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BOUNDARY VALUE PROBLEM FOR FUNCTIONAL DIFFERENTIAL INCLUSIONS ON MANIFOLDS

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Abstract. We investigate the boundary value problem for second order functional differential inclusion of the form $\frac{D}{dt}\dot{m}(t) \in F(t, m_t(\theta), \dot{m}_t(\theta))$ on a complete Riemannian manifold for a continuous curve $\varphi : [-h, 0] \to M$ and point m_1 which is non-conjugate with $\varphi(0)$ along at least one geodesic of Levi-Civita connection, where $\frac{D}{dt}$ is the covariant derivative of Levi-Civita connection and $F(t, m(\theta), X(\theta))$ is a set-valued vector field (it is either convexvalued and satisfies the upper Caratheodory condition or it is lower semicontinuous) such that: $\|F(t, m, X)\| < f(\|X\|)$ where $f : [0, \infty) \to [0, \infty)$ is an arbitrary continuous function, increasing on $[0, \infty)$. Some conditions on certain geometric characteristics, on the distance between points and on the length of time interval, under which the problem is solvable, are found. A generalization to inclusions of the same sort subjected to a non-holonomic constraint, is also presented.

1. Introduction

Let M be a finite-dimensional complete Riemannian manifold and TM be its tangent bundle with natural projection $\pi : TM \to M$. For $I = [-h, 0]$ denote by $D(I, TM)$ the space of couples $(m(\theta), X(\theta))$ where $m(\theta)$ is a continuous curve in M and $X(\theta)$ is a vector field along $m(\theta)$ being continuous from the left and having the limit from the right. Consider a set-valued mapping $F : R \times D(I, TM) \multimap TM$ such that for any $(m(\theta), X(\theta))$ the relation $\pi F(t, m(\theta), X(m(\theta))) = m(0)$ holds. We call such F *a set-valued force field*.

Specify $l > 0$. We investigate the differential inclusion of the form

$$\frac{D}{dt}\dot{m}(t) \in F(t, m_t(\theta), \dot{m}_t(\theta)), \tag{1.1}$$

where as usual for a curve $m(\cdot) : [-h, l] \to M$ and $t \in [0, l]$, we set $m_t(\theta) = m(t+\theta)$ where $\theta \in I$. We suppose that F satisfies the condition:

$$\max_{(t,m)\in I\times\Xi} \|F(t, m, X)\| \le f(\|X\|) \tag{1.2}$$

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where f is an arbitrary continuous function, increasing on $[0, \infty)$. Also we assume that F satisfies the so called upper Carathéodory condition (see Definition 3.1) and has convex closed values, or is lower semi-continuous.

The main aim of the paper is to find conditions that guarantee the solvability for some $t_1 \in (0, l)$ of the boundary value problem for (1.1) with right-hand sides as mentioned above, i.e., to find a C^0 – curve $m(t)$, $t \in [-h, t_1]$, satisfying (1.1) on $(0, t_1]$ and such that $m(t) = \varphi(t)$ for $t \in [-h, 0]$ and $m(t_1) = m_1$ where $\varphi(t)$ is a given C^1-curve with $t \in I$ and m_1 is a given point. Note that for such a solution the couple $(m_t(\theta), \dot{m}_t(\theta))$ belongs to $D(I, TM)$ for every $t \in [0, t_1]$.

It should be pointed out that even the two-point boundary value problem for ordinary second order differential equations may not be solvable at all for smooth uniformly bounded single-valued F (see, e.g., [1]) if the boundary points are conjugate along all geodesics of Levi-Civita connection joining them. That is why we suppose that the points $\varphi(0)$ and m_1 are not conjugate along at least one geodesic. We find some conditions on certain geometric characteristics of M, on t_1, and on the distance between $\varphi(0)$ and m_1, under which the problem is solvable. Note that there are examples of second order equations with non-bounded continuous right-hand sides where for a given couple of points the problem is solvable on a sufficiently small time interval but is not solvable on larger intervals. Besides, the problem can be solvable for points rather close to each other and not solvable at all for points with greater distance between them (see examples in [1]).

We construct the solutions of problem under consideration from fixed points of special integral type operators, that act in the space of continuous curves in the tangent space $T_{\varphi(0)}M$.

The special particular case of $f(x) = x^2$ was investigated and a condition for solvability was obtained in [2]. In [3] similar problem was considered for second-order differential inclusions.

Note that a single-valued continuous field $\mathfrak{f}$ is a particular case of set-valued fields F mentioned above. Thus the conditions found here for inclusion (1.1) are also valid for second order functional differential equation $\frac{D}{dt}\dot{m}(t) = \mathfrak{f}(t, m_t, \dot{m}_t)$ with continuous right-hand side. We do not formulate the results for equations separately.

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2. Technical statements

In this section we modify some constructions from [4] for the problem under consideration.

Take $m_0 \in M$, and let $v : [0, 1] \to T_{m_0}M$ be a continuous curve. It is shown in [4] that there exists a unique C^1–curve $m : [0, 1] \to M$ such that $m(0) = m_0$ and the vector $\dot{m}(t)$ is parallel along $m(\cdot)$ to the vector $v(t) \in T_{m_0}M$ at any $t \in [0, 1]$.

Denote the curve $m(t)$ constructed above from the curve $v(t)$, by the symbol $Sv(t)$. Thus we have defined a continuous operator S that sends the Banach space $C^0([0, 1], T_{m_0}M)$ of continuous maps (curves) from $[0, 1]$ to $T_{m_0}M$ into the Banach manifold $C^1([0, 1], M)$ of C^1– maps from $[0, 1]$ to M. Let a point $m_1 \in M$ be non-conjugate to the point $m_0 \in M$ along a geodesic $g(\cdot)$ of the Levi-Civita connection.

Everywhere below denote by U_R a ball in $C^0([0,t_1],T_{\varphi(0)}M)$ with center at the origin.

Lemma 2.1. *There exists a ball $U_\varepsilon \subset C^0([0,1],T_{m_0}M)$ of a radius $\varepsilon > 0$ centered at the origin such that for any curve $\hat{u}(t) \in U_\varepsilon \subset C^0([0,1],T_{m_0}M)$ there exists a unique vector $\mathbf{C}_{\hat{u}}$, belonging to a certain bounded neighborhood V of the vector $\dot{g}(0)$ in $T_{m_0}M$, that is continuous in $\hat{u}$ and such that $S(\hat{u}+\mathbf{C}_{\hat{u}})(1) = m_1$*

Introduce the notation $sup_{\mathbf{C}\in V}\|\mathbf{C}\| = C$, where V is from Lemma 2.1.

Remark 2.2. One can easily show that $\varepsilon < C$. Note that C characterizes the distance between m_0 and m_1 while ε characterizes some properties of the Riemannian geometry on M.

Lemma 2.3. *Under conditions and notation of Lemma 2.1, let $R > 0$ and $t_1 > 0$ be such that $t_1^{-1}\varepsilon > R$. Then for any curve $u(t) \in U_R \subset C^0([0,t_1],T_{m_0}M)$ there exists a unique vector C_u in a neighborhood $t_1^{-1}V$ of the vector $t_1^{-1}\dot{g}(0)$ in $T_{m_0}M$, continuously depending on u and such that $S(u+C_u)(t_1) = m_1$*

Lemmas 2.1 and 2.3 are modifications of theorem 3.3 from [4].

For the given curve $\varphi(\cdot)$ we introduce the operator $S_\varphi : C^0([0,t_1],T_{\varphi(0)}M) \to C^0([-h,t_1],M)$, defined as follows: $S_\varphi(v(\cdot))(t) = \varphi(t)$ for $t \in [-h,0]$ and $S_\varphi(v(\cdot))(t) = S(v(\cdot))(t)$ for $t \in [0,t_1]$.

Lemma 2.4. *For specified $t_1 > 0$, $R > 0$ as above and $\varphi(\cdot) \in C^1(I,M)$ all curves $S_\varphi(v+C_v)_t(\theta)$ with $v(\cdot) \in U_R \subset C^0([0,t_1],T_{\varphi(0)}M)$ take values in a compact set $\Xi \subset M$ that depends on the curve φ, ε and C, introduced above, and does not depend on t_1.*

Proof. Obviously the length of $S_\varphi(v+C_v)(\cdot)$ is a sum of lengths of $\varphi(\cdot)$ and of $S(v+C_v)(\cdot)$. Since the parallel translation preserves the norm of a vector, for any curve $v(\cdot) \in C^0([0,t_1],T_{\varphi(0)}M))$ the length of $S(v+C_v)(\cdot)$ is not greater than $\int_0^{t_1}(R+\|C_v\|)dt \le \int_0^{t_1} t_1^{-1}(\varepsilon+C)dt = \int_0^1(\varepsilon+C)dt = \varepsilon+C$. Denote $N = \sup_{t\in I}\|\dot{\varphi}(t)\|$. It is easy to see that the length of $\varphi(\cdot)$ is not greater than Nh. Hence all curves $\|S_\varphi(v+C_v)_t(\cdot)\|$ lie in a bounded subset Ξ of M. Since M is complete, by Hopf-Rinow theorem any bounded set is compact. □

Lemma 2.5. *Let the inequality $f(\varepsilon t_1^{-1}+Ct_1^{-1}) < \varepsilon t_1^{-2}$ hold where f is an arbitrary, increasing on $[0,\infty]$ function. Then there exists a small enough positive number ϕ such that $(\varepsilon t_1^{-1}-\phi) > 0$ and the inequality $f((\varepsilon t_1^{-1}-\phi)+Ct_1^{-1}) < \varepsilon t_1^{-2}-\phi t_1^{-1}$ holds.*

Proof. From the hypothesis of lemma we get $f(\varepsilon t_1^{-1}+Ct_1^{-1}) < \varepsilon t_1^{-2}$. From continuity of both sides of this inequality it follows that there exists a small enough number $\phi > 0$ such that $(\varepsilon t_1^{-1}-\phi) > 0$ and the inequality $f((\varepsilon t_1^{-1}-\phi)+Ct_1^{-1}) < \varepsilon t_1^{-2}-\phi t_1^{-1}$ holds. □

3. The main results

Everywhere below M is a complete Riemannian manifold. Denote $\|X(\cdot)\| = sup_{\theta\in I}\|X(\theta)\|$. Introduce the norm of $F(t,m,X) \in T_mM$ by usual formula:

$$\|F(t,m(\cdot),X(\cdot))\| = sup_{y\in F(t,m(\cdot),X(\cdot))}\|y\|.$$

On $D(I,TM)$ we consider Skorohod's topology (see for example [5], where it is described for the space of functions continuous from the right and having limits from the left, in our case the construction is quite analogous).

Definition 3.1. We say that $F(t, m(\theta), X(\theta)))$ satisfies upper Carathéodory conditions if:

(1) for every couple $(m(\cdot), X(\cdot))) \in D(I,TM)$ the map $F(\cdot, m(\cdot), X(\cdot))) : [0,l] \multimap T_m M$ is measurable,

(2) for almost all $t \in I$ the map $F(t,\cdot,\cdot) : D(I,TM) \multimap TM$ is upper semicontinuous.

Definition 3.2. Let $I = [0,l] \subset \mathbb{R}$. The set-valued force field $F(t, m(\theta), X(\theta)))$ is called almost lower semicontinuous if there exists a countable sequence of disjoint compact sets I_n, $I_n \subset I$ such that: (i) the measure of $I \backslash \bigcup_n I_n$ is equal to zero; (ii) the restriction of F on each $I_n \times TM$ is lower semicontinuous.

Consider a curve $\varphi(\theta) \in C^1(I,M)$ and a point $m_1 \in M$.

Theorem 3.3. *Let $\varphi(0)$ and m_1 be not conjugate along at least one geodesic of Levi-Civita connection joining them and let $F(t, m(\cdot), X(\cdot))$ satisfy the upper Caratheodory condition, have convex closed values and for a certain $t_1 > 0$ satisfy condition (1.2) on $[0,t_1] \times \Xi$ where Ξ is compact from Lemma 2.4. If*

$$f(\varepsilon t_1^{-1} + C t_1^{-1}) \leq \varepsilon t_1^{-2}, \tag{3.1}$$

there exists a solution $m(t)$ of (1.1), for which $m(t) = \varphi(t)$ for $t \in I$ and $m(t_1) = m_1$.

Proof. Since $\varphi(0)$ and m_1 are not conjugate along a geodesic of Levi-Civita connection, the numbers ε and C from Lemma 2.1 are well-posed. Denote by Θ the subset in $D(I,TM)$ such that all curves from $\pi\Theta$ belong to the compact Ξ from Lemma 2.4.

Consider a continuous curve $v : [0,t_1] \to T_{\varphi(0)}M$. Construct the C^1-curve $\gamma(t) = S_\varphi v(t)$ for $t \in [0,t_1]$.

Note that the vector filed $\dot\gamma(t)$ along $\gamma(t)$ is discontinuous at $t = 0$ but the couple $(\gamma_t(\cdot), \dot\gamma_t(\cdot))$ belongs to $D(I,TM)$ for all $t \in [0,l]$. Hence the set-valued vector field $F(t, \gamma_t(\cdot), \dot\gamma_t(\cdot))$ is well-posed for all $t \in [0,t_1]$.

Denote by Γ the operator of parallel translation of vectors along $\gamma(\cdot)$ at the point $\gamma(0) = \varphi(0)$. Apply operator Γ to all sets $F(t, \gamma_t(\theta), \dot\gamma_t(\theta))$ along $\gamma(\cdot)$. As a result for any $v(\cdot) \in C^0(I, T_{m_0}M)$ we obtain a set-valued map $\Gamma F S_\varphi v : [0,t_1] \to T_{m_0}M$ that has convex values. It follows from the results of [6] that this map satisfies upper Carathéodory conditions. Denote by $\mathcal{P}\Gamma F S_\varphi v$ the set of all measurable selections of $\Gamma F S_\varphi v$ (such selections do exist, see e.g., [7]). Define the set-valued operator $\int \mathcal{P}\Gamma F S_\varphi : C^0([0,t_1], T_{m_0}M) \multimap C^0([0,t_1], T_{m_0}M)$ by the formula

$$\int \mathcal{P}\Gamma F S_\varphi = \{ \int_0^t f(\tau) d\tau | f(\cdot) \in \mathcal{P}\Gamma F S_\varphi \}.$$

In complete analogy with [6], it can be shown that $\int \mathcal{P}\Gamma F S_\varphi$ is upper semicontinuous, has convex values and sends bounded sets from $C^0([0,t_1], T_{\varphi(0)}M)$ into compact ones.

For $\varphi(0)$ and m_1, the numbers ε and C introduced above, are well-defined. Consider the ball $U_R \in C^0([0,t_1],T_{m_0}M)$, where $R = (\varepsilon t_1^{-1} - \phi)$ and ϕ is the number from Lemma 2.5. Since $\varepsilon t_1^{-1} > R$, by Lemma 2.3 for any $v(\cdot) \in U_R$ the vector C_v is well-posed. Thus we can introduce the operator $Z : U_R \multimap C^0([0,t_1],T_{m_0}M)$ by formula:

$$Z(v) = \int \mathcal{P}\Gamma F S_\varphi(v + C_v).$$

As well as $\int \mathcal{P}\Gamma F S_\varphi v$, this operator is upper semi-continuous, convex-valued and sends bounded sets from $C^0([0,t_1],T_{m_0}M)$ into compact ones (see [1]). Since $t_1^{-1}\varepsilon - \phi > R$ and parallel translation preserves the norms of vectors, from the construction of S_φ and from Lemma 2.5 we derive that for any $v(\cdot) \in U_R$ and $t \in [0,t_1]$ the estimate

$$\|F(t, S_\varphi(v + C_v)_t(\theta), \frac{d}{d\theta}S_\varphi(v + C_v)_t(\theta))\| <$$

$$< f((\varepsilon t_1^{-1} - \phi) + Ct_1^{-1}) < (\varepsilon t_1^{-2} - \phi t_1^{-1})$$

holds. Since parallel translation preserves the norms of vectors, from the last inequality it follows that

$$\|Z(v + C_v)\| = \| \int \mathcal{P}\Gamma F S_\varphi(v + C_v)\|_{C^0([0,t_1],T_{m_0}M)} \leq (t_1^{-1}\varepsilon - \phi) = R.$$

Thus Z sends the ball U_R into itself and from the Bohnenblust-Karlin fixed point theorem (see, e.g., [7, 8]) it follows that it has a fixed point $u(\cdot) \in U_R$, i.e. $u(\cdot) \in Zu(\cdot)$. Let us show that $m(t) = S_\varphi(u(t) + C_u)$ is the desired solution. By construction we have $m(\cdot) = \varphi(\cdot)$ for $t \in [-h, 0]$ and $m(t_1) = m_1$.

Note that $\dot{u}(\cdot)$ is a selection of $\Gamma F(t, S_\varphi(u + C_u)_t(\theta), \frac{d}{d\theta}S_\varphi(u + C_u)_t(\theta))$ since u is a fixed point of Z. In other words, the inclusion $\dot{u}(t) \in \Gamma F(t, S_\varphi(u + C_u)_t(\theta), \frac{d}{d\theta}S_\varphi(u + C_u)_t(\theta))$ holds for all points t at which the derivative exists. Using the properties of the covariant derivative and the definition of u, one can show that $\dot{u}(t)$ is parallel to $\frac{D}{dt}\dot{m}(t)$ along $m(\cdot)$ and $\Gamma F(t, S_\varphi(u+C_u)_t(\theta), \frac{d}{d\theta}S_\varphi(u+C_u)_t(\theta))$ is parallel to $F(t, m_t(\theta), \dot{m}_t(\theta))$. Hence, $\frac{D}{dt}\dot{m}(t) \in F(t, m(t), \dot{m}(t))$. □

Theorem 3.4. *Let $\varphi(0)$ and m_1 be not conjugate along at least one geodesic of Levi-Civita connection joining them andlet $F(t, m(\cdot), X(\cdot))$ be almost lower semi-continuous, have closed values and for a certain $t_1 > 0$ let it satisfy condition (1.2) on $[0, t_1] \times \Xi$ where Ξ is the compact from Lemma 2.4. If (4.5) is fulfilled, there exists a solution $m(t)$ of (1.1), for which $m(t) = \varphi(t)$ for $t \in I$ and $m(t_1) = m_1$.*

Proof. Here we use the same notation as in the proof of Theorem 3.3. Notice that from the hypothesis it follows that for all $v \in C^0([0,t_1],T_{m_0}M)$ the curves from $\mathcal{P}\Gamma F S_\varphi v$ are integrable. Hence the set-valued map $\mathcal{P}\Gamma F S_\varphi v$ sends $C^0([0,t_1],T_{m_0}M)$ into $L^1(([0,t_1],\mathcal{A},\mu),T_{m_0}M)$, where $\mathcal{A}$ is Borel σ-algebra and μ is the normalized Lebesgues measure. Since F is almost lower semicontinuous, in complete analogy with [8] one can easily show that the operator $\mathcal{P}\Gamma F S_\varphi v$:

$C^0([0,t_1],T_{m_0}M) \to L^1(([0,t_1],\mathcal{A},\mu),T_{m_0}M)$ is lower semicontinuous and has decomposable images (for the definition see, e.g., [7, 9]). Then by Fryszkowski-Bressan-Colombo theorem (see, e.g., [7]) it has a continuous selection that we denote by $p\Gamma FS$.

Choose the number R as in the proof of Theorem 3.3. Then on the ball $U_R \subset C^0([0,t_1],T_{m_0}M)$ the operator

$$\mathcal{G}v = \int_0^t p\Gamma F(s, S_\varphi(v(s)+C_v), \frac{d}{ds}S_\varphi(v(s)+C_v))ds : U_R \to C^0([0,t_1],T_{m_0}M)$$

is well posed. As a corollary to Lemma 19 of [10] we obtain that G is completely continuous. Since parallel translation preserves the norm of a vector, from the construction of S and from the hypothesis for any $u \in U_R$ with given F we get:

$$\|\mathcal{G}v\| = \| \int_0^t p\Gamma F(s, S_\varphi(v(s)+C_v), \frac{d}{ds}S_\varphi(v(s)+C_v))ds\|_{C^0([0,t_1],T_{m_0}M)} \leq$$

$$\leq f\|(v(t)+C_v)\|t_1 \leq f((\varepsilon t_1^{-1}-\phi)+Ct_1^{-1})t_1 < (\varepsilon t_1^{-2}-\phi t_1^{-1})t_1 = (\varepsilon t_1^{-1}-\phi) = R$$

Hence the completely continuous operator $\mathcal{G}$ sends U_R into itself and by classical Schauder's principle it has a fixed point $u \in U_R$. Using the same arguments, as in the proof of Theorem 3.3, one can easily prove that $m(t) = S(u+C_u)(t)$ is a solution of (1.1) such that $m(t) = \varphi(t)$ for $t \in I$ and $m(t_1) = m_1$. □

Corollary 3.5. *Let $F(t,m(\cdot),X(\cdot))$ either satisfy upper Caratheodory condition and have convex closed values or be almost lower semicontinuous and have closed values. Let also $F(t,m(\cdot),X(\cdot))$ satisfy (1.2) on the entire manifold M and the points $\varphi(0)$ and m_1 be nonconjugate along a certain geodesic $g(\cdot)$ of the Levi-Civita connection. If $t_1 > 0$ is such that (4.5) is fulfilled, there exists a solution $m(t)$ of (1.1) such that $m(t) = \varphi(t)$ for $t \in I$ and $m(t_1) = m_1$.*

4. Systems with linear constraints

In this section, we show how to generalize existence theorems of previous section to systems with constraints. We refer the reader, say, to [4] for preliminary material about systems with constraints. Here we introduce only some notions necessary for understanding the constructions.

Definition 4.1. A linear constraint in the system is a smooth distribution (i.e., a subbundle of the tangent bundle) β on M.

If the distribution β is integrable, the constraint is called holonomic and nonholonomic in the other case.

Definition 4.2. A tangent vector is called admissible if it lies in the distribution β. A curve in M is admissible if all its tangent vectors are admissible.

A constraint β imposes a restriction on the motion of the system. Namely, all its solutions must be admissible.

Let $Q : TM \to \beta$ be the operator of orthogonal projection (with respect to the Riemannian metric on M) of tangent spaces on their subspaces β, i.e., we have $Q_m : T_mM \to \beta_m$ for every $m \in M$. Introduce the so-called reduced covariant derivative along a curve by the formula $\frac{\overline{D}}{dt} = Q\frac{D}{dt}$. In fact it is generated by the so-called *reduced connection* (see [4]). Below in this section we use the parallel translation of admissible vectors along admissible curves generated by reduced connection.

Definition 4.3. A least constraint non-holonomic geodesic is an admissible curve $\gamma(t)$ on M that satisfies the equation $\frac{\overline{D}}{dt}\dot{\gamma}(t) = 0$ where $\dot{\gamma} = \frac{d\gamma}{dt}$.

We investigate the differential inclusion of the form

$$\frac{\overline{D}}{dt}\dot{m}(t) \in QF(t, m(\theta), X(\theta)), \tag{4.1}$$

where right-hand side satisfies the following condition:

$$\|QF(t, m(\theta), X(\theta))\| \leq f(\|X\|) \tag{4.2}$$

on some subset in $\mathbb{R} \times M$ (a particular case – on the entire $\mathbb{R} \times M$) where $f : [0, \infty) \to [0, \infty)$ is an arbitrary function increasing on $[0, \infty)$.

In complete analogy with the construction of exponential mapping by ordinary geodesics, the least constraint non-holonomic geodesics generate the so called non-holonomic exponential mapping $exp^{\beta}_{m_0} : \beta_{m_0} \to M$.

Definition 4.4. A point $m_1 \in exp^{\beta}_{m_0}(\beta_{m_0})$ is called non-conjugate to m_0 along a least constraint geodesic $\gamma_X(t)$ where $\gamma_X(1) = m_1$ and $\dot{\gamma}_X(0) = X$, if the differential $dexp^{\beta}_{m_0}$ has at $X \in \beta_{m_0}$ the maximal rang.

Consider $m_0 \in M$, $I = [0, t_1]$, and let $v : I \to \beta_{m_0}$ be a continuous curve. It is shown in [4] that there exists unique admissible C^1–curve $m : [0, t_1] \to M$ such that $m(0) = m_0$ and the vector $\dot{m}(t)$ is parallel (with respect to reduced connection) along $m(\cdot)$ to the vector $v(t) \in T_{m_0}M$ at any $t \in [0, 1]$. We denote the curve $m(t)$ constructed in such a way from the curve $v(t)$, by the symbol $S^{\beta}v(t)$.

Let $m(t)$, $t \in I$, be an admissible C^1-curve and $X(t, m)$ an admissible vector field on M. Denote by $\Gamma^{\beta}X(t, m(t))$ the curve in $\beta_{m(0)}$ such that the vector $\Gamma^{\beta}X(t, m(t))$ at m(0) is parallel to $X(t, m(t))$ along $m(\cdot)$ with respect to reduced connection.

Assume that m_0 is not conjugate to m_1 along a least constrained geodesic γ_X. Let us specify a submanifold $N \subset M$, $m_1 \in N$, which is transversal to $exp^{\beta}_{m_0}(\beta_{m_0})$. (In other words, the sum of spaces $T_{m_1}N$ and $T_{m_1}exp^{\beta}_{m_0}(\beta_{m_0})$ coincides with $T_{m_1}M$.) Note that an example of such manifold is an open neighbourhood of m_1 in M.

Lemma 4.5 ([4])**.** *There exists a ball $U_\varepsilon \subset C^0([0,1], \beta_{m_0})$ with a radius $\varepsilon > 0$ and center at the origin such that for any curve $\hat{u}(t) \in U_\varepsilon \subset C^0([0,1], \beta_{m_0})$ there exists a unique vector $C_{\hat{u}}$, belonging to a certain bounded neighborhood V of the vector $\dot{\gamma}_X(0) \in \beta_{m_0}$, that is continuous in $\hat{u}$ and such that $S^{\beta}(\hat{u} + C_{\hat{u}})(1) \in N$*

Lemma 4.6 ([4])**.** *In conditions and notations of Lemma 4.5 let $R > 0$ and $t_1 > 0$ be such that $t_1^{-1}\hat{\varepsilon} > R$. Then for any curve $u(t) \in U_R \subset C^0([0, t_1], \beta_{m_0})$ there exists a unique vector C_u in a neighborhood $t_1^{-1}V$ of the vector $t_1^{-1}\dot{\gamma}_X(0)$ in $\beta_{m_0}M$, continuously depending on u and such that $S^{\beta}(u + C_u)(t_1) \in N$*

For the given curve $\varphi(\cdot)$ we introduce the operator $S^{\beta}_{\varphi} : C^0([0,t_1],\beta_{\varphi(0)}) \to C^0([-h,t_1],M)$, defined as follows: $S^{\beta}_{\varphi}(v(\cdot))(t) = \varphi(t)$ for $t \in [-h,0]$ and $S^{\beta}_{\varphi}(v(\cdot))(t) = S^{\beta}(v(\cdot))(t)$ for $t \in [0,t_1]$. We investigate the functional differential inclusion of the form

$$\frac{\overline{D}}{dt}\dot{m}(t) \in QF(t, m_t(\theta), \dot{m}_t(\theta)), \tag{4.3}$$

where as usual for a curve $m(\cdot) : [-h,l] \to M$ and $t \in [0,l]$, we set $m_t(\theta) = m(t+\theta)$ where $\theta \in I$ and right-hand side satisfies the following condition:

$$\|QF(t,m,X)\| \leq f(\|X\|) \tag{4.4}$$

on some subset in $\mathbb{R} \times M$ (a particular case – on the entire $\mathbb{R} \times M$) where $f : [0,\infty) \to [0,\infty)$ is an arbitrary function increasing on $[0,\infty)$.

Since the norm of orthogonal projector Q equals 1, by simple replacement of S_{φ} and Γ by S^{β}_{φ} and Γ^{β}, respectively, by use the space $C^0(I,\beta_{m_0})$ instead of $C^0(I,T_{m_0}M)$ and Lemmas 4.5, 4.6 instead of Lemmas 2.1 and 2.3 one can prove the following analogues of non-constrained Theorems 3.3 and 3.4.

Theorem 4.7. *Let $\varphi(0)$ and m_1 be not conjugate along a certain least constraint geodesic of g and let $QF(t,m(\cdot),X(\cdot))$ satisfy the upper Caratheodory condition, have convex closed values and for a certain $t_1 > 0$ satisfy condition (4.4) on $[0,t_1] \times \Xi$ where Ξ is compact from Lemma 2.4. If*

$$f(\varepsilon t_1^{-1} + Ct_1^{-1}) \leq \varepsilon t_1^{-2}, \tag{4.5}$$

there exists a solution $m(t)$ of (4.3), for which $m(t) = \varphi(t)$ for $t \in I$ and $m(t_1) = m_1$.

Theorem 4.8. *Let $\varphi(0)$ and m_1 be not conjugate along a certain least constraint geodesic of $g(\cdot)$ and let $QF(t,m(\cdot),X(\cdot))$ be almost lower semicontinuous, have closed values and for a certain $t_1 > 0$ let it satisfy condition (4.4) on $[0,t_1] \times \Xi$ where Ξ is the compact from Lemma 2.4. If (4.5) is fulfilled, there exists a solution $m(t)$ of (4.3), for which $m(t) = \varphi(t)$ for $t \in I$ and $m(t_1) = m_1$.*

Corollary 4.9. *Let $QF(t,m,X)$ either satisfy upper Caratheodory condition and have convex closed values or be almost lower semicontinuous and have closed values. Let also $QF(t,m,X)$ satisfy (4.4) on the entire manifold M and the points $\varphi(0)$ and m_1 be nonconjugate along a certain least constraint geodesic $g(\cdot)$. If $t_1 > 0$ is such that (4.5) is fulfilled, there exists a solution $m(t)$ of (4.3), for which $m(0) = m_0$ and $m(t_1) \in N$.*

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