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LMI Approach for Stability of Distributed Parameter Type Cohen-Grossberg Neural Networks with Damped Stochastic Disturbance

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Abstract. Based on Fubini theorem, the mean square exponential stability of distributed parameter type Cohen-Grossberg neural networks with damped stochastic disturbance is discussed in this paper. On the basis of the linear matrix inequality (LMI) approach, and also the Lyapunov functional method combined with the stochastic analysis, several stability criteria are derived. The proposed criteria can be checked readily by using some standard numerical packages, and no tuning of parameters is required.

AMS Subject Classifications: 35B35, 92B20

Keywords: Mean square exponential stability; Cohen-Grossberg neural networks; Damped stochastic disturbance; Brownian motion; Lyapunov functional; Linear matrix inequality

1. Introduction

Since the seminal work for Cohen-Grossberg neural networks in [1], the past two decades have witnessed the successful applications of Cohen-Grossberg neural networks in many areas such as classification, associative memory, parallel computation, especially in solving optimization problems. In practice, due to the finite speeds of the switching and transmission of signals in a network, time delays unavoidably exist in a working network, and they may lead to oscillation, instability, bifurcation or chaos of networks. Consequently, the stability analysis problems for delayed

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Cohen-Grossberg neural networks have gained considerable research attention [2-6] and references therein, where the delay type can be constant, time-varying, or distributed, and the stability criteria can be delay-dependent or delay-independent. For example, Cao and Liang [2] analyzed the boundedness and stability of a class of Cohen-Grossberg neural networks with time-varying delays by using the inequalities technique and Lyapunov method. Yuan and Cao [3] gave an analysis of global asymptotic stability (GAS) for a delayed Cohen-Grossberg neural network via nonsmooth analysis. Chen and Rong [5] discussed a class of Cohen-Grossberg neural networks with delays by constructing suitable Lyapunov functionals and in combination with the LMI approach, and derived several novel criteria guaranteeing the GAS of the equilibrium point for this system. However, neural network models may arise diffusion effect when electrons are moving in asymmetric electromagnetic field, so we must consider the space is varying with the time. Refs. [7-12] have considered the stability of neural networks with diffusion terms, which are expressed by partial differential equations. It is also common to consider the diffusion effect in biological systems such as immigration [13].

On the other hand, stochastic modelling plays an important role in many branches of science and industry. An area of particular interest has been the automatic control of stochastic systems, with consequent emphasis being placed on the analysis of stability in stochastic models [14, 15]. In [16], the authors first studied the stochastic neural networks and some algebraic criterion of almost sure exponential stability and instability are obtained. A strand of recent research in the stability of stochastic equations with delay has focussed on the speed at which convergence to an equilibrium may take place in the presence of such damped external perturbations. We highlight here the contribution of Mao [17], and Mao and Liao [18]. In these papers, the exponential stability of solutions is considered.

To the best of our knowledge, few authors have considered the mean square exponential stability of distributed parameter type or reaction-diffusion Cohen-Grossberg neural networks with damped stochastic disturbance, and LMI approach is applied by us in discussion of this problem. In this Letter, we analyze further problem of mean square exponential stability of distributed parameter type Cohen-Grossberg neural networks with damped stochastic disturbance and derive a set of simple sufficient conditions in terms of LMI, which improve and extend the earlier works in Refs. [19-22]. These possess important leading significance in the design and applications of globally stable distributed parameter type recurrent neural networks, and are of great interest in many applications.

The rest of this paper is organized as follows. In Section 2, the problem to be studied is stated and some preliminaries are presented. Based on the Lyapunov stability theory and Fubini theorem, in combination with LMI method, some mean square exponential stability criteria for the distributed parameter type Cohen-Grossberg neural networks with damped stochastic disturbance are derived in Section 3. Finally, conclusions are drawn in Section 4.

R^n n-dimensional Euclidean space;
$R^{n\times m}$ set of all $n \times m$ real matrices;
A^T, A^{-1} transpose of, inverse of any square matrix A, respectively;
I the $n \times n$ identity matrix;
$\|A\|$ norm of matrix A, i.e., $\|A\| = \sqrt{\lambda_{\max}(A^TA)}$, where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A^TA.

Denote by $L^p_{\mathcal{F}_0}([-\tau,0];R^n)$ the family of all $\mathcal{F}_0$-measurable $\mathcal{C}([-\tau,0];R^n)$-valued random variables $\varphi = \{\varphi(\theta) : -\tau \le \theta \le 0\}$ such that $\sup_{\tau\le\theta\le 0} \mathcal{E}|\varphi(\theta)|^p < \infty$ where $\mathcal{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathcal{P}$.

2. System description and preliminaries

In this paper, we consider the following distributed parameter type Cohen-Grossberg neural networks with damped stochastic disturbance

$$\begin{aligned} du_i(t,x) \quad &= \quad \Bigg[-c_i(u_i(t,x))\Bigg(d_i(u_i(t,x)) - \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big) - \sum_{j=1}^{n} a_{ij} g_j(u_j(t,x)) \\ &\qquad - \sum_{j=1}^{n} b_{ij} g_j(u_j(t-\tau_j(t),x))\Bigg)\Bigg]dt + \sum_{j=1}^{m} \sigma_{ij}(u_i(t,x))dw_j(t), \end{aligned} \tag{2.1}$$

for $i = 1,2,\cdots,n$, where $n \ge 2$ is the number of neurons in the network; $x = (x_1,x_2,\ldots,x_l)^T \in \Omega \subset R^l$, Ω is a bounded compact set with smooth boundary $\partial\Omega$ and mes$\Omega > 0$ in space R^N; $u(t) = [u_1(t,x),u_2(t,x),\cdots,u_n(t,x)]^T \in R^n$ denotes the state variable at time t and in space x; $c_i(\cdot)$ represents an amplification function, and $d_i(\cdot)$ is an appropriately behaved function. $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are the connection weight matrix and delayed connection weight matrix, Amplification functions $c_i(\cdot)$, behaved functions $d_i(\cdot)$ and activation functions $g_i(\cdot)$ are subject to certain conditions to be specified later. The time-varying delays $\tau_j(t)$ are assumed that $0 \le \tau(t) = (\tau_1(t),\tau_2(t),\cdots,\tau_n(t))^T$, $\tau^* = \max(\tau_j(t))$ and $\max(\dot\tau_j(t)) = \hat\tau < 1$ for $j = 1,\cdots,n$ and $t \ge 0$, where τ^* and $\hat\tau$ are constants.

Furthermore, $w(t) = [w_1(t),w_2(t),\cdots,w_m(t)]^T \in R^m$ is an m-dimensional Brownian motion defined on a complete probability space $(\widetilde\Omega,\mathcal{F},\mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ generated by $\{w(s) : 0 \le s \le t\}$, where we associate $\widetilde\Omega$ with the canonical space generated by $w(t)$, and denote by $\mathcal{F}_t$ the associated σ-algebra generated by $w(t)$ with the probability measure $\mathcal{P}$. For (2.1), $\sigma : R^n \times R_+ \to R^{n\times m}$ i.e. $\sigma(x,t) = (\sigma_{ij}(x,t))_{n\times m}$.

The boundary condition and initial condition are given by

$$\frac{\partial u_i}{\partial n} := \Big(\frac{\partial u_i}{\partial u_1}, \frac{\partial u_i}{\partial u_2}, \ldots, \frac{\partial u_i}{\partial u_N}\Big)^T = 0, \qquad i = 1,2,\cdots,n, \tag{2.2}$$

$$\zeta_i(x)\frac{\partial u_i(t,x)}{\partial n} + \xi_i(x)u_i(t,x) = 0, \qquad i = 1,2,\cdots,n, \tag{2.3}$$

and

$$u_i(s,x) = \phi_i(s,x), \qquad s \in [-\tau^*, 0], \quad i = 1, 2, \cdots, n, \tag{2.4}$$

where $\phi_i(s,x) \in L^2_{\mathcal{F}_0}([-\tau^*, 0]; R^n)$ is bounded and continuous on $[-\tau^*, 0] \times \Omega$.

We make the following assumptions:

(H_1) $c_i(u)$ is bounded, positive and continuous; furthermore $0 < \underline{\alpha}_i \leq c_i(u) \leq \overline{\alpha}_i,\ i = 1, 2, \cdots, n$;

(H_2) $d_i(u)$ is continuous, i.e., $d_i^{'}(u_i) \geq r_i > 0,\ r = \min_{1 \leq i \leq n}(r_i),\ i = 1, 2, \cdots, n$;

(H_3) The activation function $g_i(x_i)$ is bounded and there exist constant $l_i > 0$ such that

$$|g_i(x) - g_i(y)| \leq l_i|x - y|, \qquad i = 1, 2, \cdots, n,$$

for arbitrary $x, y \in R$.

For further deriving the mean square exponential stability condition of the system (2.1), the following definition and lemmas are needed.

Definition 2.1. *System (2.1) is said to be pth moment exponentially stable if there exists a pair of positive constants λ and c such that*

$$\mathcal{E}\Big(\int_\Omega |u(t,x)|^p dx\Big) \leq c\mathcal{E}\Big(\int_\Omega |\phi(t,x)|^p dx\Big)e^{-\lambda}, \qquad t \geq 0 \tag{2.5}$$

holds for any ϕ. In this case

$$\lim_{t\to\infty} \sup \frac{1}{t} \log\Big[\mathcal{E}\Big(\int_\Omega |u(t,x)|^p dx\Big)\Big] \leq -\lambda. \tag{2.6}$$

The left-hand side of (2.5) is called the pth moment Lyapunov exponent of the solution. When $p = 2$ it is usually called the exponential stability in mean square.

Lemma 2.1. ([23]) *Let $M(t), t \geq 0$ be a local martingale with $M(0) = 0$ a.s. Let $X(t), t \geq 0$ be an nonnegative $\mathcal{F}_t$-adapted process such that $\mathcal{E}(X(0)) < \infty$. If*

$$X(t) \leq X(0) + M(t) \qquad \forall t \geq 0$$

almost surely, then

$$\lim_{t\to\infty} \sup X(t) < \infty \qquad a.s.$$

Lemma 2.2. *Let $x \in R^n$, $y \in R^n$, and $\varepsilon > 0$. Then we have $x^T y + y^T x \leq \varepsilon x^T x + \varepsilon y^T y$.*

Proof. The proof follows from the inequality $(\varepsilon^{1/2}x - \varepsilon^{-1/2}y)^T(\varepsilon^{1/2}x - \varepsilon^{-1/2}y) \geq 0$ immediately. $\square$

Lemma 2.3. *(Boyd et al. [24]) Given constant matrices* $\Sigma_1, \Sigma_2, \Sigma_3$, *where* $\Sigma_1 = \Sigma_1^T$ *and* $0 < \Sigma_2 = \Sigma_2^T$, *then*

$$\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$$

if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0 \quad \textit{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$

Let $C^{2,1}(R^n \times R_+; R_+)$ denote the family of all nonnegative functions $V(u,t)$ on $R^n \times R_+$ which are continuously twice differentiable in y and once differentiable in t. For each $V \in C^{2,1}(R^n \times R_+; R_+)$, define an operator $\mathcal{L}V$, associated with the stochastic neural network (2.1), from $R^n \times R^n \times R_+$ to R by

$$\begin{aligned} \mathcal{L}V &= V_t(u,t) + V_u(u,t)\Bigg[- C(u(t,x))\Bigg(D(u(t,x)) - \sum_{i=1}^{n}\sum_{k=1}^{N} \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big) \\ &\quad -AG(u(t,x)) - BG(u(t-\tau(t),x))\Bigg)\Bigg] \\ &\quad +\frac{1}{2}\,\mathrm{trace}\,\sigma^T(u(t,x))V_{uu}(u,t)\sigma(u(t,x)), \end{aligned}$$

where

$V_t(u,t) = \frac{\partial V(u,t)}{\partial t}$, $\quad V_u(u,t) = \Big(\frac{\partial V(u,t)}{\partial u_1}, \cdots, \frac{\partial V(u,t)}{\partial u_n}\Big)$,

$V_{uu}(u(t),t) = \Big(\frac{\partial^2 V(u,t)}{\partial u_i \partial u_j}\Big)_{n\times n}$,

$C(u(t,x)) = \mathrm{diag}\Big(c_1(u_1(t,x)), \cdots, c_n(u_n(t,x))\Big) \in R^{n\times n}$,

$D(u(t,x)) = \Big(d_1(u_1(t,x)), \cdots, d_n(u_n(t,x))\Big)^T \in R^n$,

$G(u(t,x)) = \Big(g_1(u_1(t,x)), \cdots, g_n(u_n(t,x))\Big)^T \in R^n$,

$G(u(t-\tau(t),x)) = \Big(g_1(u_1(t-\tau_1(t),x)), \cdots, g_n(u_n(t-\tau_n(t),x))\Big)^T \in R^n$.

Let us stress that $\mathcal{L}V$ is defined on $R^n \times R^n \times R_+$ while V on $R^n \times R_+$. Let $C(R^n; R_+)$ denote the family of all continuous functions from R^n to R_+ while $C([-T,0]; R^n)$, $C(R; R_+)$, etc.

3. Main results

In this section, we shall present a mean square exponential stability criterion of the system (2.1). The main theorem given below shows that the stability criterion can be expressed in terms of the feasibility of a linear matrix inequality.

Theorem 3.1. *Suppose $(H_1) - (H_3)$ hold. Assume that there exist a pair of positive constants δ and γ such that*

$$\operatorname{trace} \sigma^T(u(t,x))\sigma(u(t,x)) \leq \delta e^{-\gamma u(t,x)}. \tag{3.1}$$

Assume also that there exists a positive definite matrix Q and positive diagonal matrices P, Λ such that

$$\Theta_1 = \begin{bmatrix} -2P\Gamma + \Lambda & PA & PB \\ A^T P & Q - \Lambda L^{-2} & 0 \\ B^T P & 0 & -(1-\hat{\tau})Q \end{bmatrix} < 0, \tag{3.2}$$

where $\Gamma = \operatorname{diag}(r_1, r_2, \cdots, r_n)$, $L = \operatorname{diag}(l_1, l_2, \cdots, l_n)$. Then the distributed parameter type Cohen-Grossberg neural network (2.1) with damped stochastic disturbance is globally exponentially stable in mean square, and the mean square Lyapunov exponential estimate is:

$$\lim_{T\to+\infty} \frac{1}{T} \log(\|u(T,x)\|^2) \leq -\alpha. \tag{3.3}$$

Proof. For convenience, denote $u_i = u_i(t,x)$. Consider the following Lyapunov functional:

$$V(u,t) = \int_\Omega \left[2\sum_{i=1}^n p_i \int_0^{u_i} \frac{s}{c_i(s)} ds + \int_{t-\tau(t)}^t G^T(u(s,x))QG(u(s,x))ds \right] dx. \tag{3.4}$$

Then we have

$$\int_\Omega \frac{1}{\overline{\alpha}} \sum_{i=1}^n p_i u_i^2(t,x) dx \leq V(u,t) \leq \int_\Omega \left[\frac{1}{\underline{\alpha}} \sum_{i=1}^n p_i u_i^2(t,x) + \tau^* \|Q\| G^2(u(t,x)) \right] dx, \tag{3.5}$$

or

$$\int_\Omega \frac{1}{\overline{\alpha}} u^T(t,x)Pu(t,x)dx \leq V(x,t) \leq \int_\Omega \left[\frac{1}{\underline{\alpha}} u^T(t,x)Pu(t,x) + \tau^* \|Q\| G^2(u(t,x)) \right] dx. \tag{3.6}$$

Applying the It$\hat{o}$'s formula to $V(u(t),t)$ and using the assumptions we derive that:

$$
\begin{aligned}
& dV(u(t),t) \\
= \;& \int_\Omega \Big\{ -2\sum_{i=1}^n p_i u_i \Big(d_i(u_i) - \sum_{k=1}^N \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big) \\
& - \sum_{j=1}^n a_{ij} g_j(u_j(t,x)) - \sum_{j=1}^n b_{ij} g_j(u_j(t-\tau_j(t),x))\Big) + G^T(u(t,x))QG(u(t,x)) \\
& -(1-\dot{\tau}(t))G^T(u(t-\tau(t),x))QG(u(t-\tau(t),x))\Big\} dxdt \\
& + \int_\Omega \operatorname{trace}\sigma^T(u(t,x))\sigma(u(t,x))dx + 2\int_\Omega \sum_{i=1}^n p_i u_i \sum_{j=1}^m \sigma_{ij}(u_i)dw_j(t)dx \\
\le \;& \int_\Omega \Big\{ 2\sum_{i=1}^n p_i u_i \sum_{k=1}^N \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big) \\
& -u^T(t,x)(P\Gamma+\Gamma^T P)u(t,x) + 2u^T(t,x)PAG(u(t,x)) \\
& +2u^T(t,x)PBG(u(t-\tau(t),x)) + G^T(u(t,x))QG(u(t,x)) \\
& -(1-\dot{\tau}(t))G^T(u(t-\tau(t),x))QG(u(t-\tau(t),x))\Big\} dxdt \\
& + \int_\Omega \delta e^{-\gamma u(t,x)}dx + 2\int_\Omega \sum_{i=1}^n p_i u_i \sum_{j=1}^m \sigma_{ij}(u_i)dw_j(t)dx \\
= \;& \int_\Omega \Big\{ 2\sum_{i=1}^n p_i u_i \sum_{k=1}^N \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big) - u^T(t,x)(P\Gamma+\Gamma^T P-\Lambda)u(t,x) \\
& -u^T(t,x)\Lambda u(t,x) + 2u^T(t,x)PAG(u(t,x)) \\
& +2u^T(t,x)PBG(u(t-\tau(t),x)) + G^T(u(t,x))QG(u(t,x)) \\
& -(1-\dot{\tau}(t))G^T(u(t-\tau(t),x))QG(u(t-\tau(t),x))\Big\} dxdt \\
& + \int_\Omega \delta e^{-\gamma u(t,x)}dx + 2\int_\Omega \sum_{i=1}^n p_i u_i \sum_{j=1}^m \sigma_{ij}(u_i)dw_j(t)dx \\
\le \;& \int_\Omega \Big\{ 2\sum_{i=1}^n p_i u_i \sum_{k=1}^N \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big) - u^T(t,x)(P\Gamma+\Gamma^T P-\Lambda)u(t,x) \\
& -G^T(u(t,x))\Lambda L^{-2}G(u(t,x)) + 2u^T(t,x)PAG(u(t,x)) \\
& +2u^T(t,x)PBG(u(t-\tau(t),x)) + G^T(u(t,x))QG(u(t,x)) \\
& -(1-\hat{\tau})G^T(u(t-\tau(t),x))QG(u(t-\tau(t),x))\Big\} dxdt \\
& + \int_\Omega \delta e^{-\gamma u(t,x)}dx + 2\int_\Omega \sum_{i=1}^n p_i u_i \sum_{j=1}^m \sigma_{ij}(u_i)dw_j(t)dx. \qquad (3.7)
\end{aligned}
$$

From the boundary condition (2.2), we get

$$\begin{aligned}
\int_\Omega \sum_{i=1}^{n}\sum_{k=1}^{N} u_i \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big)dx &= \sum_{i=1}^{n}\int_\Omega u_i \nabla\Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big)_{k=1}^{N} dx \\
&= \sum_{i=1}^{n}\int_\Omega \nabla\left(u_i D_{ik}\frac{\partial u_i}{\partial x_k}\right)_{k=1}^{N} dx \\
&\quad -\sum_{i=1}^{n}\int_\Omega \Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big)_{k=1}^{N} \nabla u_i dx \\
&= \sum_{i=1}^{n}\int_{\partial\Omega} \left(u_i D_{ik}\frac{\partial u_i}{\partial x_k}\right)_{k=1}^{N} dx \\
&\quad -\sum_{i=1}^{n}\sum_{k=1}^{N}\int_\Omega D_{ik}\Big(\frac{\partial u_i}{\partial x_k}\Big)^2 dx \\
&= -\sum_{i=1}^{n}\sum_{k=1}^{N}\int_\Omega D_{ik}\Big(\frac{\partial u_i}{\partial x_k}\Big)^2 dx \\
&\le 0 \qquad (3.8)
\end{aligned}$$

in which $\nabla = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_N}\Big)^T$ is the gradient operator, and

$$\Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big)_{k=1}^{N} = \Big(D_{i1}\frac{\partial u_i}{\partial x_1}, D_{i2}\frac{\partial u_i}{\partial x_2}, \cdots, D_{il}\frac{\partial u_i}{\partial x_N}\Big)^T.$$

Substituting (3.7) into (3.6), it follows that

$$\begin{aligned}
dV(u(t),t) &\le \int_\Omega \Big[Z_1^T(t,x)\Theta_1 Z_1(t,x)\Big]dxdt + \int_\Omega \delta e^{-\gamma u(t,x)}dx \\
&\quad +2\int_\Omega \sum_{i=1}^{n} p_i u_i \sum_{j=1}^{m}\sigma_{ij}(u_i)dw_j(t)dx \\
&\le \lambda_{\max}(\Theta_1)\int_\Omega \Big[u^2(t,x) + G^2(u(t,x)) + G^2(u(t-\tau(t),x))\Big]dxdt \\
&\quad + \int_\Omega \delta dx + 2\int_\Omega \sum_{i=1}^{n} p_i u_i \sum_{j=1}^{m}\sigma_{ij}(u_i)dw_j(t)dx, \qquad (3.9)
\end{aligned}$$

where

$Z_1(t,x) = \Big[u^T(t,x) \quad G^T(u(t,x)) \quad G^T(u(t-\tau(t),x))\Big]^T$,

Θ_1 is shown in (3.2).

Taking $\alpha \in (0, -\lambda_{\max}(\Theta_1))$ and applying Itô's formula once again we have that

$$
\begin{aligned}
d\Big[e^{\alpha t}V(u,t)\Big] &= e^{\alpha t}\Big[\alpha V(u,t)dt + dV(u,t)\Big] \\
&\leq \alpha e^{\alpha t}\int_\Omega \left[\frac{\max(p_i)}{\underline{\alpha}}u^2(t,x) + \tau^*\|Q\|G^2(u(t,x))\right]dxdt \\
&\quad + e^{\alpha t}\Bigg[\lambda_{\max}(\Theta_1)\int_\Omega \Big(u^2(t,x) + G^2(u(t,x)) \\
&\qquad + G^2(u(t-\tau(t),x))\Big)dxdt \\
&\quad + \int_\Omega \delta dx + 2\int_\Omega \sum_{i=1}^n p_i u_i \sum_{j=1}^m \sigma_{ij}(u_i)dw_j(t)dx\Bigg]. \qquad (3.10)
\end{aligned}
$$

Integrating (3.10) from 0 to T, we obtain

$$
\begin{aligned}
e^{\alpha T}V(u,T) &\leq \rho + \Big(\lambda_{\max}(\Theta_1) + \alpha\frac{\max(p_i)}{\underline{\alpha}}\Big)\int_\Omega\int_0^T e^{\alpha t}u^2(t,x)dtdx \\
&\quad + \Big(\lambda_{\max}(\Theta_1) + \tau^*\|Q\|\Big)\int_\Omega\int_0^T e^{\alpha t}G^2(u(t,x))dtdx \\
&\quad + \lambda_{\max}(\Theta_1)\int_\Omega\int_0^T e^{\alpha t}G^2(u(t-\tau(t),x))dtdx + M(T), \quad (3.11)
\end{aligned}
$$

where

$\rho = \int_\Omega |\phi(x)|^2 dx$,

$M(T) = 2\int_\Omega\int_0^T e^{\alpha t}\sum_{i=1}^n p_i u_i(t,x)\sum_{j=1}^m \sigma_{ij}(u_i(t,x))dw_j(t)dtdx + \int_\Omega \delta T dx.$

Obviously, $M(T)$ is a continuous martingale satisfying Lemma 2.1. From Lemma 2.1, there exits $C > 0$ such that

$$
e^{\alpha T}V(u,T) \leq C < \infty. \qquad (3.12)
$$

Thus, from Definition 2.1, we know that the system (2.1) is globally exponentially stable in mean square, and the mean square Lyapunov exponential estimate (3.3) holds. □

Theorem 3.2. *Suppose $(H_1) - (H_3)$ hold. Assume that there exist a pair of positive constants δ and γ such that*

$$
\operatorname{trace}\sigma^T(u(t,x))\sigma(u(t,x)) \leq \delta e^{-\gamma u(t,x)}. \qquad (3.13)
$$

Assume also that there exists a positive definite matrix Q and positive diagonal matrices P, Λ such that

$$
\Theta_2 = \begin{bmatrix} (1,1) & PB \\ B^TP & -(1-\hat{\tau})Q \end{bmatrix} < 0, \qquad (3.14)
$$

where $(1,1) = -P\Gamma - \Gamma^T P + PAA^T P + L^T L + L^T QL$, $\Gamma = \mathrm{diag}(r_1, r_2, \cdots, r_n)$, $L = \mathrm{diag}(l_1, l_2, \cdots, l_n)$. *Then the distributed parameter type Cohen-Grossberg neural network (2.1) with damped stochastic disturbance is globally exponentially stable in mean square, and the mean square Lyapunov exponential estimate is:*

$$\lim_{T\to+\infty} \frac{1}{T} \log(\|u(T,x)\|^2) \le -\alpha. \tag{3.15}$$

Proof. We still consider the Lyapunov functional $V(u,t)$ used in Theorem 3.1. Applying the Itô's formula to $V(u(t),t)$ and using the assumptions we derive that:

$$\begin{aligned} dV(u,t) \quad \le \quad & \int_\Omega \Bigg\{ 2\sum_{i=1}^n p_i u_i \sum_{k=1}^N \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big) \\ & -u^T(t,x)(P\Gamma + \Gamma^T P - \Lambda)u(t,x) - G^T(u(t,x))\Lambda L^{-2}G(u(t,x)) \\ & +2u^T(t,x)PAG(u(t,x)) + 2u^T(t,x)PBG(u(t-\tau(t),x)) \\ & +G^T(u(t,x))QG(u(t,x)) \\ & -(1-\hat{\tau})G^T(u(t-\tau(t),x))QG(u(t-\tau(t),x)) \Bigg\} dxdt \\ & + \int_\Omega \delta e^{-\gamma u(t,x)}dx + 2\int_\Omega \sum_{i=1}^n p_i u_i \sum_{j=1}^m \sigma_{ij}(u_i)dw_j(t)dx. \end{aligned} \tag{3.16}$$

By Lemma 2.2, we obtain the following inequality

$$\begin{aligned} 2u^T(t,x)PAG(u(t,x)) \quad & \le \quad u^T(t,x)PAA^T Pu(t,x) + G^T(u(t,x))G(u(t,x)) \\ & \le \quad u^T(t,x)(PAA^T P + L^T L)u(t,x). \end{aligned} \tag{3.17}$$

Substituting (3.8) and (3.17) into (3.16), we get

$$\begin{aligned} dV(u,t) \quad \le \quad & \int_\Omega u^T(t,x)\Big(-P\Gamma - \Gamma^T P + PAA^T P + L^T L + L^T QL\Big)u(t,x)dx \\ & + \int_\Omega \delta e^{-\gamma u(t,x)}dx + \int_\Omega 2u^T(t,x)PBG(u(t-\tau(t),x))dx \\ & - \int_\Omega (1-\hat{\tau})G^T(u(t-\tau(t),x))QG(u(t-\tau(t),x))dx \\ & +2\int_\Omega \sum_{i=1}^n p_i u_i \sum_{j=1}^m \sigma_{ij}(u_i)dw_j(t)dx \\ \le \quad & -\int_\Omega Z_2^T(t,x)\Theta_2 Z_2(t,x)dx + \int_\Omega \delta e^{-\gamma u(t,x)}dx \\ & +2\int_\Omega \sum_{i=1}^n p_i u_i \sum_{j=1}^m \sigma_{ij}(u_i)dw_j(t)dx, \end{aligned} \tag{3.18}$$

where $Z_2(t,x) = \Big[u^T(t,x) \quad G^T(u(t-\tau(t),x))\Big]^T$.

The following proof is similar to Theorem 3.1. We omit it and the proof is completed. □

In the following part, we shall study the mean square exponential stability of distributed parameter type Cohen-Grossberg neural network (2.1) when the stochastic disturbance is linear, i.e.,

$$\begin{aligned} du_i(t,x) &= \Bigg[- c_i(u_i(t,x))\Bigg(d_i(u_i(t,x)) - \sum_{k=1}^{N} \frac{\partial}{\partial x_k}\Big(D_{ik}\frac{\partial u_i}{\partial x_k}\Big) - \sum_{j=1}^{n} a_{ij} g_j(u_j(t,x)) \\ &\quad - \sum_{j=1}^{n} b_{ij} g_j(u_j(t-\tau_j(t),x))\Bigg)\Bigg] dt + \sum_{j=1}^{m} \sigma_{ij}(t) u_i(t,x) dw_j(t). \end{aligned} \tag{3.19}$$

Similarly, we can obtain the following theorem.

Theorem 3.3. *Suppose $(H_1)-(H_3)$ hold. Assume that there exist a pair of positive constants δ and γ such that*

$$\operatorname{trace} \sigma^T(t)\sigma(t) \le \delta e^{-\gamma t}. \tag{3.20}$$

Assume also that there exists a positive definite matrix Q and positive diagonal matrices P, Λ such that

$$\Theta_3 = \begin{bmatrix} -2P\Gamma + \Lambda + \delta I & PA & PB \\ A^T P & Q - \Lambda L^{-2} & 0 \\ B^T P & 0 & -(1-\hat{\tau})Q \end{bmatrix} < 0, \tag{3.21}$$

where $\Gamma = \operatorname{diag}(r_1, r_2, \cdots, r_n)$, $L = \operatorname{diag}(l_1, l_2, \cdots, l_n)$. Then the neural network (3.19) is globally exponentially stable in mean square, and the mean square Lyapunov exponential estimate is:

$$\lim_{T\to+\infty} \frac{1}{T} \log(\|u(T,x)\|^2) \le -\alpha. \tag{3.22}$$

Proof. The foregoing proofs are similar to Theorem 3.1. From (3.20), we have

$$\begin{aligned} \int_\Omega \operatorname{trace}(\sigma(t)u(t,x))^T(\sigma(t)u(t,x))dx &\le \int_\Omega \sum_{i=1}^{n} u_i^2(t,x) \operatorname{trace} \sigma^T(t)\sigma(t) dx \\ &\le \int_\Omega \delta e^{-\gamma t} \sum_{i=1}^{n} u_i^2(t,x) dx \\ &\le \int_\Omega \delta \sum_{i=1}^{n} u_i^2(t,x) dx. \end{aligned} \tag{3.23}$$

Therefore, combining (3.7)-(3.8) and (3.23), we derive

$$\begin{aligned} dV(u(t),t) &\leq \int_{\Omega}\Big[Z_1^T(t,x)\Theta_3 Z_1(t,x)\Big]dxdt \\ &\quad +2\int_{\Omega}\sum_{i=1}^{n}p_i u_i\sum_{j=1}^{m}\sigma_{ij}(t)u_i dw_j(t)dx \end{aligned} \tag{3.24}$$

The following proof is similar to Theorem 3.1. We omit it and the proof is completed. □

Consider the following stochastic neural network

$$\begin{aligned} du_i(t,x) &= \Bigg[-c_i(u_i(t,x))\Bigg(d_i(u_i(t,x))-\beta_i(t)\Delta u_i(t,x)-\sum_{j=1}^{n}a_{ij}g_j(u_j(t,x)) \\ &\quad -\sum_{j=1}^{n}b_{ij}g_j(u_j(t-\tau_j(t),x))\Bigg)\Bigg]dt+\sum_{j=1}^{m}\sigma_{ij}(u_i(t,x))dw_j(t), \end{aligned} \tag{3.25}$$

where the boundary condition and initial condition are shown in (2.2)-(2.4); $\beta_i(t)>0$ $(t>0, i=1,2,\cdots,n)$ are continuous functions with low boundedness; Δ is the Laplace operator.

Theorem 3.4. *Suppose $(H_1)-(H_3)$ hold. Assume that there exist a pair of positive constants δ and γ such that*

$$\mathrm{trace}\,\sigma^T(u(t,x))\sigma(u(t,x))\leq \delta e^{-\gamma u(t,x)}. \tag{3.26}$$

Assume also that there exists a positive definite matrix Q and positive diagonal matrices P, Λ such that

$$\Theta_1=\begin{bmatrix} -2P\Gamma+\Lambda & PA & PB \\ A^TP & Q-\Lambda L^{-2} & 0 \\ B^TP & 0 & -(1-\hat{\tau})Q \end{bmatrix}<0, \tag{3.27}$$

where $\Gamma=\mathrm{diag}(r_1,r_2,\cdots,r_n)$, $L=\mathrm{diag}(l_1,l_2,\cdots,l_n)$. Then the distributed parameter type Cohen-Grossberg neural network (3.25) with damped stochastic disturbance is globally exponentially stable in mean square, and the mean square Lyapunov exponential estimate is:

$$\lim_{T\to+\infty}\frac{1}{T}\log(\|u(T,x)\|^2)\leq -\alpha. \tag{3.28}$$

In fact, from Green formula and Robin boundary condition (2.3), we get that

$$\begin{aligned} \int_{\Omega}\Delta u_i(t,x)u_i(t,x)dx &= \int_{\partial\Omega}\Big[u_i(t,x)\frac{\partial u_i(t,x)}{\partial n}-u_i(t,x)\frac{\partial u_i(t,x)}{\partial n}\Big]ds \\ &\quad -\lambda_0\int_{\Omega}u_i^2(t,x)dx\leq 0. \end{aligned} \tag{3.29}$$

Then using the similar proofs of Theorem 3.1, Theorem 3.4 can be obtained.

Remark 3.1. Stochastic neutral networks have been widely studied in the literature, see [16, 19-22]. However, all of them considered the stability stochastic neutral networks without distributed parameter and damped. Moreover, if $c_i(u_i(t,x)) \equiv 1$ and $d_i(u_i(t,x))$ is linear, i.e., $d_i(u_i(t,x)) = r_i u_i(t,x)$, then the system (2.1) without distributed parameter has been studied in [19]. Note that different from the commonly used matrix norm theories (such as the M-matrix method), LMIs can be easily solved by using the Matlab LMI toolbox, and no tuning of parameters is required [25]. So the results obtained in this paper improve and extend the earlier works.

4. One numerical example

A simple example is presented here in order to illustrate the usefulness of our main results.

Example 4.1. Consider a two-neuron Cohen-Grossberg neural network with damped stochastic disturbance:

$$\begin{aligned} du_i(t,x) \;=\; & \Bigg[- c_i(u_i(t,x))\Bigg(d_i(u_i(t,x)) - \frac{\partial}{\partial x}\Big(D_i \frac{\partial u_i}{\partial x}\Big) - \sum_{j=1}^{n} a_{ij} g_j(u_j(t,x)) \\ & - \sum_{j=1}^{n} b_{ij} g_j(u_j(t-\tau_j(t),x))\Bigg)\Bigg] dt + \sigma_i(u_i(t,x))dw(t), \; i = 1,2, \end{aligned} \tag{4.1}$$

for $n = 2$, where $D_1 = t^2x^4$, $D_2 = 2x^2$, $c_1(u_1(t,x)) = 2 + g(u_1(t,x)), c_2(u_2(t,x)) = 2+g(u_2(t,x)), d_1(u_1(t,x)) = 6u_1(t,x), d_2(u_2(t,x)) = 5u_2(t,x)$, $\sigma_1(u_1(t,x)) = e^{-u(t,x)}$, $\sigma_2(u_2(t,x)) = 2e^{-u(t,x)}$, $\tau_1(t),x) = \tau_2(t),x) = 0.5\sin(t)$, $g(u) = 0.5(|u+1| - |u-1|)$.

So note that $\underline{\alpha} = 1, \overline{\alpha} = 3$,

$$\Gamma = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix},\; A = \begin{pmatrix} 3 & -2 \\ -3 & 1 \end{pmatrix},\; B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\; L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\; \hat{\tau} = 0.5.$$

So we have

$$\operatorname{trace} \sigma^T(u(t,x))\sigma(u(t,x)) \le 5e^{-2u(t,x)}. \tag{4.2}$$

In Theorem 2.1, by solving the LMI (3.2) using the Matlab Toolbox, a feasible solution is

$$P = 10^{-10} \times \begin{pmatrix} 0.7399 & 0 \\ 0 & 0.7399 \end{pmatrix},$$

$$Q = 10^{-9} \times \begin{pmatrix} 0.1423 & 0.1035 \\ 0.1035 & 0.1882 \end{pmatrix},$$

$$\Lambda = 10^{-9} \times \begin{pmatrix} 0.4138 & 0 \\ 0 & 0.4138 \end{pmatrix}.$$

Therefore, all conditions of Theorem 3.1 are satisfied, which implies the Cohen-Grossberg neural network (4.1) is globally asymptotically stable in mean square.

Using Theorem 3.3 with $\delta = 5$, by solving the LMI (3.21) using the Matlab Toolbox, a feasible solution is

$$P = \begin{pmatrix} 0.1083 & 0 \\ 0 & 0.1083 \end{pmatrix},$$

$$Q = \begin{pmatrix} 2.7905 & 0 \\ 0 & 2.7905 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 4.4909 & 0 \\ 0 & 4.4909 \end{pmatrix}.$$

Therefore, all conditions of Theorem 3.3 are satisfied, which implies the Cohen-Grossberg neural network (4.1) is globally asymptotically stable in mean square. However, when $A \equiv 0$, $D_i \equiv 0$, $\underline{\alpha}_1 r_1 = 6 < L_1(\overline{\alpha}_1|b_{11}| + \overline{\alpha}_2|b_{21}|) = 9$, so Theorem 3.5 in [5] is not applicable.

5. Conclusions

In this paper, the mean square exponential stability problem is considered for a class of distributed parameter type Cohen-Grossberg neural networks with damped stochastic disturbance. On the basis of the LMI approach, and also the Lyapunov functional method combined with the Fubini theorem and conducting the stochastic analysis, several stability criteria are derived. The proposed criteria can be checked readily by using some standard numerical packages, and improve and extend the earlier works.

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