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PRE-I_{sn}-OPEN SETS AND SOME NOTIONS RELATED TO PRE-I-CONVERGENCE

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Abstract. In this article, we define a new notion of open sets which are namely pre-I_{sn}-open sets. Furthermore, we show and prove some properties on pre-I-convergence and pre-I-neighbourhood spaces.

1. Introduction and preliminaries

Kuratowski [6] in 1933 introduced the concept of ideal on a topological space (X, τ). An ideal I is a collection of non-empty sets of X which satisfies the following (i) if $A \in I$ and $B \subset A$, then $B \in I$ and (ii) if $A \in I$ and $B \in I$, then $A \cup B \in I$. Given a topological space (X, τ) with an ideal I is called ideal topological space and it is denoted by (X, τ, I). On the other hand, the notion of I-convergence was originality introduced by Kostyrko et. al. [5] in 2000 as a generalization of statistical convergence which is based on the structure of the ideal I of subset of natural numbers $\mathbb{N}$. Later, in 2011, Hazarika [4] used the notion of I-convergence for defining the concept of an ideal convergence in topological group. Taking into account the notion of I-convergence, Granados in 2020 [2] introduced the concept of pre-I-convergence, besides he showed some notions related to pre-I-irresolute functions on pre-I-convergence. Otherwise, the notion of pre-I-open sets were introduced by Dontchev [1] in 1996. On the other hand, the study of open sets in sequences I-convergence has been recently studied by many authors. In 2019, Zhou et al. [10], studied I-open sets on I-convergence. Besides, Lin [8] in 2020, complemented this notion, they defined and studied I_{sn}-open sets and showed some characterizations.

In this paper, we define and study the notion of pre-I_{sn}-open sets, which are defined taking into account the concepts of sequences pre-I-eventually and pre-I-sequential neighbourhood. Furthermore, we prove some characterizations on pre-I_{sn}-neighbourhood. Throughout this paper, $\mathbb{N}$ denotes the set of natural number.

Definition 1.1. Let I be a family of non-empty sets of $\mathbb{N}$ and consider the following statements:

(1) If $A, B \in I$, then $A \cup B \in I$.
(2) If $B \subset A$ and $A \subset I$, then $B \in A$.
(3) $I \subseteq \{\emptyset\}$ and $\mathbb{N} \notin I$.
(4) I is a cover of $\mathbb{N}$.

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If the family I of $\mathbb{N}$ satisfies conditions (1) and (2), I is an ideal on $\mathbb{N}$ [6]. If I satisfies conditions (1), (2) and (3), I is a non-trivial on $\mathbb{N}$ [5]. If I satisfies conditions (1), (2), (3) and (4), I is an admissible ideal on $\mathbb{N}$ [5].

Definition 1.2. ([5]) A sequence $\{x_n\}$ of X is said to be I-convergent to a point $x_0 \in X$, provides of a neighbourhood V of x_0, we have that $\{n \in \mathbb{N} : x_n \notin V\} \in I$.

Definition 1.3. ([2]) A sequence $\{x_n\}$ of X is said to be pre-I-convergent to a point $x_0 \in X$, provides of a pre-neighbourhood V of x_0, we have that $\{n \in \mathbb{N} : x_n \notin V\} \in I$, this will be denoted by p-I-$\lim_{n\to\infty} x_n = x_0$ or $x_n \to^{pI} x_0$ and the point x_0 is called p-I-limit of the sequence $\{x_n\}$.

Definition 1.4. ([2]) Let I be an ideal on $\mathbb{N}$ and X be a topological space, then X is said to be a pre-I-sequential space if each pre-I-closed set in X is closed.

Definition 1.5. Zhou [11] defined the following notions:

1. Let P be a subset of X. A sequence $\{x_n\}$ in X is said to be I-eventually in P, if the set $\{n \in \mathbb{N} : x_n \notin P\} \in I$.
2. Let P be a subset of X. Then, P is said to be I-sequential neighbourhood of a point x of X if every sequence which is I-convergent to x is I-eventually in V.
3. Let P be a subset of X. Then, P is said to be I_{sn}-open, if P is a I-sequential neighbourhood of x for each $x \in P$.

2. Pre-I_{sn}-open sets

In this section, we introduce the notion of pre-I_{sn}-open spaces. Moreover, we study some of their properties.

Definition 2.1. Let V be an open subset of X. A sequence $\{x_n\}$ in X is said to be pre-I-eventually in V, if the set $\{n \in \mathbb{N} : x_n \notin V\} \in I$.

Definition 2.2. Let V be a subset of X. Then, V is said to be pre-I-sequential neighbourhood of a point x_0 of X if every sequence which is pre-I-convergent to x_0 is pre-I-eventually in V.

Definition 2.3. Let A be a subset of X. Then, V is said to be pre-I_{sn}-open, if V is a pre-I-sequential neighbourhood of x_0 for each $x_0 \in V$.

Remark 2.4. Let U a subset of X. Then, U is called pre-I_{sn}-closed if the complement of $X - U$ is a pre-I_{sn}-open set.

Definition 2.5. Let V be a subset of X. Then, V is said to be pre-I_{sn}-neighbourhood of x_0, if there exits a pre-I_{sn}-open set A of X such that $x_0 \in A \subset V$.

Definition 2.6. Let U be a subset of X. Then, U is said to be pre-I-closed if any sequence $\{x_n\}$ in U with $x_n \to^{pI} x_0$ in X, the p-lim point $x_0 \in U$.

Remark 2.7. Let U be a subset of X. Then, V is called pre-I-open, if the complement of $X - V$ is a pre-I-closed set.

Now, show results taking into account the previously definitions are shown.

Lemma 2.8. *pre-I-convergence implies I-convergence.*

Proof. Let V an open set of (X, τ), then V is a pre-open set. Since $\{x_n\}$ is a pre-I-convergent sequence, we have that $\{n \in \mathbb{N} : x_n \notin V\} \in I$. Therefore, by the Definition 1.2, $\{x_n\}$ is a I-convergent sequence. □

Lemma 2.9. *Every pre-I_{sn}-open set is I_{sn}-open set.*

Proof. Let V be a pre-I_{sn}-open set, then there exits a sequence $\{x_n\}$ which is pre-I-convergent, by the Lemma 2.8, $\{x_n\}$ is I-convergent. Now, since $\{x_n\}$ is pre-I-convergent, then $\{x_n\}$ is pre-I-eventually and hence by the Definition 1.5, $\{x_n\}$ is I-eventually. Therefore, this proofs that V is I_{sn}-open. □

Remark 2.10. Every I_{sn}-open set is not pre-I_{sn}-open set.

This is followed by, first: I-convergence does not always imply pre-I-convergence and second: If we have a set V which is not open and $\{n \in \mathbb{N} : x_n \notin V\} \in I$, by the Definition 1.5 V is I-eventually, but it is not pre-I-eventually.

Lemma 2.11. *([2]) Let I be an ideal on $\mathbb{N}$ and X be a topological space. If a sequence $(X_n : x \in \mathbb{N})$ pre-I-convergent to a point $x \in X$ and $(y_n : n \in \mathbb{N})$ is a sequence in X with $\{n \in \mathbb{N} : x_n \neq y_n\} \in I$, then the sequence $(y_n : n \in \mathbb{N})$ pre-I-convergent to $x \in X$*

Theorem 2.12. *Let X be a topological space and V a subset of X, if we have the following conditions:*

(1) *V is a open set of X.*
(2) *V is pre-I_{sn}-open set of X.*
(3) *V is a pre-I-open set of X.*
(4) *$\{n \in \mathbb{N} x_n \in V\} \notin I$ for each sequence $\{x_n\}$ in X with $x_n \to^{pI} x_0$.*

Then, the have that:

$$(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4).$$

Proof. (1) $\Rightarrow$ (2): Consider V be an open set of X. If a point $x_0 \in V$ and a sequence $\{x_n\}$ in X is pre-I-convergent to x_n. Then, $n \in \mathbb{N} : x_n \notin V\} \in I$, this means that the sequence $\{x_n\}$ is pre-I-eventually in V. Hence, V is a pre-I-sequential neighbourhood of x_0. Therefore, this shows that the set V is a pre-I_{sn}-open set of X.
(2) $\Rightarrow$ (3): Suppose that the set V is not a pre-I-open set of X, then the complement $X - V$ is not a pre-I-closed set of X, this implies that there exists a sequence $\{x_n\}$ in $X - V$ and a point $x_0 \in V$ with $x_n \to^{pI} x_0$, thus $\{n \in \mathbb{N} : x_n \notin V\} = \mathbb{N} \notin I$, in consequence the sequence $\{x_n\}$ is not pre-I-eventually in V. Therefore, V is not a pre-I_{sn}-open set of X.
(3) $\Leftrightarrow$ (4): We begin prove (3) $\Rightarrow$ (4). Suppose that V is a pre-I-open set of X and let $(x_n : n \in \mathbb{N})$ be a sequence in X satisfying $x_n \to^{pI} x \in V$. Now, choose $N_0 = \{n \in \mathbb{N} : x_n \in V\}$. If $N_0 \in I$, then $N_0 \neq \mathbb{N}$ and so $V \neq X$. Now, take a point $a \in X - V$ and define the sequence $(y_n : n \in \mathbb{N})$ in X by $y_n = a, n \in N_0$, thus $y_n = x_n$, $n \notin N_0$. By Lemma 2.11 , the sequence $(y_n : n \in \mathbb{N})$ pre-I-converges to x. Therefore, we can see that $X - V$ is pre-I-closed and $(y_n)_{n \in \mathbb{N}} \subseteq X - V$, and hence $x \in X - V$ and this is a contradiction. Therefore, $N_0 \notin I$.
The prove of (4) $\Rightarrow$ (3) is followed taking into account that I is an admissible ideal. □

Remark 2.13. If I is and ideal of all never dense sets and let X be a topological space and V a subset of X, if we have the following conditions:

(1) V is a open set of X.
(2) V is pre-I_{sn}-open set of X.
(3) V is a pre-I-open set of X.
(4) V is a pre-open set of X.
(5) $\{n \in \mathbb{N} x_n \in V\} \notin I$ for each sequence $\{x_n\}$ in X with $x_n \to^{pI} x_0$.

Then, the have that:

$$\begin{array}{ccccccc} (1) & \to & (2) & \to & (3) & \leftrightarrow & (4) \\ & & & & \updownarrow & & \updownarrow \\ & & & & & (5) & \end{array}$$

The prove is similar to the Theorem 2.12, taking into account that pre-I-open sets and semi-open sets are equivalent.

Lemma 2.14. *([2]) Let X be a topological space. If a sequence $\{x_n\}$ in X is pre-I-convergent to a point $x \in X$, and $\{y_n\}$ is a sequence in X with $\{n \in \mathbb{N} : x_n \neq y_n\} \in I$, then the sequence $\{y_n\}$ is pre-I-convergent to $x \in X$.*

Theorem 2.15. *Let X be a topological space. The, the following statements hold:*

(1) *If $Y \subset X$ and V is pre-I_{sn}-open (resp. pre-I_{sn}-closed, pre-I-open, pre-I-closed) subset of X, then $A \cap Y$ is a pre-I_{sn}-open (resp. pre-I_{sn}-closed, pre-I-open, pre-I-closed) subset of the subspace Y.*
(2) *If Y is a pre-I_{sn}-open subset of X and V is a pre-I-open (resp. pre-I_{sn}-open) subset of the subspace Y, then V is a pre-I-open (resp. pre-I_{sn}-subst of X).*
(3) *If Y is a pre-I-closed subset of X and V is a pre-I-closed subset of the subspace Y, then V is a pre-I-closed subset of X.*

Proof. (1) We will prove the cases for pre-I_{sn}-open and pre-I-open subsets. Let $Y \subset X$. If $\{x_n\}$ is a sequence in Y with $\{x_n \to^{spI} x_0 \in Y$ and P is a neighbourhood of x_0 in X, then $\{n \in \mathbb{N} : x_n \notin P\} = \{n \in \mathbb{N} : x_n \notin P \cap Y\} \in I$. Therefore, the sequence $\{x_n \to^{pI} x_0 \in X$.
Consider that V is a pre-I_{sn}-open subset of X. Now, let $\{x_n\}$ be a sequence in Y with $\{x_n \to^{pI} x_0 \in V \cap Y$. Then, the sequence $x_n \to^{pI} x_0$ in X. Since V is a pre-I_{sn}-open subset of X, the set $\{n \in \mathbb{N} : x_n \notin V \cap Y\} = \{n \in \mathbb{N} : x_n \notin V\} \in I$, this implies that the sequence $\{x_n\}$ is pre-I-eventually in $V \cap Y$ is a pre-I_{sn}-open subset of the subspace Y.
Consider that V is a pre-I-open subset of X. Now, let $\{x_n\}$ be a sequence in Y with $x_n \to^{pI} x_0 \in V \cap Y$. Then, the sequence $x_n \to^{sI} x_0$ in X, and $\{n \in \mathbb{N} : x_n \in V \cap Y\} = \{n \in \mathbb{N} : x_n \in V\} \notin I$, by part (4) of the Theorem 2.12 . Hence, $V \cap Y$ is a pre-I-open subset of the subspace Y.

The cases for pre-I_{sn}-closed and pre-I-closed subsets are proved analogous by complement sets.

(2) Consider that Y is an open subset of X, then Y is a pre-I_{sn}-open subset of X. Now, let V be a pre-I-open subset of the subspace Y. If V is not pre-I-open subset of X, then $X - V$ is not a pre-I-closed subset of

X, hence there exits a sequence $\{x_n\}$ in $X - V$ and a point $x \in V$ with $x_n \rightarrow^{pI} x_0$ in X. By $V \neq Y$, choose a point $y \in Y - A$ and define a sequence $\{y_n\}$ in Y as follows: $y_n = x_n$, if $x_n \in Y$. $y_n = y$, if $x_n \notin Y$, It is well known that Y is a pre-I-sequential neighbourhood of x, then we have that $\{n \in \mathbb{N} : x_n \neq y_n\} = \{n \in \mathbb{N} : x_n \notin Y\} \in I$. Therefore, by Lemma 2.14, the sequence $\{y_n\}$ is pre-I-convergent to x in the subspace Y. Now, by part (4) of the Theorem 2.12, we have that $\emptyset = \{n \in \mathbb{N} : y_n \in V\} \notin I$, and this is a contradiction. Hence, this proves that V is a pre-I-open subset of X.

Let V be an open subset of Y, then V is a pre-I_{sn}-open subset of the subspace Y. If $x \in V$ and $\{x_n\}$ is a sequence with $x_n \rightarrow^{pI} x_0$ in X, then Y is a pre-I-sequential neighbourhood of x in X, and $\{n \in \mathbb{N} : x_n \notin y\} \in I$. Suppose that $Y \neq V$ and choose a point $y \in Y - V$. Now, define a sequence $\{y_n\} \in Y$ as follows: $y_n = x_n$, if $x_n \in Y$. $y_n = y$ If $x_n \notin Y$. Thus, $\{n \in \mathbb{N} : x_n \notin y_n\} = \{n \in \mathbb{N} : x_n \notin Y\} \in Y$. Now, by the Lemma 2.14, $\{y_n\}$ is pre-I-convergent to x in X, in consequence $\{y_n\}$ is pre-I-convergent to x in the subspace Y as well. Therefore, $\{n \in \mathbb{N} : x_n \notin V\} = \{n \in \mathbb{N} : x_n \notin Y\} \in I$, this means that the sequence $\{x_n\}$ is pre-I-eventually in V and then pre-I-sequential neighbourhood of x in X. Therefore, this proves that V is a pre-I_{sn}-open subset of X.

(3) Consider that Y is a pre-I-closed subset of X and V is a pre-I-closed subset of the subspace Y. If a sequence $\{x_n\}$ in V is pre-I-convergent to a point x in X, then $x \in Y$. Therefore, the sequence $\{x_n\}$ is pre-I-convergent to x in the subspace Y, thus $x \in V$. In consequence, V is a pre-I-closed subset of X.

□

Definition 2.16. Let V be a subset of X, then

(1) $[V]_{PI_s} = \{x_0 \in X :$ there exits a sequence $\{x_n\}$ in V with $x_n \rightarrow^{pI} x_0\}$.
(2) $(V)_{PI_s} = \{x_0 \in X :$ there exits no sequence $\{x_n$ in $X - V$ with $x_n \rightarrow^{pI} x_0\}$.
(3) $[V]_{PI_{sn}} = \{x_0 \in X :$ if W is a pre-I-sequential neighbourhood of x_0, then $W \cap V \neq \emptyset$.
(4) $(V)_{PI_{sn}} = \{x_0 \in X : V$ is a pre-I-sequential neighbourhood of $x\}$.

Remark 2.17. $[V]_{PI_s}$ and $(V)_{PI_s}$ denote pre-I-hull and pre-I-kernel of the set V in X. besides, $[V]_{PI_{sn}}$ and $(V)_{PI_{sn}}$ denote pre-I_{sn}-closure and pre-I_{sn}-interior of V in X.

Theorem 2.18. *Let X be a topological space. If $V, U \subset X$. Then, the following statements hold:*

(1) $[V]_{PI_s} = X - (X - V)_{PI_s}$.
(2) $[V]_{PI_{sn}} = X - (X - A)_{PI_{sn}}$.
(3) $Int(V) \subset (V)_{PI_{sn}} \subset (V)_{PI_s} \subset V \subset [V]_{PI_s} \subset [V]_{PI_{sn}} \subset Cl(V)$.
(4) $(V \cap U)_{PI_{sn}} = (V)_{PI_{sn}} \cap (U)_{PI_{sn}}$ *and* $[V \cup U]_{PI_{sn}} = [V]_{PI_{sn}} \cup [U]_{PI_{sn}}$.

Proof. We begin proving (2): If $x \in (X - V)_{PI_{sn}}$, then $X - V$ is a pre-I-sequential neighbourhood of x_0 and $(X - V) \cap V = \emptyset$, and then, $x_0 \notin [V]_{PI_{sn}}$. This implies

that $[V]_{PI_{sn}} \subset X - (X - A)_{PI_{sn}}$. Now, if $x_0 \notin [A]_{PI_{sn}}$, then there exits a pre-I-sequential neighbourhood W of x_0 with $W \cap V = \emptyset$, thus $W \subset X - V$, hence $x \in (X - A)_{PI_{sn}}$. Therefore, $X - (X - V)_{PI_{sn}} \subset [V]_{PI_{sn}}$. This proves that $[V]_{PI_{sn}} = X - (X - V)_{PI_{sn}}$. The proof of (1) is similar to the proof of (2).
We continue proving (3): By the Theorem 2.12, it results that $Int(V) \subset (V)_{PI_{sn}}$. If $x_0 \in (V)_{PI_{sn}} - (V)_{PI}$, then there exits a sequence $\{x_n\}$ in $X - V$ with $x_n \to^{pI} x_0$, and so, $\mathbb{N} = \{n \in \mathbb{N} : x_n \notin V\} \in I$, which is a contradiction. Therefore, this implies that $(V)_{PI_{sn}} \subset (V)_{PI_s}$. If $x_0 \in X - V$, since the constant sequence $x_0, x_0, ...$ is pre-I-convergent to x_0, $x_0 \notin (V)_{pI_s}$. This proves that $(V)_{PI_s} \subset V$. By part (1) and (2) of this Theorem, it results that $V \subset [V]_{PI_s} \subset [V]_{PI_{sn}} \subset Cl(V)$.
Finally, we prove (4): We will only prove $(V \cap U)_{PI_{sn}} = (V)_{PI_{sn}} \cap (U)_{PI_{sn}}$. It is clear that $(V \cap U)_{PI_{sn}} \subset (V)_{PI_{sn}} \cap (U)_{PI_{sn}}$. Otherwise, consider that $x \in (V)_{PI_{sn}} \cap (U_{PI_{sn}})$ and a sequence $\{x_n\}$ in X is pre-I-convergent to the point x_0. Then, V and U are pre-I-sequential neighbourhoods of x_0, thus $\{n \in \mathbb{N} : x_n \notin V\} \in I$ and $\{n \in \mathbb{N} : x_n \notin U\} \in I$. It follows that $\{n \in \mathbb{N} : x_n \notin V \cap U\} = \{n \in \mathbb{N} : x_n \notin V\} \cup \{n \in \mathbb{N} : x_n \notin U\} \in I$. This implies that the set $V \cap U$ is a pre-I-sequential neighbourhoods of x_0. this means, $x_0 \in (U \cap V)_{PI_{sn}}$. Therefore, $(V \cap U)_{PI_{sn}} = (V)_{PI_{sn}} \cap (U)_{PI_{sn}}$. □

Remark 2.19. It is an open problem: If $(V \cap U)_{PI_s} = (V)_{PI_s} \cap (U)_{PI_s}$ and $[V \cup U]_{PI_s} = [V]_{PI_s} \cup [U]_{PI_s}$ for any subsets V and U of X.

3. Pre-I-neighbourhood spaces

In this section, we introduce the notion of pre-I-neighbourhood spaces. Besides, we study some of their properties and prove some equivalent conditions of the transformations among various neighbourhoods defined by pre-I-convergence.

Remark 3.1. Any family of pre-I_{sn}-open subsets of a topological space is closed under arbitrary unions. Indeed, let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of pre-I_{sn}-open subsets of a topological space X. Then, $(\bigcup_{\lambda\in\Lambda} V_\lambda)_{PI_{sn}} \subset \bigcup_{\lambda\in\Lambda} V_\alpha = \bigcup_{\alpha\in\Lambda}(V_\alpha)_{PI_{sn}} \subset (\bigcup_{\alpha\in\Lambda} V_\alpha)_{PI_{sn}}$. Therefore, $\bigcup_{\alpha\in\Lambda} V_\alpha = (\bigcup_{\alpha\in\Lambda} V_\alpha)_{PI_{sn}}$, this means that the set $\bigcup_{\alpha\in\Lambda} V_\alpha$ is a pre-I_{sn}-open subset of X.

Remark 3.2. $\tau_{PI_{sn}} = \{V \subset X : V = (V)_{PI_{sn}}\}$ is a topology on X, this is followed by part (4) of the Theorem 2.18.

Definition 3.3. Let (X, τ) be a topological space, then:

1. X is said to be pre-I-neighbourhood space provided a subset V of X is pre-I-open if and only if $V = (V)_{PI_{sn}}$.
2. The family $\tau_{PI_{sn}}$ is said to be pre-I_{sn}-open topology induced by the topology τ and the ideal I, and the topological space $(X, \tau_{PI_{sn}})$ is called pre-I_{sn}-open topological space induced by the space (X, τ) or a pre-I_{sn}-coreflection of the space (X, τ) which is denoted by $X_{PI_{sn}}$.

Lemma 3.4. *Both topological spaces (X, τ) and $(X, \tau_{PI_{sn}})$ have the same pre-I-convergent sequences*

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X. It is followed that $\tau \subset \tau_{PI_{sn}}$ that if $x_n \to^{pI} x_0$ in $\tau_{PI_{sn}}$, then $x_n \to^{pI} x_0$ in τ. Conversely, suppose that $x_n \to^{pI} x_0$

in τ and $x \in V \in \tau_{pI_{sn}}$, where V is an open set in τ. Then, V is a pre-I-sequential neighbourhood of x, therefore the sequence $\{x_n\}$ is pre-I-eventually in V, thus $\{n \in \mathbb{N} : x_n \notin V\} \in I$. This implies that, $x_n \to^{pI} \tau_{PI_{sn}}$. □

Theorem 3.5. *If a topology σ of a set of X contains each pre-I-open subset of a topological space (X, τ), then both spaces (X, τ) and (X, σ) have the same pre-I-convergent sequences if and only if $\sigma = \tau_{PI_{sn}}$*

Proof. Let (X, τ) and (X, σ) be two topological spaces such that they have the same pre-I-convergent sequences. Since the family σ contains each pre-I-open subsets of (X, τ), by the Theorem 2.12, $\tau_{PI_{sn}} \subset \sigma$.

Otherwise, if $V \in \sigma$, then V is a pre-I_{sn}-open set of (X, σ). Since both spaces (X, τ) and (X, σ) have the same pre-I-convergent sequences, they have the same pre-I_{sn}-open subsets, and hence $V \in \tau_{PI_{sn}}$. Therefore, this proves that $\sigma = \tau_{PI_{sn}}$.

Now, consider $\sigma = \tau_{PI_{sn}}$. Since both spaces (X, τ) and $(X, \tau_{PI_{sn}})$ have the same pre-I-convergent sequences, both spaces (X, τ) and (X, σ) have the same pre-I-convergent sequences. If V is a pre-I-open subset of (X, σ), then V is a pre-I-open subset of (X, τ). And then, σ contains each pre-I-open subset of (X, τ). □

Remark 3.6. The topological space (X, σ) of the above Theorem, it is a pre-I-sequential space.

Lemma 3.7. *Let (X, τ) be a topological space. Then, the following statements hold:*

1. *X is a pre-I-sequential space if and only if X is a pre-I-neighbourhood space and $\tau = \tau_{PI_{sn}}$.*
2. *X is a pre-I-neighbourhood space if and only if any pre-I-neighbourhood of each point is a pre-I_{sn}-neighbourhood of the point in X.*
3. *X is a pre-I-sequential space if and only if any pre-I-neighbourhood of each point is a neighbourhood of the point X.*

Proof. We begin proving (1): Let X be a pre-I-sequential space. If V is a pre-I-open subsets of X, then V is open in X, by the Theorem 2.12, V is pre-I_{sn}-open, hence X is a pre-I-neighbourhood space. Besides, we have that $\tau \subset \tau_{PI_{sn}}$. If $V \in \tau_{PI_{sn}}$, then $V = (V)_{PI_{sn}}$, and by the Theorem 2.12, V is a pre-I-open subsets of X, therefore V is open in X, and hence $V \in \tau$. This proves that $\tau = \tau_{PI_{sn}}$.

The proof of (2) and (3) are followed directly by the definitions. □

Remark 3.8. Taking into account that we have seen so far and since that any family of pre-I_{sn}-open subsets of a topological space X is closed under arbitrary unions, a subset V of X is pre-I_{sn}-open if and only if V is a pre-I_{sn}-neighbourhood of x_0 for each $x_0 \in V$. Then, it would be easily to check that the following statements are equivalent for a topological space (X, τ):

1. $\tau = \tau_{PI_{sn}}$.
2. Every pre-I_{sn}-open subset of X is open.
3. Any pre-I_{sn}-neighbourhood of each point is a neighbourhood of the point in X.

Furthermore, it would be easily to check that the following statements are equivalent for a topological space (X, τ) as well:

(1) Any pre-I-sequential neighbourhood of each point is a neighbourhood of the point in X.
(2) For each $V \subset X$, then $Int(V) = (V)_{PI_{sn}}$, and $Cl(V) = [A]_{PI_{sn}}$.
(3) For each $V \subset X$, then $(A)_{PI_{sn}}$ is open, and $[A]_{PI_{sn}}$ is closed in X.

Remark 3.9. If we have a pre-I-neighbourhood, the following statements are open problems for future work:

(1) Being hereditary with respect to subspaces.
(2) Being hereditary with respect to pre-I-open (resp. pre-I-closed) subspaces.
(3) Being preserved by topological sums.
(4) We could not find an ideal I on $\mathbb{N}$ and a topological space X such that X is not a pre-I-neighbourhood space.

The following results show some results which were found on continuous functions.

Definition 3.10. Let I be an ideal on $\mathbb{N}$, $(X,\tau),(Y,\sigma)$ be a topological spaces and $f:(X,\tau)\to(Y,\sigma)$ be a function, then:

(1) f is said to be pre-I_{sn}-continuous if V is a pre-I_{sn} open subset of Y, then $f^{-1}(V)$ is a pre-I_{sn}-open subset of X.
(2) [2] f is said to be preserving pre-I-convergence provided for each sequences $(x_n : n \in \mathbb{N})$ in X with $x_n \to^{pI} x$, the sequence $(f(x_n) : n \in \mathbb{N})$ pre-I-converges to $f(x)$.

Theorem 3.11. *Let X and Y be two topological spaces and $f : X \to Y$ be a function. Then, the following statements are equivalent:*

(1) *f preserving pre-I-convergence.*
(2) *f is a pre-I_{sn}-continuous function.*
(3) *If W is a pre-I_{sn}-closed subset of Y, then $f^{-1}(W)$ is a pre-I_{sn}-closed subset of X.*
(4) *$f([V]_{PI_{sn}}) \subset [f(V)]_{PI_{sn}}$ for each $V \subset X$.*
(5) *If A is open and pre-I-sequential neighbourhood of a point $y \in Y$ and $x \in f^{-1}(y)$, then $f^{-1}(A)$ is open and semi-I-sequential neighbourhood of $x \in X$.*

Proof. (1)⇒(5): Let A be an open set and be a pre-I-sequential neighbourhood of a point $y \in Y$ and $x \in f^{-1}(y)$. Consider a sequence $\{x_n\}$ in X is pre-I-convergent to the point $x \in X$. Since f is a preserving pre-I-convergence function, the sequence $\{f(x_n)$ in Y is pre-I-convergent to $f(x)$. Hence, $\{n \in \mathbb{N} : x_n \notin f^{-1}(A)\} = \{n \in \mathbb{N} : f(x_n) \notin A\} \in I$. Since $f^{-1}(A)$ is open, then $\{x_n\}$ is pre-I-eventually in $f^{-1}(A)$. Therefore, $f^{-1}(A)$ is a pre-I-sequential neighbourhood of x.

(5)⇒(4): Let $V \subset X$. Consider that $x \in [V]_{PI_{sn}} \subset X$. If A is a pre-I-sequential neighbourhood of $f(x)$ in Y, by part (5) of this Theorem, $f^{-1}(A)$ is a pre-I-sequential neighbourhood of x in X, hence $f^{-1}(A)\cap V \neq \emptyset$, this implies that $A \cap f(V) \neq \emptyset$, and then $f(x) \in [f(V)]_{PI_{sn}}$. Therefore, $f([V]_{PI_{sn}}) \subset [f(V)]_{PI_{sn}}$.

(4)⇒(3): Let W be a pre-I_{sn}-closed subset of Y. Then, by part (4) of this Theorem, $f([f^{-1}(W)]_{PI_{sn}}]) \subset [f(f^{-1}(W))]_{PI_{sn}} \subset [W]_{PI_{sn}} = W$, this implies that $[f^{-1}(W)]_{PI_{sn}} \subset f^{-1}(W)$. Therefore, this proves that $f^{-1}(W)$ is a pre-I_{sn}-closed subset of X.

(3)⇒(2): Let V be a pre-I_{sn}-open subset of Y, thus $Y - V$ is a pre-I_{sn}-closed subset of Y. By part (3) of this Theorem, $f^{-1}(Y - V) = X - f^{-1}(V)$ is a pre-I_{sn}-closed subset of X, and then $f^{-1}(V)$ is a pre-I_{sn}-open subset of X. Therefore, this proves that f is a pre-I_{sn}-continuous function.

(2)⇒(1): Consider that a sequence $x_n \to^{pI} x_0$ in X and V be a open subset of Y with $f(x) \in V$. Since V is pre-I_{sn}-open subset of Y, then by part (2) of this Theorem, $f^{-1}(V)$ is a pre-I_{sn}-open subset of X and $x \in f^{-1}(V)$. Hence, the sequence $\{x_n\}$ is pre-I-eventually in $f^{-1}(V)$ and so, $\{n \in \mathbb{N} : f(x_n) \notin V\} = \{n \in \mathbb{N} : x_n \notin f^{-1}(V)\} \in I$, besides V and $f^{-1}(V)$ are pre-open, therefore this implies that $f(x_n) \to^{pI} f(x_0)$ in Y. In consequence, f is preserving pre-I-convergence. □

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