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Generalized Quasilinearization Method and Cubic Convergence for Integro-Differential Equations

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Abstract. In this paper we develop the generalized quasilinearization method for partial integro-differential equations of parabolic type. We consider the situation when the nonlinearties satisfy a regularity, a monotonicity, and a Lipschitz condition. Using the natural upper and lower solutions we develop two sequences whose elements are solutions of simpler nonlinear differential equations, and the sequences converge uniformly and monotonically to the unique solution of the nonlinear integro-differential equation.We further prove that the rate of convergence is cubic. As an application a numerical example is presented.

AMS Subject Classifications: 35K57, 35K60

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1. Introduction

The method of quasilinearization [1, 2] combined with the technique of upper and lower solutions has been extended recently to a wide variety of nonlinear problems. It has been referred to as a generalized quasilinearization method. See [3, 7, 9] for details and [10, 11] for applications.

In the nuclear reactor model if the effect of the temperature feedback is taken into consideration the neutron flux $u \equiv u(t, x)$ is governed by a Volterra type integro-differential equation. On the other hand, in the study of nerve propagation, a simplified Hodgkin-Huxley model (see [15]) for the propagation of a voltage pulse through a

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nerve axon is governed by a similar Volterra type integro-differential equation. They also occur in structured population models. Motivated by above models we consider nonlinear parabolic integro-differential equations in this paper. In [16] the authors obtained by using generalized quasilinearization method a quadratic order of convergence for nonlinear integro-differential equations of parabolic type. They have considered situations when the forcing function is convex and they have used linear iterates to obtain the solution of the nonlinear integro-differential equation. In addition, the rate of convergence is quadratic. However, in [4] they have used generalized monotone method and in [12] extended quasilinearization method to obtain higher order of convergence for ordinary differential equations. See [6, 7] for monotone method for a variety of nonlinear problems. In this paper we extend the above results when the second derivative of the forcing function is nondecreasing in u and satisfies a one sided Lipschitz condition in u. Using an appropriate iterative scheme and lower and upper solutions under suitable conditions, we obtain natural sequences which converge to the unique solution of the nonlinear integro-differential equations of Volterra's type and the rate of convergence is cubic. Finally, we provide a numerical example to demonstrate the applicability of generalized quasilinearization method we have developed here to solve nonlinear parabolic integro-differential equations. For recent results on higher order of convergence see [13, 14].

2. Preliminaries

In this section we list the assumptions and recall some known existence and comparison theorems which we need in our main result.

Let us consider a nonlinear second order parabolic integro-differential equation of the form

$$\begin{aligned} \mathcal{L}u &= f(t,x,u(t,x)) + \int_0^t g(t,x,s,u(s,x))ds \qquad \text{in} \quad Q_T, \\ u(t,x) &= \Phi(t,x), \qquad x \in \partial\Omega, \\ u(0,x) &= u_0(x), \qquad x \in \Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded domain in R^m with boundary $\partial\Omega \in C^{2+\gamma}$ ($\gamma \in (0,1)$) and closure $\overline{\Omega}$, $Q_T = (0,T) \times \Omega$, $\overline{Q}_T = [0,T] \times \overline{\Omega}$, $T > 0$. Let $\mathcal{L}$ be a second order differential operator defined by

$$\mathcal{L} = \frac{\partial}{\partial t} - L, \tag{2.2}$$

where

$$L = \sum_{i,j=1}^{m} a_{i,j}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(t,x)\frac{\partial}{\partial x_i}. \tag{2.3}$$

Here we recall some known auxiliary results and list the following assumptions for convenience which will be needed for our main result.

(A_0) (i) For each $i,j = 1,\ldots,m$, $a_{i,j}, b_j \in C^{\frac{\gamma}{2},\gamma}[\overline{Q}_T, R]$, and $\mathcal{L}$ is strictly uniformly parabolic in $\overline{Q}_T$, that means $a_{i,j}$, b_j are *Hölder* continuous of order $\frac{\gamma}{2}$ and γ in t and x respectively;

(*ii*) $\partial\Omega$ belongs to the class $C^{2+\gamma}$, that means the boundary is *Hölder* continuous of order $2+\gamma$;

(*iii*) $f \in C^{\frac{\gamma}{2},\gamma}[[0,T]\times\overline{\Omega}\times R, R]$, $g \in C^{\frac{\gamma}{2},\gamma}[[0,T]\times\overline{\Omega}\times R^2, R]$ that is $f(t,x,u)$, $g(t,x,u)$ are *Hölder* continuous in t and (x,u) with exponents $\frac{\gamma}{2}$ and γ, respectively;

(*iv*) $\Phi \in C^{1+\frac{\gamma}{2},2+\gamma}[[0,T]\times\partial\Omega, R]$ and $u_0(x) \in C^{2+\gamma}[\overline{\Omega}, R]$; (Note that Φ and $u_0(x)$ are *Hölder* continuous in t and x of the appropriate order mentioned above.)

(*v*) $u_0(x) = \Phi(0,x), \quad \Phi_t = Lu_0 + f(0,x,u_0)$ for $t=0$ and $x \in \partial\Omega$.

We need the following definition.

Definition 2.1. *The functions* α_0 , $\beta_0 \in C^{1,2}[\overline{Q}_T, R]$ *with* $g(t,x,u)$ *nondecreasing in* u *are said to be lower and upper solutions of (2.1), respectively, if*

$$\begin{aligned}
\mathcal{L}\alpha_0 &\le f(t,x,\alpha_0(t,x)) + \int_0^t g(t,x,s,\alpha_0(s,x))ds \qquad in \quad Q_T,\\
\alpha_0(t,x) &\le \Phi(t,x), \qquad x \in \partial\Omega,\\
\alpha_0(0,x) &\le u_0(x), \qquad x \in \Omega,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}\beta_0 &\ge f(t,x,\beta_0(t,x)) + \int_0^t g(t,x,s,\beta_0(s,x))ds \qquad in \quad Q_T,\\
\beta_0(t,x) &\ge \Phi(t,x), \qquad x \in \partial\Omega,\\
\beta_0(0,x) &\ge u_0(x), \qquad x \in \Omega.
\end{aligned}$$

Next we recall a known existence theorem for (2.1) which we need in our main results.

Theorem 2.1. *Assume that* (A_0) *holds. Then (2.1) has a unique smooth solution* $u(t,x) \in C^{1+\frac{\gamma}{2},2+\gamma}[\overline{Q}_T, R]$.

See [5] for details.
Also we recall positivity and comparison theorems which we need to prove the monotonicity and the order of convergence in our main result.

Theorem 2.2. *Let* $u(t,x) \in C^{\frac{1+\gamma}{2},1+\gamma}[\overline{Q}_T, R]$ *be such that*

$$\begin{aligned}
\mathcal{L}u + cu &\ge 0 \quad in \quad Q_T,\\
u(t,x) &\ge 0, \quad x \in \partial\Omega,\\
u(0,x) &\ge 0, \quad x \in \Omega,
\end{aligned}$$

and $c \equiv c(t,x)$ *is a bounded function in* Q_T. *Then* $u(t,x) \ge 0$ *in* $\overline{Q}_T$.

See [15] for details.

Theorem 2.3. *Assume that*

(i) $f_u(t,x,u)$ *and* $g_u(t,x,s,u)$ *are bounded functions with* $g(t,x,s,u)$ *nondecreasing in* u *on* $\overline{Q}_T$.

(ii) $\alpha(t,x)$ *and* $\beta(t,x)$ *satisfy*

$$\begin{array}{lcll} \mathcal{L}\alpha & \leq & f(t,x,\alpha(t,x)) + \int_0^t g(t,x,s,\alpha(s,x))ds & \text{in} \quad Q_T, \\ \mathcal{L}\beta & \geq & f(t,x,\beta(t,x)) + \int_0^t g(t,x,s,\beta(s,x))ds & \text{in} \quad Q_T, \end{array}$$

with

$$\begin{array}{lcll} \alpha(t,x) & \leq & \beta(t,x), & x \in \partial\Omega, \\ \alpha(0,x) & \leq & \beta(0,x), & x \in \Omega. \end{array}$$

Then $\alpha(t,x) \leq \beta(t,x)$ *on* $\overline{Q}_T$.

See [16] for details.
The next comparison result follows from Lemma 6.2 in [3] and [8] .

Theorem 2.4. *Suppose that*

(i) $g(t,x,s,u)$ *is monotone nondecreasing in* u *for each fixed point (t,x,s),*

(ii) $\alpha(t,x)$ *satisfies*

$$\begin{array}{rcll} \mathcal{L}\alpha & \leq & f(t,x,\alpha(t,x)) + \int_0^t g(t,x,s,\alpha(s,x))ds & \text{in} \quad Q_T, \\ \alpha(t,x) & = & 0, \qquad x \in \partial\Omega, & \\ \alpha_0(0,x) & = & u_0(x), \qquad x \in \Omega, & \end{array}$$

(iii) $r(t)$ *is the solution of the following ordinary integro-differential equation*

$$\begin{array}{rcl} r' & = & h_1(t,r) + \int_0^t h_2(t,s,r))ds, \\ r(0) & = & \max\{\max_{x\in\Omega} u_0(x), 0\}, \end{array}$$

where

$$h_1(t,r) \geq \max_{x\in\Omega} f(t,x,r) \quad \text{and} \quad h_2(t,s,r) \geq \max_{x\in\Omega} g(t,x,s,r).$$

Then $\alpha(t,x) \leq r(t)$ *on* $\overline{Q}_T$.

3. Main Results

In this section we extend the method of generalized quasilinearization to (2.1) with cubic order of convergence. This has been achieved under weaker assumptions than the usual convexity assumption to the nonlinear integro-differential equations. We obtain cubic convergence when the nonlinearity of the iterates is quadratic. This is precisely our main result, which we state below.

Theorem 3.1. *Assume that all of* (A_0) *holds except* (*iii*)*; further assume that*

(*i*) α_0, β_0 *are lower and upper solutions of (2.1) with* $\alpha_0(t,x) \leq \beta_0(t,x)$ *on* $\overline{Q}_T$.

(*ii*) $\dfrac{\partial^l f(t,x,u)}{\partial u^l}$, $\dfrac{\partial^l g(t,x,s,u)}{\partial u^l}$ *exist and are bounded functions on* $\overline{Q}_T$ *for* $l = 0,1,2$ *such that* $\dfrac{\partial f^l(t,x,u)}{\partial u^l}, \dfrac{\partial^l g(t,x,s,u)}{\partial u^l} \in C^{\frac{\gamma}{2},\gamma}[Q_T \times R, R]$.

(*iii*) *Also* g *is a nondecreasing function in* u *on* $\overline{Q}_T$ *such that*

$$g_u(\alpha_0) \geq g_{uu}(\beta_0)(\beta_0 - \alpha_0)$$

and

$$0 \leq \frac{\partial^2 f(t,x,\eta_1)}{\partial u^2} - \frac{\partial^2 f(t,x,\eta_2)}{\partial u^2} \leq M_1(\eta_1 - \eta_2) \quad \text{on } \overline{Q}_T,$$

$$0 \leq \frac{\partial^2 g(t,x,\xi_1)}{\partial u^2} - \frac{\partial^2 g(t,x,\xi_2)}{\partial u^2} \leq M_2(\xi_1 - \xi_2) \quad \text{on } \overline{Q}_T,$$

whenever

$$\alpha_0(t,x) \leq \eta_2(t,x) \leq \eta_1(t,x) \leq \beta_0(t,x),$$
$$\alpha_0(t,x) \leq \xi_2(t,x) \leq \xi_1(t,x) \leq \beta_0(t,x).$$

Then there exist monotone sequences $\{\alpha_n(t,x)\}$, $\{\beta_n(t,x)\}$, $n \geq 0$ *which converge uniformly and monotonically to the unique solution of (2.1) and the convergence is of order* 3.

Proof. Let us first consider the following equations:

$$\begin{aligned}
\mathcal{L}w &= F_1(t,x,\alpha;w) + \textstyle\int_0^t G_1(t,x,s,\alpha(s,x);w(s,x))ds \\
&= \sum_{i=0}^{2} \frac{\partial^i f(t,x,\alpha)}{\partial u^i}\frac{(w-\alpha)^i}{i!} \\
&\quad + \int_0^t \sum_{i=0}^{2} \frac{\partial^i g(t,x,s,\alpha(s,x))}{\partial u^i}\frac{(w(s,x)-\alpha(s,x))^i}{i!}ds \qquad \text{in} \quad Q_T, \\
w(t,x) &= \Phi(t,x), \qquad x \in \partial\Omega, \\
w(0,x) &= u_0(x), \qquad x \in \Omega,
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\mathcal{L}v &= F_2(t,x,\beta;v) + \textstyle\int_0^t G_2(t,x,s,\beta(s,x);v(s,x))ds \\
&= \sum_{i=0}^{2} \frac{\partial^i f(t,x,\beta)}{\partial u^i}\frac{(v-\beta)^i}{i!} \\
&\quad + \int_0^t \sum_{i=0}^{2} \frac{\partial^i g(t,x,s,\beta(s,x))}{\partial u^i}\frac{(v(s,x)-\beta(s,x))^i}{i!}ds \qquad \text{in} \quad Q_T, \\
v(t,x) &= \Phi(t,x), \qquad x \in \partial\Omega, \\
v(0,x) &= u_0(x), \qquad x \in \Omega,
\end{aligned} \tag{3.2}$$

where $\alpha(t,x) \le v, w \le \beta(t,x)$ and $\alpha(0,x) \le u_0(x) \le \beta(0,x)$.

Initially, we prove (α_0, β_0) are lower and upper solutions of (3.1) and (3.2), respectively.

Let $\alpha = \alpha_0$ and $\beta = \beta_0$ in (3.1). Then we have

$$\begin{aligned} \mathcal{L}\alpha_0 &\le f(t,x,\alpha_0) + \textstyle\int_0^t g(t,x,s,\alpha_0(s,x))ds \\ &= F_1(t,x,\alpha_0;\alpha_0) + \textstyle\int_0^t G_1(t,x,s,\alpha_0(s,x);\alpha_0(s,x))ds, \\ \alpha_0(t,x) &\le \Phi(t,x), \qquad x \in \partial\Omega, \\ \alpha_0(0,x) &\le u_0(x), \qquad x \in \Omega, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathcal{L}\beta_0 &\ge f(t,x,\beta_0) + \textstyle\int_0^t g(t,x,s,\beta_0(s,x))ds \\ &= \sum_{i=0}^{1} \frac{\partial^i f(t,x,\alpha_0)}{\partial u^i} \frac{(\beta_0-\alpha_0)^i}{i!} + \frac{\partial^2 f(t,x,\ \xi_1)}{\partial u^2} \frac{(\beta_0-\alpha_0)^2}{(2)!} \\ &\quad + \int_0^t \left[\sum_{i=0}^{1} \frac{\partial^i g(t,x,s,\alpha_0)}{\partial u^i} \frac{(\beta_0-\alpha_0)^i}{i!} + \frac{\partial^2 g(t,x,s,\ \xi_2)}{\partial u^2} \frac{(\beta_0-\alpha_0)^2}{(2)!} \right] ds \\ &\ge \sum_{i=0}^{2} \frac{\partial^i f(t,x,\alpha_0)}{\partial u^i} \frac{(\beta_0-\alpha_0)^i}{i!} + \int_0^t \left[\sum_{i=0}^{2} \frac{\partial^i g(t,x,s,\alpha_0)}{\partial u^i} \frac{(\beta_0-\alpha_0)^i}{i!} \right] ds \\ &= F_1(t,x,\alpha_0;\beta_0) + \textstyle\int_0^t G_1(t,x,s,\alpha_0;\beta_0)ds, \\ \beta_0(t,x) &\ge \Phi(t,x), \qquad x \in \partial\Omega, \\ \beta_0(0,x) &\ge u_0(x), \qquad x \in \Omega, \end{aligned} \tag{3.4}$$

where $\alpha_0 \le \xi_1, \xi_2 \le \beta_0$.

By (3.3) and (3.4) we can conclude that α_0 and β_0 are the lower and upper solutions of (3.1). To apply Theorem 2.1 we need to verify (iii) of (A_0) relative to the equation (3.1). For $\eta \in C^{\frac{1+\gamma}{2},1+\gamma}[\overline{Q_T}, R]$ such that $\alpha_0(x,t) \le w(x,t), \eta(t,x) \le \beta_0(t,x)$ on $\overline{Q}_T$ we have

$$\begin{aligned} F_1(t,x,\eta;w) &= \sum_{i=0}^{2} \frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i} \frac{[w(t,x)-\eta(t,x)]^i}{i!} \\ &= \sum_{i=0}^{2} \frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i} \frac{\sum_{j=0}^{i}(-1)^j \binom{i}{j} w^{i-j}(t,x)\eta^j(t,x)}{i!} \\ &= \sum_{i=0}^{2}\sum_{j=0}^{i} \frac{(-1)^j\binom{i}{j}}{i!} \frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i} w^{i-j}(t,x)\eta^j(t,x) \\ &= \sum_{i=0}^{2}\sum_{j=0}^{i} K_{i,j} d_{i,j}(t,x) w^{i-j}(t,x), \end{aligned}$$

where $K_{i,j} = \frac{(-1)^j\binom{i}{j}}{i!}$ and $d_{i,j}(t,x) = \frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i}\eta^j(t,x)$. We need to prove that $d_{i,j}(t,x)$ belongs to $C^{\frac{\gamma}{2},\gamma}[\overline{Q_T}, R]$ for $i,j = 0,1,2$. We will only show the details for one term $d_{i,j}(t,x)$ when $|\eta| \leq C_1$ and $|\frac{\partial^i f}{\partial u^i}| \leq C_2$.

$$\begin{aligned}
|d_{i,j}(t,x) - d_{i,j}(\bar{t},x)| &= \Big|\frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i}\eta^j(t,x) - \frac{\partial^i f(\bar{t},x,\eta(\bar{t},x))}{\partial u^i}\eta^j(\bar{t},x)\Big| \\
&\leq \Big|\frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i}\eta^j(t,x) - \frac{\partial^i f(\bar{t},x,\eta(\bar{t},x))}{\partial u^i}\eta^j(t,x)\Big| \\
&+\Big|\frac{\partial^i f(\bar{t},x,\eta(\bar{t},x))}{\partial u^i}\eta^j(t,x) - \frac{\partial^i f(\bar{t},x,\eta(\bar{t},x))}{\partial u^i}\eta^j(\bar{t},x)\Big| \\
&= |\eta^j(t,x)|\Big|\frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i} - \frac{\partial^i f(\bar{t},x,\eta(\bar{t},x))}{\partial u^i}\Big| \\
&+\Big|\frac{\partial^i f(\bar{t},x,\eta(\bar{t},x))}{\partial u^i}\Big||\eta(t,x) - \eta(\bar{t},x)|\Big|\sum_{l=0}^{j-1}\eta^{j-l-1}(t,x)\eta^l(\bar{t},x)\Big| \\
&\leq C_1^j C_t(\frac{\partial^i f}{\partial u^i})(|t-\bar{t}|^{\frac{\gamma}{2}} + C_t(\eta)|t-\bar{t}|^{\frac{1+\gamma}{2}}) \\
&\quad +jC_1^2 C_2 C_t(\eta)|t-\bar{t}|^{\frac{1+\gamma}{2}} \\
&\leq C_t(F)|t-\bar{t}|^{\frac{\gamma}{2}},
\end{aligned}$$

where $C_t(F)$ depends on C_1, C_2, $C_t(\frac{\partial^i f}{\partial u^i})$, $C_t(\eta)$, and T . This shows that $F_1(t,x,\alpha;w)$ is Hölder continuous in t with exponent $\frac{\gamma}{2}$. Similarly, we can prove that $F_1(t,x,\alpha;w)$ is Hölder continuous in (x,w) with exponent γ. That is:

$$\begin{aligned}
|d_{i,j}(t,x) - d_{i,j}(t,\bar{x})| &= \Big|\frac{\partial^i f(t,x,\eta(t,x))}{\partial u^i}\eta^j(t,x) - \frac{\partial^i f(t,\bar{x},\eta(t,\bar{x}))}{\partial u^i}\eta^j(t,\bar{x})\Big| \\
&\leq C_1^j C_x(\tfrac{\partial^i f}{\partial u^i})(\|x-\bar{x}\|^\gamma + C_x(\eta)\|x-\bar{x}\|^{1+\gamma}) \\
&\quad +jC_1^2 C_2 C_x(\eta)\|x-\bar{x}\|^{1+\gamma} \\
&\leq C_{x,w}(F)|x-\bar{x}|^\gamma,
\end{aligned}$$

where $C_{x,w}(F)$ depends on C_1, C_2, $C_x(\frac{\partial^i f}{\partial u^i})$, and $C_x(\eta)$. Hence $F_1(t,x,\alpha;w)$ is Hölder continuous in t and (x,w) with exponents $\frac{\gamma}{2}$ and γ, respectively. The proof that $G_1(t,x,\alpha;w)$ is Hölder continuous in t and (x,w) with exponents $\frac{\gamma}{2}$ and γ, respectively, follows the same lines. Similar conclusions hold for $F_2(t,x,\beta;v)$ and $G_2(t,x,\beta;v)$. It follows by Theorem 2.1 that there exists a unique solution α_1 of (3.1). One can prove that $\alpha_0 \leq \alpha_1 \leq \beta_0$. Let $\mu = \alpha_1 - \alpha_0$. Then it follows that

$$\begin{aligned}
\mathcal{L}\mu &= \mathcal{L}(\alpha_1 - \alpha_0) \\
&\geq F_1(t,x,\alpha_0;\alpha_1) + \textstyle\int_0^t G_1(t,x,s,\alpha_0;\alpha_1)ds \\
&\quad -F_1(t,x,\alpha_0;\alpha_0) - \textstyle\int_0^t G_1(t,x,s,\alpha_0;\alpha_0)ds \\
&= F_{1u}(t,x,\alpha_0;\xi_1)\mu + \textstyle\int_0^t G_{1u}(t,x,s,\alpha_0;\xi_2)\mu ds, \\
\mu(t,x) &= 0, \qquad x \in \partial\Omega, \\
\mu(0,x) &= 0, \qquad x \in \Omega.
\end{aligned}$$

Using this and applying Theorem 2.2 one can obtain $\mu \geq 0$ or $\alpha_0 \leq \alpha_1$.

Next let set $\mu = \beta_0 - \alpha_1$. Then

$$\begin{aligned}
\mathcal{L}\mu &= \mathcal{L}(\beta_0 - \alpha_1) \\
&\geq F_1(t,x,\alpha_0;\beta_0) + \int_0^t G_1(t,x,s,\alpha_0;\beta_0)ds \\
&\quad -F_1(t,x,\alpha_0;\alpha_1) - \int_0^t G_1(t,x,s,\alpha_0;\alpha_1)ds \\
&= F_{1_u}(t,x,\alpha_0;\xi_1)\mu + \int_0^t G_{1_u}(t,x,s,\alpha_0;\xi_2)\mu ds, \\
\mu(t,x) &= 0, \qquad x \in \partial\Omega, \\
\mu(0,x) &= 0, \qquad x \in \Omega.
\end{aligned}$$

By Theorem 2.2 and above inequalities one can conclude that $\mu \geq 0$ or $\alpha_1 \leq \beta_0$. Similarly we will prove now that (α_0, β_0) are lower and upper solutions of (3.2). Set $\alpha = \alpha_0$ and $\beta = \beta_0$ in (3.2). Then we get

$$\begin{aligned}
\mathcal{L}\beta_0 &\geq f(t,x,\beta_0) + \int_0^t g(t,x,s,\beta_0(s,x))ds \\
&= F_2(t,x,\beta_0;\beta_0) + \int_0^t G_2(t,x,s,\beta_0(s,x);\beta_0(s,x))ds, \\
\beta_0(t,x) &\geq \Phi(t,x), \qquad x \in \partial\Omega, \\
\beta_0(0,x) &\geq u_0(x), \qquad x \in \Omega,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\mathcal{L}\alpha_0 &\leq f(t,x,\alpha_0) + \int_0^t g(t,x,s,\alpha_0(s,x))ds \\
&= \sum_{i=0}^{1} \frac{\partial^i f(t,x,\beta_0)}{\partial u^i}\frac{(\alpha_0-\beta_0)^i}{i!} + \frac{\partial^2 f(t,x,\ \xi_1)}{\partial u^2}\frac{(\alpha_0-\beta_0)^2}{(2)!} \\
&\quad + \int_0^t \Big[\sum_{i=0}^{1} \frac{\partial^i g(t,x,s,\beta_0)}{\partial u^i}\frac{(\alpha_0-\beta_0)^i}{i!} + \frac{\partial^2 g(t,x,s,\ \xi_2)}{\partial u^2}\frac{(\alpha_0-\beta_0)^2}{(2)!}\Big]ds \\
&\leq \sum_{i=0}^{2} \frac{\partial^i f(t,x,\beta_0)}{\partial u^i}\frac{(\alpha_0-\beta_0)^i}{i!} + \int_0^t \sum_{i=0}^{2} \frac{\partial^i g(t,x,s,\beta_0)}{\partial u^i}\frac{(\alpha_0-\beta_0)^i}{i!}ds \\
&= F_2(t,x,\beta_0;\alpha_0) + \int_0^t G_2(t,x,s,\beta_0;\alpha_0)ds, \\
\alpha_0(t,x) &\leq \Phi(t,x), \qquad x \in \partial\Omega, \\
\alpha_0(0,x) &\leq u_0(x), \qquad x \in \Omega,
\end{aligned} \tag{3.6}$$

where $\alpha_0 \leq \xi_1, \xi_2 \leq \beta_0$.

One can conclude that α_0 and β_0 are the lower and upper solutions of (3.2) considering (3.5) and (3.6). By Theorem 2.1 there exists a unique solution β_1 of (3.2). We show that $\alpha_0 \leq \beta_1 \leq \beta_0$. Let $\mu = \beta_0 - \beta_1$. Then we have

$$\begin{aligned}
\mathcal{L}\mu &= \mathcal{L}(\beta_0 - \beta_1) \\
&\geq F_2(t,x,\beta_0;\beta_0) + \int_0^t G_2(t,x,s,\beta_0;\beta_0)ds \\
&\quad -F_2(t,x,\beta_0;\beta_1) - \int_0^t G_2(t,x,s,\beta_0;\beta_1)ds \\
&= F_{2_u}(t,x,\beta_0;\xi_1)\mu + \int_0^t G_{2_u}(t,x,s,\beta_0;\xi_2)\mu ds, \\
\mu(t,x) &= 0, \qquad x \in \partial\Omega, \\
\mu(0,x) &= 0, \qquad x \in \Omega.
\end{aligned}$$

Applying again Theorem 2.2 we can conclude that $\mu \geq 0$ or $\beta_0 \geq \beta_1$.

Next set $\mu = \beta_1 - \alpha_0$. Then

$$\begin{aligned}
\mathcal{L}\mu &= \mathcal{L}(\beta_1 - \alpha_0) \\
&\geq F_2(t,x,\beta_0;\beta_1) + \textstyle\int_0^t G_2(t,x,s,\beta_0;\beta_1)ds \\
&\quad - F_2(t,x,\beta_0;\alpha_0) - \textstyle\int_0^t G_2(t,x,s,\beta_0;\alpha_0)ds \\
&= F_{2u}(t,x,\beta_0;\xi_1)\mu + \textstyle\int_0^t G_{2u}(t,x,s,\beta_0;\xi_2)\mu ds, \\
\mu(t,x) &= 0, \qquad x \in \partial\Omega, \\
\mu(0,x) &= 0, \qquad x \in \Omega.
\end{aligned}$$

Using again Theorem 2.2 we can obtain that $\mu \geq 0$ or $\beta_1 \geq \alpha_0$. Hence $\alpha_0 \leq \beta_1 \leq \beta_0$.

Next we prove that $\beta_1 \geq \alpha_1$. We can get

$$\begin{aligned}
f(t,x,\alpha_1) &+ \textstyle\int_0^t g(t,x,s,\alpha_1(s,x))ds \\
&= \sum_{i=0}^{1} \frac{\partial^i f(t,x,\alpha_0)}{\partial u^i}\frac{(\alpha_1-\alpha_0)^i}{i!} + \frac{\partial^2 f(t,x,\ \xi_1)}{\partial u^2}\frac{(\alpha_1-\alpha_0)^2}{(2)!} \\
&\quad + \int_0^t \Big[\sum_{i=0}^{1} \frac{\partial^i g(t,x,s,\alpha_0)}{\partial u^i}\frac{(\alpha_1-\alpha_0)^i}{i!} + \frac{\partial^2 g(t,x,s,\ \xi_2)}{\partial u^2}\frac{(\alpha_1-\alpha_0)^2}{(2)!}\Big]ds \\
&\geq \sum_{i=0}^{2} \frac{\partial^i f(t,x,\alpha_0)}{\partial u^i}\frac{(\alpha_1-\alpha_0)^i}{i!} + \int_0^t \Big[\sum_{i=0}^{2} \frac{\partial^i g(t,x,s,\alpha_0)}{\partial u^i}\frac{(\alpha_1-\alpha_0)^i}{i!}\Big]ds \qquad (3.7) \\
&= F_1(t,x,\alpha_0;\alpha_1) + \textstyle\int_0^t G_1(t,x,s,\alpha_0;\alpha_1) \\
&= \mathcal{L}\alpha_1, \\
\alpha_1(t,x) &= \Phi(t,x), \qquad x \in \partial\Omega, \\
\alpha_1(0,x) &= u_0(x), \qquad x \in \Omega,
\end{aligned}$$

and

$$\begin{aligned}
f(t,x,\beta_1) &+ \textstyle\int_0^t g(t,x,s,\beta_1(s,x))ds \\
&= \sum_{i=0}^{1} \frac{\partial^i f(t,x,\beta_0)}{\partial u^i}\frac{(\beta_1-\beta_0)^i}{i!} + \frac{\partial^2 f(t,x,\ \xi_1)}{\partial u^2}\frac{(\beta_1-\beta_0)^2}{(2)!} \\
&\quad + \int_0^t \Big[\sum_{i=0}^{1} \frac{\partial^i g(t,x,s,\beta_0)}{\partial u^i}\frac{(\beta_1-\beta_0)^i}{i!} + \frac{\partial^2 g(t,x,s,\ \xi_2)}{\partial u^2}\frac{(\beta_1-\beta_0)^2}{(2)!}\Big]ds \\
&\leq \sum_{i=0}^{2} \frac{\partial^i f(t,x,\beta_0)}{\partial u^i}\frac{(\beta_1-\beta_0)^i}{i!} + \int_0^t \Big[\sum_{i=0}^{2} \frac{\partial^i g(t,x,s,\beta_0)}{\partial u^i}\frac{(\beta_1-\beta_0)^i}{i!}\Big]ds \qquad (3.8) \\
&= F_2(t,x,\beta_0;\beta_1) + \textstyle\int_0^t G_2(t,x,s,\beta_0;\beta_1) \\
&= \mathcal{L}\beta_1, \\
\beta_1(t,x) &= \Phi(t,x), \qquad x \in \partial\Omega, \\
\beta_1(0,x) &= u_0(x), \qquad x \in \Omega.
\end{aligned}$$

By (3.7) and (3.8) together with Theorem 2.3 one can obtain that $\beta_1 \geq \alpha_1$.

Hence we have $\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0$. Using this inequality and the method of mathematical induction, one can show that

$$\alpha_0 \leq \alpha_1 \leq ... \leq \alpha_n \leq \beta_n \leq ... \leq \beta_1 \leq \beta_0 \qquad \text{for all } n.$$

Let u be any solution of (2.1) such that $\alpha_0 \leq u \leq \beta_0$ with $\alpha_0(0) \leq u_0 \leq \beta_0(0)$ on $\overline{Q}_T$. Suppose for some u , we have $\alpha_n \leq u \leq \beta_n$ on $\overline{Q}_T$. Set $\Phi_1 = u - \alpha_{n+1}$, $\Phi_2 = \beta_{n+1} - u$ so that

$$\begin{aligned}
\mathcal{L}\Phi_1 &= \mathcal{L}u - \mathcal{L}\alpha_{n+1} \\
&= f(t,x,u) + \textstyle\int_0^t g(t,x,s,u(s,x))ds \\
&\quad - \sum_{i=0}^{2} \frac{\partial^i f(t,x,\alpha_n)}{\partial u^i} \frac{(\alpha_{n+1}-\alpha_n)^i}{i!} \\
&\quad - \int_0^t \sum_{i=0}^{2} \frac{\partial^i g(t,x,s,\alpha_n(s,x))}{\partial u^i} \frac{(\alpha_{n+1}-\alpha_n)^i}{i!} ds \\
&\geq f(t,x,u) - \ f(t,x,\alpha_{n+1}) + \textstyle\int_0^t [g(t,x,s,u) - \ g(t,x,s,\alpha_{n+1}(s,x))]ds \\
&\geq f_u(t,x,\eta_1)\Phi_1 + \textstyle\int_0^t [g_u(t,x,s,\eta_2)\Phi_1]ds, \\
\Phi_1(t,x) &= 0, \qquad x \in \partial\Omega, \\
\Phi_1(0,x) &= 0, \qquad x \in \Omega,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\Phi_2 &= \mathcal{L}\beta_{n+1} - \mathcal{L}u \\
&= -f(t,x,u) - \textstyle\int_0^t g(t,x,s,u(s,x))ds \\
&\quad + \sum_{i=0}^{2} \frac{\partial^i f(t,x,\beta_n)}{\partial u^i} \frac{(\beta_{n+1}-\beta_n)^i}{i!} \\
&\quad + \ \textstyle\int_0^t \sum_{i=0}^{2} \frac{\partial^i g(t,x,s,\beta_n(s,x))}{\partial u^i} \frac{(\beta_{n+1}-\beta_n)^i}{i!} ds \\
&\geq -f(t,x,u) + \ f(t,x,\beta_{n+1}) \\
&\quad + \textstyle\int_0^t [-g(t,x,s,u(s,x)) + \ g(t,x,s,\beta_{n+1}(s,x))]ds \\
&\geq f_u(t,x,\eta_3)\Phi_2 + \textstyle\int_0^t [g_u(t,x,s,\eta_4)\Phi_2]ds, \\
\Phi_2(t,x) &= 0, \qquad x \in \partial\Omega, \\
\Phi_2(0,x) &= 0, \qquad x \in \Omega,
\end{aligned}$$

where $\eta_1,\ \eta_2$ are between u and α_{n+1}, and $\eta_3,\ \eta_4$ are between u and β_{n+1}. It is clear that $\alpha_{n+1} \leq u \leq \beta_{n+1}$ by Theorem 2.2. Since $\alpha_0 \leq u \leq \beta_0$, this proves by induction that $\alpha_n \leq u \leq \beta_n$ for all n. From this we can conclude

$$\alpha_0 \leq \alpha_1 \leq ... \leq \alpha_n \leq u \leq \beta_n \leq ... \leq \beta_1 \leq \beta_0.$$

Since $\{\alpha_n(t,x)\}$ and $\{\beta_n(t,x)\}$ are in $C^{1+\frac{\gamma}{2},2+\gamma}[\overline{Q}_T, R]$, one can show that these sequences converge to (ρ, r) using the same technique as in [15].

That is

$$\lim_{n\to\infty} \alpha_n(t,x) = \rho(t,x) \leq u \leq r(t,x) = \lim_{n\to\infty} \beta_n(t,x).$$

Now we need to prove that $\rho(t,x) \geq r(t,x)$. From (3.1) and (3.2) we get

$$\begin{aligned}
\mathcal{L}\rho(t,x) &= F_1(t,x,\rho;\rho) \ + \textstyle\int_0^t G_1(t,x,s,\rho(s,x);\rho(s,x))ds \\
&= f(t,x,\rho) + \textstyle\int_0^t g(t,x,s,\rho(s,x))ds, \\
\rho(t,x) &= \Phi(t,x), \qquad x \in \partial\Omega, \\
\rho(0,x) &= u_0(x), \qquad x \in \Omega,
\end{aligned}$$

and

$$\begin{aligned} \mathcal{L}r(t,x) &= F_2(t,x,r;r) \ + \textstyle\int_0^t G_2(t,x,s,r(s,x);r(s,x))ds \\ &= f(t,x,r) + \textstyle\int_0^t g(t,x,s,r(s,x))ds, \\ r(t,x) &= \Phi(t,x), \qquad x \in \partial\Omega, \\ r(0,x) &= u_0(x), \qquad x \in \Omega. \end{aligned}$$

Setting $\Theta = r(t,x) - \rho(t,x)$, we get

$$\begin{aligned} \mathcal{L}\Theta &= \mathcal{L}r - \mathcal{L}\rho \\ &= f(t,x,r) + \textstyle\int_0^t g(t,x,s,r(s,x))ds - f(t,x,\rho) - \int_0^t g(t,x,s,\rho(s,x))ds \\ &\le L_1(r-\rho) + \textstyle\int_0^t L_2(r-\rho)ds \\ &\le L_1\Theta + \textstyle\int_0^t L_2\Theta ds, \qquad L_1,\ L_2 \ge 0, \\ \Theta(t,x) &= 0, \qquad x \in \partial\Omega, \\ \Theta(0,x) &= 0, \qquad x \in \Omega, \end{aligned}$$

using assumptions (iii) of the hypotesis. Now by Theorem 2.2 we can conclude that $r(t,x) \le \rho(t,x)$. This proves $r(t,x) = \rho(t,x) = u(t,x)$ is the unique solution of (2.1). Hence $\{\alpha_n(t,x)\}$ and $\{\beta_n(t,x)\}$ converge uniformly and monotonically to the unique solution of (2.1).

Let us consider the order of convergence of $\{\alpha_n(t,x)\}$ and $\{\beta_n(t,x)\}$ to the unique solution $u(t,x)$ of (2.1). To do this, set

$$p_n(t,x) = u(t,x) - \alpha_n(t,x) \ge 0,$$

$$q_n(t,x) = \beta_n(t,x) - u(t,x) \ge 0.$$

Using the definitions for α_n, β_n, the Taylor expansion with Lagrange remainder, and the Mean Value Theorem, we obtain

$$\begin{aligned} \mathcal{L}p_{n+1} &= \mathcal{L}u - \mathcal{L}\alpha_{n+1} \\ &= f(t,x,u) + \textstyle\int_0^t g(t,x,s,u(s,x))ds \\ &\quad - \sum_{i=0}^{2} \frac{\partial^i f(t,x,\alpha_n)}{\partial u^i} \frac{(\alpha_{n+1}-\alpha_n)^i}{i!} \\ &\quad - \int_0^t \sum_{i=0}^{2} \frac{\partial^i g(t,x,s,\alpha_n(s,x))}{\partial u^i} \frac{(\alpha_{n+1}-\alpha_n)^i}{i!} ds \\ &= f(t,x,u) - \ f(t,x,\alpha_{n+1}) + \frac{\partial^2 f(t,x,\xi_1)}{\partial u^2} \frac{(\alpha_{n+1}-\alpha_n)^2}{(2)!} \\ &\quad - \frac{\partial^2 f(t,x,\alpha_n)}{\partial u^2} \frac{(\alpha_{n+1}-\alpha_n)^2}{(2)!} + \int_0^t \Big[g(t,x,s,u) - g(t,x,s,\alpha_{n+1}(s,x)) \\ &\quad + \frac{\partial^2 g(t,x,s,\xi_2(s,x))}{\partial u^2} \frac{(\alpha_{n+1}-\alpha_n)^2}{(2)!} \\ &\quad - \frac{\partial^2 g(t,x,s,\alpha_n(s,x))}{\partial u^2} \frac{(\alpha_{n+1}-\alpha_n)^2}{(2)!} \Big] ds \\ &\le f_u(t,x,\eta_1)(u-\alpha_{n+1}) + \frac{M_1}{(2)!}(\xi_1-\alpha_n)(\alpha_{n+1}-\alpha_n)^2 \\ &\quad + \int_0^t \Big[g_u(t,x,s,\eta_2)(u-\alpha_{n+1}) + \frac{M_2}{(2)!}(\xi_2-\alpha_n)(\alpha_{n+1}-\alpha_n)^2 \Big] ds \end{aligned}$$

$$
\begin{aligned}
&\le K_1 p_{n+1} + K_2 p_n^3 + \int_0^t [K_3 p_{n+1} + K_4 p_n^3] ds, \\
p_{n+1}(t,x) &= 0, \qquad x \in \partial\Omega, \\
p_{n+1}(0,x) &= 0, \qquad x \in \Omega,
\end{aligned}
$$

where $\alpha_n \le \xi_1, \xi_2 \le \alpha_{n+1}$, $\alpha_{n+1} \le \eta_1, \eta_2 \le u$, $|f_u| \le K_1$, $\frac{M_1}{(2)!} = K_2$, $|g_u| \le K_3$, and $\frac{M_2}{(2)!} = K_4$.

Let $r(t)$ be the solution of the following ordinary integro-differential equation:

$$r'(t) = K_1 r(t) + K_3 \int_0^t r(s) ds + (K_2 + K_4 T) \max_{\Omega} p_n^3, \qquad r(0) = 0.$$

Now computing the solution of the above equation, we get

$$r(t) \le \frac{2exp(\sqrt{K_1^2 + 4K_3}\ T)}{\sqrt{K_1^2 + 4K_3}} [(K_2 + K_4 T) \max_{\Omega} p_n^3].$$

One can see that

$$\int_0^t K_4 p_n^3 ds \le K_4 T \max_{\Omega} p_n^3.$$

It follows that $p_{n+1}(t,x) \le r(t)$ by Theorem 2.4. Hence

$$\max_{\overline{Q}_T} |p_{n+1}(t,x)| \le [(K_2 + K_4 T)] \Big[\frac{2exp(\sqrt{K_1^2 + 4K_3}\ T)}{\sqrt{K_1^2 + 4K_3}} \Big] \max_{\overline{Q}_T} |p_n^3(t,x)|.$$

Similarly, one can obtain that

$$\max_{\overline{Q}_T} |q_{n+1}(t,x)| \le [(K_2 + K_4 T)] \Big[\frac{2exp(\sqrt{K_1^2 + 4K_3}\ T)}{\sqrt{K_1^2 + 4K_3}} \Big] \max_{\overline{Q}_T} |q_n^3(t,x)|.$$

Hence the order of convergence of the sequences $\{\alpha_n(t,x)\}$, $\{\beta_n(t,x)\}$ is cubic. □

4. Numerical Results

In this section we demonstrate the applications of the main result which we have developed in Section 3. Let us consider the following example:

$$
\begin{aligned}
u_t - u_{xx} &= u^4 - 9u + \sin^2 t + \int_0^t [u^3(s,x) + 6u(s,x)] ds, \qquad 0 \le x, t \le 1 \\
u(0,t) &= u(1,t) = 0, \qquad 0 \le t \le 1 \\
u(0,x) &= \sin(\pi x), \qquad 0 \le x \le 1.
\end{aligned}
\tag{4.1}
$$

Choosing $\alpha_0(t,x) \equiv 0$ and $\beta_0(t,x) \equiv 1$, we have

$$\begin{aligned} 0 &\le \sin^2 t, \quad 0 \le t \le 1, \\ 0 &\ge 1 - 9 + \sin^2 t + 7t, \quad 0 \le t \le 1, \\ 0 &\le 1, \qquad 0 \le t \le 1, \\ 0 &\le \sin(\pi x) \le 1, \quad 0 \le x \le 1. \end{aligned}$$

Hence $\alpha_0(t,x) \equiv 0$ and $\beta_0(t,x) \equiv 1$ are natural lower and upper solutions for (4.1) respectively. Denote

$$f(t,x,u) = u^4(t,x) - 9u(t,x) + \sin^2 t,$$

$$g(t,x,u) = u^3(t,x) + 6u(t,x).$$

It is true that

$$g_u(0) = 3(0)^2 + 6 \ge g_{uu}(1)(1-0) = 6(1)(1-0),$$

$$0 \le f_{uu}(t,x,u_1) - f_{uu}(t,x,u_2) \le 24(u_1 - u_2), \quad u_1 \ge u_2,$$
$$0 \le g_{uu}(t,x,u_1) - g_{uu}(t,x,u_2) \le 6(u_1 - u_2), \quad u_1 \ge u_2.$$

Hence we can apply iterates of Theorem 3.1 with the Lipschitzian constants $M_1 = 24$ and $M_2 = 6$ to find the approximate solution of the equation (4.1). After only three iterates of α and β we can derive the approximate solution of (4.1) as shown in the following table for $t = 0.5$:

Table of Three α, β - Iterates and the Solution

x	$\alpha_1(t)$	$\alpha_2(t)$	$\alpha_3(t)$	u	$\beta_3(t)$	$\beta_2(t)$	$\beta_1(t)$
0.1	0.0050893	0.0050893	0.0050893	0.0050893	0.0050893	0.0051042	0.0614934
0.1	0.0085013	0.0085026	0.0085026	0.0085026	0.0085026	0.0085225	0.1046710
0.3	0.0106567	0.0106573	0.0106573	0.0106573	0.0106573	0.0107025	0.1257870
0.4	0.0118672	0.0118694	0.0118694	0.0118694	0.0118694	0.0119073	0.1447620
0.5	0.0122466	0.0122471	0.0122471	0.0122471	0.0122471	0.0123077	0.1435990
0.6	0.0118866	0.0118888	0.0118888	0.0118888	0.0118888	0.0119275	0.1451940
0.7	0.0106907	0.0106913	0.0106913	0.0106913	0.0106913	0.0107380	0.1265440
0.8	0.0085404	0.0085417	0.0085417	0.0085417	0.0085417	0.0085631	0.1056010
0.9	0.0051180	0.0051188	0.0051188	0.0051188	0.0051188	0.0051348	0.0622288

On the Figure 1 we can see the α-iterates (with unbroken line) and the β-iterates (with broken line) for $t = 0.5$.

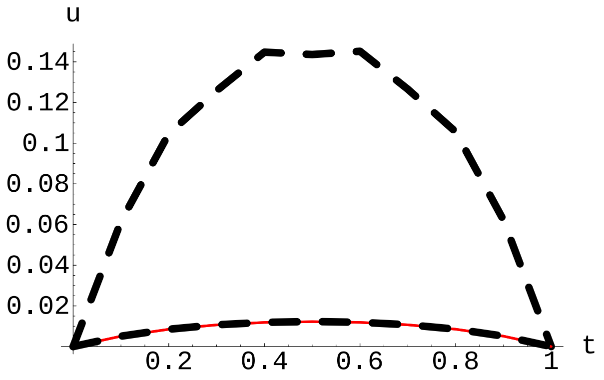

Figure 1.

The graph on the Figure 2 shows the approximate solution of (4.1) using the finite-difference method and Mathematica for each iterate.

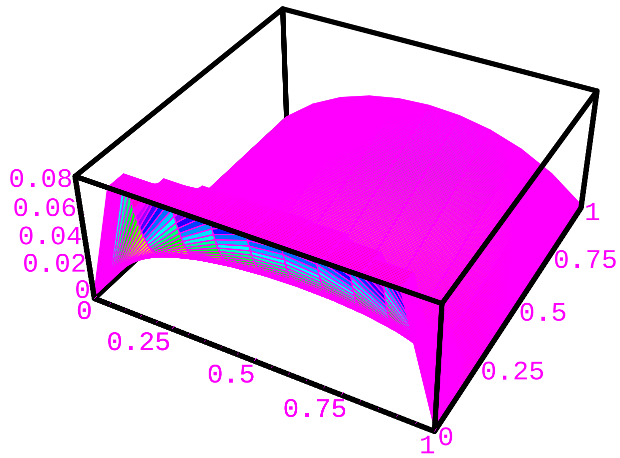

Figure 2.

Since the convergence of the iterates is of order 3 we obtained the approximate solution very fast, in three steps only.

Remark 4.1. The above result can be extended to include the situation when

$$f(t, x, u) = f_1(t, x, u) + f_2(t, x, u),$$

where $f_1(t, x, u)$ satisfies the hypothesis of the theorem whereas $f_2(t, x, u)$ satisfies

$$0 \;\geq\; \frac{\partial^2 f_2(t, x, \zeta_1)}{\partial u^2} - \frac{\partial^2 f_2(t, x, \zeta_2)}{\partial u^2} \;\geq\; -M_3(\zeta_1 - \zeta_2) \qquad \text{on } \overline{Q}_T$$

for $\alpha_0(t, x) \leq \zeta_2(t, x) \leq \zeta_1(t, x) \leq \beta_0(t, x)$.

Conclusions

In the above theorem we assumed that the 2nd derivative of the functions $f(t, x, u)$ and $g(t, x, u)$ with respect to u are nondecreasing and one-sided Lipschitzian with respect to u. We have developed iterates of nonlinearity of order 2 which converge rapidly (order 3) to the unique solution of nonlinear integro-differential equation of parabolic type. The error in this numerical computation of solution can be made as small as possible. We demonstrate the application of the theoretical result with numerical simulation.

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