assignment is "bi-functoriell", so that the category b-STOP of b-supertopological spaces and related maps can be considered as a subcategory of pLOSP. In this connexion we refer to [12] and note that Efremovič proximity spaces also can be dealt with.

Definition 5.4. A conic supergrill space $(X, \mathcal{B}^X, N)$ is called proximal iff N is linked, dense and presymmetric by satisfying (l), (d) and (psy), respectively, e.g.

(l) $B_1 \cup B_2 \in \mathcal{B}^X$ and $\rho \in N(B_1 \cup B_2)$ imply $\{F\} \in N(B_1) \cup N(B_2) \forall F \in \rho$;

(d) $B \subset X$ and $cl_N(B) \in \mathcal{B}^X$ imply $N(cl_N(B)) = N(B)$;

(psy) $B \in \mathcal{B}^X \backslash \{\varnothing\}$ and $\rho \in N(B)$ imply $\{B\} \in \cap\{N(F) : F \in \rho \cap \mathcal{B}^X\}$.

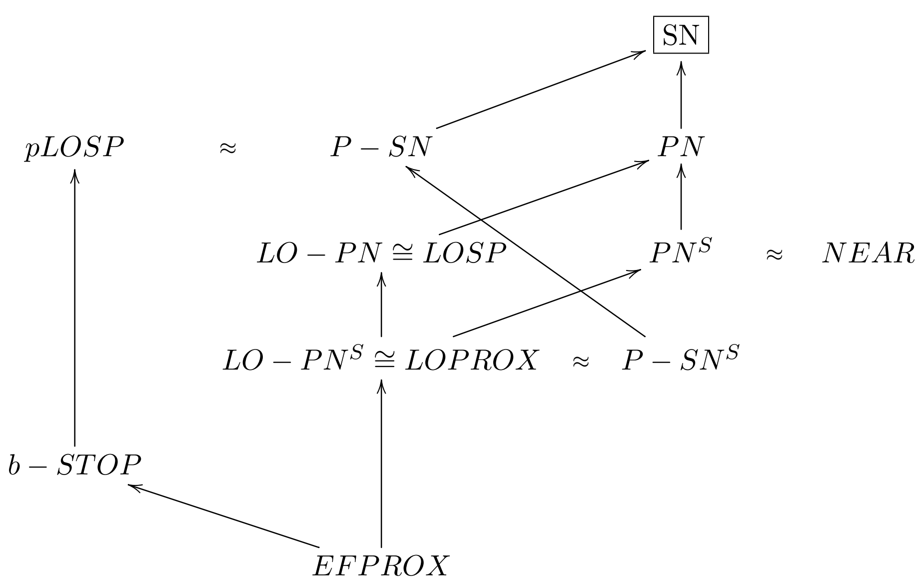
We denote by P-SN the corresponding full subcategory of CG-SN (see also 2.7.).

Theorem 5.5. *The category pLOSP is isomorphic to the category P-SN.*

Proof. According to 1.2.(iii) and 5.1 respectively we conversely set for a proximal supernear space $(Y, \mathcal{B}^Y, M)$ $B_{P_M} A$ iff $\{A\} \in M(B)$ for each $B \in \mathcal{B}^X, A \subset X$. □

Remark 5.6. According to [10] we claim that in the "saturated" case the category LOPROX of LODATO proximity spaces and p-maps is isomorphic to the category LO-PN^S as well as to the category P-SN^S. Hence, LO-PN^S and P-SN^S are isomorphic, too.

Diagram 5.7. (Some relationship between former mentioned categories).



Remark 5.8. Now, it seems to be of interest to characterize those proximal supernear spaces, whichever are induced by a topological space Y such that B-near collections are described by the fact that the closure of its members in Y meets the corresponding one of B in Y. But we will solve this problem in a forthcoming paper and only mention here that its solution is leading us to a further generalization of LODATO's theorem, moreover Doîtchinov's result also can be dealt with.

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