RELATIONS BETWEEN MODEL FUNCTIONS IN DIFFERENT REPRESENTATIONS OF SOLUTIONS TO A FIRST ORDER GENERAL LINEAR ELLIPTIC SYSTEM

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Abstract. In this paper, some new second kind representations of solutions to a linear general first order uniform elliptic system in a simply connected plane domain G are established. In complex notation, we write system as $\partial_{\bar z}w + q_1(z)\partial_z w + q_2(z)\partial_{\bar z}\overline{w} + A(z)w + B(z)\overline{w} = 0$, where $w = w(z)$ is the desired complex function, $\partial_{\bar z} = 1/2(\partial/\partial x + i\partial/\partial y)$, $\partial_z = 1/2(\partial/\partial x - i\partial/\partial y)$, stand for Sobolev's derivatives, $q_1(z)$ and $q_2(z)$ are given measurable complex functions satisfying the condition of uniform ellipticity of the system: $|q_1(z)|+|q_2(z)| \le q_0 = \text{const} < 1$, $z \in \overline{G}$, and $A(z)$, $B(z)$, $R(z) \in L_p(\overline{G})$, $p > 2$, are also given complex functions. Some inequalities between model functions in first kind and second kind representations of solutions to this system are obtained. Such inequalities are useful for an investigation of solutions to the correspondently quasilinear system.

1. Introduction. Statement of Results

Everywhere in this paper we denote by $D \equiv D_z = \{z : |z| < 1\}$ the unit disk of the complex z-plane E, $z = x+iy$, $i^2 = -1$; $\Gamma = \partial D$; $\overline{D} = D \cup \Gamma$; by G the simply connected bounded domain of the complex ζ-plane; $\partial G = \mathcal{L}$; $\overline{G} = G \cup \mathcal{L}$.

We use the following functional spaces with the standard norms: $L_p(\overline{D})$ is the space of functions integrable to the power $p \ge 1$ in $\overline{D}$; $W_p^k(\overline{D})$, $k = 0, 1, \ldots$, $p \ge 1$, is the class of functions having in $\overline{D}$ weak Sobolev's derivatives up to order k integrable to the power p, $W_p^0(\overline{D}) \equiv L_p(\overline{D})$; $C_\alpha^k(\overline{D})$, $k = 0, 1, \ldots$, $0 < \alpha \le 1$, is the space of functions having continuous partial derivatives up to order k in $\overline{D}$ that are Hölder continuous with exponent α, $C_\alpha^0(\overline{D}) \equiv C_\alpha(\overline{D})$.

The notation $C_\alpha^k(\overline{G})$, $L_p(\overline{G})$, $W_p^k(\overline{G})$, $C_\alpha^k(\mathcal{L})$ has the similar sense. The detailed definitions of these spaces and norms can be found in [10].

Also we use the Banach space $W_p^{k-\frac{1}{p}}(\mathcal{L})$ of traces of functions from $W_p^k(\overline{G})$ (see more in [8]).

We say that a contour $\mathcal{L} \in C_\alpha^k$, $k \ge 1$, $0 < \alpha \le 1$ ($W_p^{l-\frac{1}{p}}(\mathcal{L})$, $l \ge 2$, $p > 2$), if there exists a homeomorphic mapping $\zeta = f(z)$ of the circle Γ on $\mathcal{L}$ of class $C_\alpha^k(\Gamma)$

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$(W_p^{l-\frac{1}{p}}(\Gamma))$ such, that $f'(z) \neq 0$. Observe that the invers mapping $z = f^{-1}(\zeta)$ is of class $C^k_\alpha(\mathcal{L})$ $(W_p^{l-\frac{1}{p}}(\mathcal{L}))$. In this case the mapping $\zeta = f(z)$ (as well as the invers) is called a diffeomorfism of class C^k_α $(W_p^{l-\frac{1}{p}})$ between the contours Γ and $\mathcal{L}$. By analogy, we define diffeomorphisms of arbitrary contours of corresponding smoothness.

We denote by $A^k_\alpha(\overline{D}) = A^k_\alpha \subset C^k_\alpha(\overline{D})$, $0 < \alpha < 1$, (respectively $A^k_p(\overline{D}) = A^k_p \subset W^k_p(\overline{D})$, $p > 2$) the closed subspace of holomorphic functions. The notation $A^k_\alpha(\overline{G})$ $(A^k_p(\overline{G}))$ has the similar sense.

We will use the next integral operators (in all definitions we mean $f \in L_p(\overline{G})$, $p > 2$):

$$T_G f(\zeta) \equiv Tf(\zeta) = -\frac{1}{\pi}\iint\limits_G \frac{f(t)}{t-\zeta}\,dxdy,\ t = x+iy. \tag{1.1}$$

$$\begin{aligned} T_n f(\zeta) = Tf(\zeta) + \frac{1}{\pi}\iint\limits_G f(t)P(\zeta, t, \zeta_1, \dots, \zeta_n)\,dxdy \equiv \\ \equiv Tf(\zeta) + P_n f(\zeta),\ t = x+iy, \end{aligned} \tag{1.2}$$

where $\zeta_k \in \overline{G}$, $k = 1, \dots, n$, are any different fixed points,

$$\begin{aligned} &P(\zeta, t, \zeta_1, \dots, \zeta_n) = \\ &= \sum_{k=1}^{n} \frac{(\zeta-\zeta_1)\dots(\zeta-\zeta_{k-1})(\zeta-\zeta_{k+1})\dots(\zeta-\zeta_n)}{(\zeta_k-\zeta_1)\dots(\zeta_k-\zeta_{k-1})(\zeta_k-\zeta_{k+1})\dots(\zeta_k-\zeta_n)} \cdot \frac{1}{t-\zeta_k}. \end{aligned}$$

We note, that

$$T_n f(\zeta_k) = 0,\ k = 1, \dots, n.$$

In the case $G = D$ we put

$$T^* f(z) = -\frac{1}{\pi}\iint\limits_D \left[\frac{f(t)}{t-z} + \frac{z\overline{f(t)}}{1-\bar{t}z}\right] dxdy,\ t = x+iy, \tag{1.3}$$

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$$\mathrm{Re}\{T^* f(z)\}\Big|_{z\in\Gamma} = 0;$$

$$
\begin{aligned}
&T_n^* f(z) = -\frac{1}{\pi}\iint\limits_D \left[\frac{f(t)}{t-z} + \frac{z^{2n+1}\overline{f(t)}}{1-\bar{t}z}\right] dxdy- \\
&-\sum_{k=1}^{2n+1} \frac{(1-z\bar{z}_1)\dots(1-z\bar{z}_{k-1})(1-z\bar{z}_{k+1})\dots(1-z\bar{z}_{2n+1})}{(\bar{z}_k-\bar{z}_1)\dots(\bar{z}_k-\bar{z}_{k-1})(\bar{z}_k-\bar{z}_{k+1})\dots(\bar{z}_k-\bar{z}_{2n+1})}\times \\
&\times\frac{1}{\pi}\iint\limits_D \frac{\overline{f(t)}}{\bar{t}-\bar{z}_k}\, dxdy+ \\
&+\sum_{k=1}^{2n+1} \frac{(z-z_1)\dots(z-z_{k-1})(z-z_{k+1})\dots(z-z_{2n+1})}{(z_k-z_1)\dots(z_k-z_{k-1})(z_k-z_{k+1})\dots(z_k-z_{2n+1})}\times \\
&\times\frac{1}{\pi}\iint\limits_D \frac{f(t)}{t-z_k}\, dxdy,\ t=x+iy,
\end{aligned}
\tag{1.4}
$$

where $n \geq 0$ is an integer; z_k, $k=1,\dots,2n+1$, are any different fixed points on the contour Γ. We note, that

$$
\mathrm{Re}\{z^{-n}T_n^*f(z)\}\Big|_{z\in\Gamma} = 0,\ T_n^*f(z_k)=0,\ k=1,\dots,2n+1.
$$

We consider in $\overline{G}$ the general linear elliptic first-order system in complex notation

$$
\partial_{\bar{\zeta}} w + q_1(\zeta)\partial_\zeta w + q_2(\zeta)\partial_{\bar{\zeta}}\overline{w} + A(\zeta)w + B(\zeta)\overline{w} = 0, \tag{1.5}
$$

where $\zeta=\xi+i\eta$, $w=w(\zeta)$ is the desired complex function, $\partial_{\bar{\zeta}} = 1/2(\partial/\partial\xi + i\partial/\partial\eta)$, $\partial_\zeta = 1/2(\partial/\partial\xi - i\partial/\partial\eta)$, stand for Sobolev's derivatives, $q_1(\zeta)$ and $q_2(\zeta)$ are given measurable complex functions satisfying the condition of uniform ellipticity of System (1.5)

$$
|q_1(\zeta)| + |q_2(\zeta)| \leq q_0 = \mathrm{const} < 1,\ \zeta \in \overline{G}, \tag{1.6}
$$

and $A(\zeta)$, $B(\zeta) \in L_p(\overline{G})$, $p>2$, are also given complex functions.

If $G=D$, we wright z instead of ζ.

In the paper [1] under assumption of rectifiability of $\mathcal{L}=\partial G$, for any solution $w(\zeta)\in W_p^1(\overline{G})$, $p>2$, to System (1.5) the next *first kind representations* where obtained.

With the operator T:

$$
w(\zeta) = f(\zeta)e^{T_G\omega(\zeta)}, \tag{1.7}
$$

where $f=f(\zeta)$ is the single-defined solution to the Beltrami equation

$$
\partial_{\bar{\zeta}} f + q(\zeta)\partial_\zeta f = 0,\ q(\zeta) = q_1(\zeta) + q_2(\zeta)\frac{\partial_{\bar{\zeta}}\overline{w}}{\partial_\zeta w}; \tag{1.8}
$$

$\omega = \omega(\zeta)$ is the solution to the singular integral equation

$$\begin{gathered}\omega(\zeta) + q(\zeta)\Pi\omega(\zeta) = h(\zeta),\\ \Pi\omega(\zeta) = \partial_\zeta T\omega(\zeta) = -\frac{1}{\pi}\iint\limits_G \frac{\omega(t)}{(t-\zeta)^2}\,dxdy,\ t = x+iy,\\ h(\zeta) = \begin{cases} -\left(A + B\dfrac{\overline{w}}{w}\right), & \text{åñëè } w(\zeta) \neq 0,\\ -(A+B), & \text{åñëè } w(\zeta) = 0.\end{cases}\end{gathered} \tag{1.9}$$

We note, that $\omega \in L_s(\overline{G})$, $f \in W_s^1(\overline{G})$, where $2 < s \leq 2+\varepsilon \leq p$ and $\varepsilon > 0$ is generally speaking sufficiently small.

With the operator T^*:

$$w(z) = f^*(z)e^{T^*\omega(z)},\ |w(z)| = |f^*(z)|\Big|_{z\in\Gamma}, \tag{1.10}$$

where everything is similar to the previous one, only $G = D$ and instead of the operator Π appears singular integral operator $\Pi^* = \partial_z T^*$.

In [1] Bojarski gives the scheme of the building of different representations like (1.7). We use here some other representations of such type (the substantiation of ones see below).

With the operator T_n:

$$w(\zeta) = f_n(\zeta)e^{T_n\omega(\zeta)},\ w(\zeta_k) = f_n(\zeta_k),\ k = 1,\dots,n, \tag{1.11}$$

where $f_n(\zeta)$ is the solution to Equation (1.8), and $\omega(\zeta)$ is the solution to integral equation like (1.9), but instead of the operator Π there is the operator $\Pi_n = \partial_\zeta T_n$.

With the operator T_n^*:

$$w(z) = f_n^*(z)e^{T_n^*\omega(z)},\ w(z_k) = f_n^*(z_k),\ k = 1,\dots,2n+1, \tag{1.12}$$

where everything is similar to the previous one, only $G = D$ and instead of the operator Π appears the singular integral operator $\Pi_n^* = \partial_z T_n^*$.

Using the same scheme one can build first kind representations which do not consist $\dfrac{\partial_{\bar\zeta}\overline{w}}{\partial_\zeta w}$, but ones consist only $\dfrac{\overline{w}}{w}$. We construct one such representation using the operator T. With all other operators, representations of this class are built similarly.

$$w(\zeta) = \widetilde{f}(\zeta)e^{T\omega(\zeta)}, \tag{1.13}$$

where the function $\omega(\zeta)$ is the solution to the singular integral equation

$$\omega + q_1\Pi\omega + q_2\cdot\frac{\overline{w}}{w}\cdot\overline{\Pi\omega} = h; \tag{1.14}$$

the function $h(\zeta)$ is defined by (1.9); the function $\widetilde{f}(\zeta)$ is the solution to the generalized Beltrami equation

$$\partial_{\bar\zeta} f + q_1\partial_\zeta f + q_2\cdot e^{\overline{T\omega}-T\omega}\cdot\partial_{\bar\zeta}\overline{f} = 0. \tag{1.15}$$

There is well known the Pompeiu formula [10, p.p. 41, 57, 69]: if $w(\zeta) \in W_p^1(\overline{G})$, $p > 2$, $\partial G = \mathcal{L} \in C^1$, then

$$w(\zeta) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{w(\tau)}{\tau - \zeta} d\tau - \frac{1}{\pi} \iint\limits_{G} \frac{\partial w}{\partial \bar{\tau}} \cdot \frac{dxdy}{\tau - \zeta}, \; \tau = x + iy. \tag{1.16}$$

Let $w(\zeta) \in W_p^1(\overline{G})$, $p > 2$, is a solution to the equation (1.5). Using (1.16) one can wright *the second kind representation* of $w(\zeta)$:

$$\Omega(w) = \Phi(\zeta), \;\; \text{ãäå} \;\; \Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{w(\tau)}{\tau - \zeta} d\tau, \tag{1.17}$$

where

$$\Omega(w) \equiv w(\zeta) + T(q_1 \partial_\tau w + q_2 \partial_{\bar{\tau}} \overline{w} + Aw + B\overline{w})(\zeta). \tag{1.18}$$

The next representations are obvious corollaries of (1.17):

$$\Omega_n(w) = \Phi_n(\zeta), \tag{1.19}$$

where

$$\Omega_n(w) \equiv w(\zeta) + T_n(q_1 \partial_\tau w + q_2 \partial_{\bar{\tau}} \overline{w} + Aw + B\overline{w})(\zeta); \tag{1.20}$$

$$\Omega^*(w) = \Phi^*(z), \tag{1.21}$$

where

$$\Omega^*(w) \equiv w(z) + T^*(q_1 \partial_\tau w + q_2 \partial_{\bar{\tau}} \overline{w} + Aw + B\overline{w})(z); \tag{1.22}$$

$$\Omega_n^*(w) = \Phi_n^*(z), \tag{1.23}$$

where

$$\Omega_n^*(w) \equiv w(z) + T_n^*(q_1 \partial_\tau w + q_2 \partial_{\bar{\tau}} \overline{w} + Aw + B\overline{w})(z). \tag{1.24}$$

It is not difficult to write out expressions for holomorphic functions $\Phi_n(\zeta)$, $\Phi^*(z)$, $\Phi_n^*(z)$. Further, we will not need them, so we will not write them out.

Definition 1.1. Functions $f(\zeta)$, $f_n(\zeta)$, $f^*(z)$, $f_n^*(z)$, $\widetilde{f}(\zeta)$, $\Phi(\zeta)$, $\Phi_n(\zeta)$, $\Phi^*(z)$, $\Phi_n^*(z)$ appearing in representations of solutions w to Equation (1.5) we will call model functions for the solution w.

In the partial case $q_1 = q_2 \equiv 0$ (generalized analytic functions in the sense of I.N. Vekua) all model functions are holomorphic.

Second kind representations for solutions to System (1.5) were investigated in the papers [2] – [5], [8]. The following results were obtained.

Theorem 1.2 ([2])**.** *If $q_1(z), q_2(z) \in C(\overline{D})$, $A(z), B(z) \in L_p(\overline{D})$, $p > 2$, then the operator* (1.24) *Ω_n^* is a real linear isomorphism of the Banach space $W_p^1(\overline{D})$.*[1]

If $q_1(z), q_2(z)$ are only bounded and measurable, then there exists such number $s : 2 < s \leq p$, generally speaking close enough to two, that the operator Ω_n^* is a real linear isomorphism of the Banach space $W_s^1(\overline{D})$.

[1] In [2], the assumption of continuity almost everywhere of the coefficients q_1, q_2 are erroneously, although the proof uses the continuity of these coefficients. The example 3 in [5] shows that the assumption of continuity almost everywhere is insufficient.

Theorem 1.3 ([2]). *If $q_1(z), q_2(z), A(z), B(z) \in C^k_\alpha(\overline{D})$ $(W^{k+1}_p(\overline{D}))$, $k \geq 0$, $0 < \alpha < 1$ $(p > 2)$, then the operator* (1.24) Ω^*_n *is a real linear isomorphism of the Banach space $C^{k+1}_\alpha(\overline{D})$ $(W^{k+2}_p(\overline{D}))$.*

Theorem 1.4 ([4]). *If $\partial G = \mathcal{L} \in C^1_\alpha$, $0 < \alpha < 1$, $q_1(\zeta), q_2(\zeta) \in C(\overline{G})$, $A(\zeta)$, $B(\zeta) \in L_p(\overline{G})$, $p > 2$, then the operator* (1.18) Ω *is a real linear isomorphism of the Banach space $W^1_p(\overline{G})$.*

If $q_1(\zeta), q_2(\zeta)$ are only bounded and measurable, then there exists such number $s : 2 < s \leq p$, generally speaking close enough to two, that the operator Ω is a real linear isomorphism of the Banach space $W^1_s(\overline{G})$.

Theorem 1.5 ([4], [8]). *If $\partial G = \mathcal{L} \in C^{k+1}_\alpha$, $(W^{k+2-1/p})$, $q_1(\zeta), q_2(\zeta), A(\zeta), B(\zeta) \in C^k_\alpha(\overline{G})$ $(W^{k+1}_p(\overline{G}))$, $k \geq 0$, $0 < \alpha < 1$ $(p > 2)$, then the operator* (1.18) Ω *is a real linear isomorphism of the Banach space $C^{k+1}_\alpha(\overline{G})$ $(W^{k+2}_p(\overline{G}))$.*

Using these results, we will prove here the next similar assertions for Ω_n and Ω^*.

Theorem 1.6. *If $\partial G = \mathcal{L} \in C^1_\alpha$, $0 < \alpha < 1$, $q_1(\zeta), q_2(\zeta) \in C(\overline{G})$, $A(\zeta), B(\zeta) \in L_p(\overline{G})$, $p > 2$, then the operator* (1.20) Ω_n *is a real linear isomorphism of the Banach space $W^1_p(\overline{G})$.*

If $q_1(\zeta), q_2(\zeta)$ are only bounded and measurable, then there exists such number $s : 2 < s \leq p$, generally speaking close enough to two, that the operator Ω_n is a real linear isomorphism of the Banach space $W^1_s(\overline{G})$.

Theorem 1.7. *If $\partial G = \mathcal{L} \in C^{k+1}_\alpha$, $(W^{k+2-1/p})$, $q_1(\zeta), q_2(\zeta), A(\zeta), B(\zeta) \in C^k_\alpha(\overline{G})$ $(W^{k+1}_p(\overline{G}))$, $k \geq 0$, $0 < \alpha < 1$ $(p > 2)$, then the operator* (1.20) Ω_n *is a real linear isomorphism of the Banach space $C^{k+1}_\alpha(\overline{G})$ $(W^{k+2}_p(\overline{G}))$.*

Theorem 1.8. *If $q_1(z), q_2(z) \in C(\overline{D})$, $A(z), B(z) \in L_p(\overline{D})$, $p > 2$, then the operator* (1.22) Ω^* *is a real linear isomorphism of the Banach space $W^1_p(\overline{D})$.*

If $q_1(z), q_2(z)$ are only bounded and measurable, then there exists such number $s : 2 < s \leq p$, generally speaking close enough to two, that the operator Ω^* is a real linear isomorphism of the Banach space $W^1_s(\overline{D})$.

Theorem 1.9. *If $q_1(z), q_2(z), A(z), B(z) \in C^k_\alpha(\overline{D})$ $(W^{k+1}_p(\overline{D}))$, $k \geq 0$, $0 < \alpha < 1$ $(p > 2)$, then the operator* (1.22) Ω^* *is a real linear isomorphism of the Banach space $C^{k+1}_\alpha(\overline{D})$ $(W^{k+2}_p(\overline{D}))$.*

Now we formulate the main results about relations between model functions.

Theorem 1.10. *Let $w(z) \in W^1_p(\overline{G})$ be a solution to Equation* (1.5). *In the assumptions of Theorem 1.4 the next inequalities are valid:*

$$c \cdot \|\Phi_n(\zeta)\|_{W^1_p(\overline{G})} \leq \|\Phi(\zeta)\|_{W^1_p(\overline{G})} \leq C \cdot \|\Phi_n(\zeta)\|_{W^1_p(\overline{G})}, \tag{1.25}$$

where the constants $C \geq c > 0$ do not depend on w.

Theorem 1.11. *Let $w(z) \in C^{k+1}_\alpha(\overline{G})$ $(W^{k+2}_p(\overline{G}))$ be a solution to Equation* (1.5). *In the assumptions of Theorem 1.5 the next inequalities are valid:*

$$\begin{aligned} &c_1 \cdot \|\Phi_n(\zeta)\|_{C^{k+1}_\alpha(\overline{G})} \leq \|\Phi(\zeta)\|_{C^{k+1}_\alpha(\overline{G})} \leq C_1 \cdot \|\Phi_n(\zeta)\|_{C^{k+1}_\alpha(\overline{G})}; \\ &c_2 \cdot \|\Phi_n(\zeta)\|_{W^{k+2}_p(\overline{G})} \leq \|\Phi(\zeta)\|_{W^{k+2}_p(\overline{G})} \leq C_2 \cdot \|\Phi_n(\zeta)\|_{W^{k+2}_p(\overline{G})}, \end{aligned} \tag{1.26}$$

where the constants $C_1 \geq c_1 > 0$ *è* $C_2 \geq c_2 > 0$ *do not depend on* w.

Theorem 1.12. *Let* $w(z) \in W_p^1(\overline{D})$ *be a solution to Equation* (1.5). *In the assumptions of Theorem 1.2 the next inequalities are valid:*

$$\begin{aligned} c_3 \cdot \|\Phi_n^*(z)\|_{W_p^1(\overline{D})} &\leq \|\Phi(z)\|_{W_p^1(\overline{D})} \leq C_3 \cdot \|\Phi_n^*(z)\|_{W_p^1(\overline{D})};\\ c_4 \cdot \|\Phi^*(z)\|_{W_p^1(\overline{D})} &\leq \|\Phi(z)\|_{W_p^1(\overline{D})} \leq C_4 \cdot \|\Phi^*(z)\|_{W_p^1(\overline{D})};\\ c_5 \cdot \|\Phi_n^*(z)\|_{W_p^1(\overline{D})} &\leq \|\Phi^*(z)\|_{W_p^1(\overline{D})} \leq C_5 \cdot \|\Phi_n^*(z)\|_{W_p^1(\overline{D})}, \end{aligned} \tag{1.27}$$

where the constants $C_j \geq c_j > 0$, $j = 3, 4, 5$, *do not depend on* w.

Theorem 1.13. *Let* $w(z) \in C_\alpha^{k+1}(\overline{D})$ $(W_p^{k+2}(\overline{D}))$ *be a solution to Equation* (1.5). *In the assumptions of Theorem 1.3 the next inequalities are valid:*

$$\begin{aligned} c_6 \cdot \|\Phi_n^*(z)\|_{C_\alpha^{k+1}(\overline{D})} &\leq \|\Phi(z)\|_{C_\alpha^{k+1}(\overline{D})} \leq C_6 \cdot \|\Phi_n^*(z)\|_{C_\alpha^{k+1}(\overline{D})};\\ c_7 \cdot \|\Phi^*(z)\|_{C_\alpha^{k+1}(\overline{D})} &\leq \|\Phi(z)\|_{C_\alpha^{k+1}(\overline{D})} \leq C_7 \cdot \|\Phi^*(z)\|_{C_\alpha^{k+1}(\overline{D})};\\ c_8 \cdot \|\Phi_n^*(z)\|_{C_\alpha^{k+1}(\overline{D})} &\leq \|\Phi^*(z)\|_{C_\alpha^{k+1}(\overline{D})} \leq C_8 \cdot \|\Phi_n^*(z)\|_{C_\alpha^{k+1}(\overline{D})}; \end{aligned} \tag{1.28}$$

$$\begin{aligned} c_9 \cdot \|\Phi_n^*(z)\|_{W_p^{k+2}(\overline{D})} &\leq \|\Phi(z)\|_{W_p^{k+2}(\overline{D})} \leq C_9 \cdot \|\Phi_n^*(z)\|_{W_p^{k+2}(\overline{D})};\\ c_{10} \cdot \|\Phi^*(z)\|_{W_p^{k+2}(\overline{D})} &\leq \|\Phi(z)\|_{W_p^{k+2}(\overline{D})} \leq C_{10} \cdot \|\Phi^*(z)\|_{W_p^{k+2}(\overline{D})};\\ c_{11} \cdot \|\Phi_n^*(z)\|_{W_p^{k+2}(\overline{D})} &\leq \|\Phi^*(z)\|_{W_p^{k+2}(\overline{D})} \leq C_{11} \cdot \|\Phi_n^*(z)\|_{W_p^{k+2}(\overline{D})}, \end{aligned} \tag{1.29}$$

where the constants $C_j \geq c_j > 0$, $j = 6, \ldots, 11$, *do not depend on* w.

Theorem 1.14. *Let* $w(\zeta) \in W_p^1(\overline{G})$ *be a solution to Equation* (1.5). *In the assumptions of Theorem 1.4 for the model functions* $f(\zeta)$ *and* $\Phi(\zeta)$ *of the solution* $w(\zeta)$ *from* (1.7) *and* (1.17), *and for* $f_n(\zeta)$ *and* $\Phi_n(\zeta)$ *from* (1.11) *and* (1.19), *and* $\widetilde{f}(\zeta)$ *and* $\Phi(\zeta)$ *from* (1.13) *and* (1.17), *the next inequalities are valid:*

$$\begin{aligned} c_{12} \cdot \|f(\zeta)\|_{W_s^1(\overline{G})} &\leq \|\Phi(\zeta)\|_{W_s^1(\overline{G})} \leq C_{12} \cdot \|f(\zeta)\|_{W_s^1(\overline{G})};\\ c_{13} \cdot \|f_n(\zeta)\|_{W_s^1(\overline{G})} &\leq \|\Phi_n(\zeta)\|_{W_s^1(\overline{G})} \leq C_{13} \cdot \|f_n(\zeta)\|_{W_s^1(\overline{G})};\\ c_{14} \cdot \|\widetilde{f}(\zeta)\|_{W_s^1(\overline{G})} &\leq \|\Phi(\zeta)\|_{W_s^1(\overline{G})} \leq C_{14} \cdot \|\widetilde{f}(\zeta)\|_{W_s^1(\overline{G})}, \end{aligned} \tag{1.30}$$

where $s : 2 < s \leq p$ *is sufficiently close to two, and the constants* $C_j \geq c_j > 0$, $j = 12, 13, 14$, *do not depend on* $w(\zeta)$.

If $q_2(\zeta) = B(\zeta) \equiv 0$, *then we can put* $s = p$; *otherwise, generally speaking,* $s < p$.

Theorem 1.15. *Let* $w(z) \in W_p^1(\overline{D})$ *be a solution to Equation* (1.5). *In the assumptions of Theorem 1.2 for the model functions* $f^*(z)$ *and* $\Phi^*(z)$, $f_n^*(z)$ *and* $\Phi_n^*(z)$ *of the solution* $w(z)$ *from* (1.10), (1.21) *and* (1.12), (1.23) *correspondently, the next inequalities are valid:*

$$\begin{aligned} c_{15} \cdot \|f^*(z)\|_{W_s^1(\overline{D})} &\leq \|\Phi^*(z)\|_{W_s^1(\overline{D})} \leq C_{15} \cdot \|f^*(z)\|_{W_s^1(\overline{D})};\\ c_{16} \cdot \|f_n^*(z)\|_{W_s^1(\overline{D})} &\leq \|\Phi_n^*(z)\|_{W_s^1(\overline{D})} \leq C_{16} \cdot \|f_n^*(z)\|_{W_s^1(\overline{D})};\\ c_{17} \cdot \|f_n^*(z)\|_{W_s^1(\overline{D})} &\leq \|f^*(z)\|_{W_s^1(\overline{D})} \leq C_{17} \cdot \|f_n^*(z)\|_{W_s^1(\overline{D})}, \end{aligned}$$

where $s : 2 < s \le p$ is sufficiently close to two, and the constants, $C_j \ge c_j > 0$, $j = 15, 16, 17$, do not depend on $w(z)$.

If $q_2(z) = B(z) \equiv 0$, then we can put $s = p$; otherwise, generally speaking, $s < p$.

Theorem 1.16. *Let $w(\zeta) \in C_\alpha^{k+1}(\overline{G})$ $(W_p^{k+2}(\overline{G}))$ be a solution to Equation (1.5), $q_2(\zeta) = B(\zeta) \equiv 0$. Then, in the assumptions of Theorem 1.5 and the former notation for model functions, the next inequalities are valid:*

$$c_{18} \cdot \|f(\zeta)\|_{C_\alpha^{k+1}(\overline{G})} \le \|\Phi(\zeta)\|_{C_\alpha^{k+1}(\overline{G})} \le C_{18} \cdot \|f(\zeta)\|_{C_\alpha^{k+1}(\overline{G})};$$

$$c_{19} \cdot \|f_n(\zeta)\|_{C_\alpha^{k+1}(\overline{G})} \le \|\Phi_n(\zeta)\|_{C_\alpha^{k+1}(\overline{G})} \le C_{19} \cdot \|f_n(\zeta)\|_{C_\alpha^{k+1}(\overline{G})};$$

$$c_{20} \cdot \|\widetilde{f}(\zeta)\|_{C_\alpha^{k+1}(\overline{G})} \le \|\Phi(\zeta)\|_{C_\alpha^{k+1}(\overline{G})} \le C_{20} \cdot \|\widetilde{f}(\zeta)\|_{C_\alpha^{k+1}(\overline{G})},$$

$$c_{21} \cdot \|f(\zeta)\|_{W_p^{k+2}(\overline{G})} \le \|\Phi(\zeta)\|_{W_p^{k+2}(\overline{G})} \le C_{21} \cdot \|f(\zeta)\|_{W_p^{k+2}(\overline{G})};$$

$$c_{22} \cdot \|f_n(\zeta)\|_{W_p^{k+2}(\overline{G})} \le \|\Phi_n(\zeta)\|_{W_p^{k+2}(\overline{G})} \le C_{22} \cdot \|f_n(\zeta)\|_{W_p^{k+2}(\overline{G})};$$

$$c_{23} \cdot \|\widetilde{f}(\zeta)\|_{W_p^{k+2}(\overline{G})} \le \|\Phi(\zeta)\|_{W_p^{k+2}(\overline{G})} \le C_{23} \cdot \|\widetilde{f}(\zeta)\|_{W_p^{k+2}(\overline{G})},$$

where the constants $C_j \ge c_j > 0$, $j = 18, \dots, 23$, do not depend on $w(\zeta)$.

Theorem 1.17. *Let $w(z) \in C_\alpha^{k+1}(\overline{D})$ $(W_p^{k+2}(\overline{D}))$ be a solution to Equation (1.5), $q_2(z) = B(z) \equiv 0$. Then, in the assumptions of Theorem 1.3 and the former notation for model functions, the next inequalities are valid:*

$$c_{24} \cdot \|f^*(z)\|_{C_\alpha^{k+1}(\overline{D})} \le \|\Phi^*(z)\|_{C_\alpha^{k+1}(\overline{D})} \le C_{24} \cdot \|f^*(z)\|_{C_\alpha^{k+1}(\overline{D})};$$

$$c_{25} \cdot \|f_n^*(z)\|_{C_\alpha^{k+1}(\overline{D})} \le \|\Phi_n^*(z)\|_{C_\alpha^{k+1}(\overline{D})} \le C_{25} \cdot \|f_n^*(z)\|_{C_\alpha^{k+1}(\overline{D})};$$

$$c_{26} \cdot \|f_n^*(z)\|_{C_\alpha^{k+1}(\overline{D})} \le \|f(z)\|_{C_\alpha^{k+1}(\overline{D})} \le C_{26} \cdot \|f_n^*(z)\|_{C_\alpha^{k+1}(\overline{D})},$$

$$c_{27} \cdot \|f^*(z)\|_{W_p^{k+2}(\overline{D})} \le \|\Phi^*(z)\|_{W_p^{k+2}(\overline{D})} \le C_{27} \cdot \|f^*(z)\|_{W_p^{k+2}(\overline{D})};$$

$$c_{28} \cdot \|f_n^*(z)\|_{W_p^{k+2}(\overline{D})} \le \|\Phi_n^*(z)\|_{W_p^{k+2}(\overline{D})} \le C_{28} \cdot \|f_n^*(z)\|_{W_p^{k+2}(\overline{D})};$$

$$c_{29} \cdot \|f_n^*(z)\|_{W_p^{k+2}(\overline{D})} \le \|f(z)\|_{W_p^{k+2}(\overline{D})} \le C_{29} \cdot \|f_n^*(z)\|_{W_p^{k+2}(\overline{D})},$$

where the constants $C_j \ge c_j > 0$, $j = 24, \dots, 29$, do not depend on $w(z)$.

2. Auxiliary Statements

The next assertions are valid (see [10, Ch. 1, §6, §8], [1], [2], [6]).

Lemma 2.1. *If $\partial G = \mathcal{L} \in C_\alpha^{k+1}$, $0 < \alpha < 1$, $k \ge 0$, ($W_p^{k+1-\frac{1}{p}}$, $p > 2$, at $k \ge 1$, and C_α^1 at $k = 0$), then the singular integral operator Π maps continuously the Banach space $C_\alpha^k(\overline{G})$ ($W_p^k(\overline{G})$) into itself. Herewith $\|\Pi\|_{L_2} = 1$ and for any q_0 : $0 < q_0 < 1$ there exists $s_0 = s_0(q_0) > 2$ such, that $q_0\|\Pi\|_{L_s} < 1$ at $\forall s : 2 < s < s_0$.*

The same statement is true for operators Π^* and Π_n^*.

Lemma 2.2. *If $\partial G = \mathcal{L} \in C_\alpha^{k+1}$, $0 < \alpha < 1$, $k \ge 0$, ($W_p^{k+1-\frac{1}{p}}$, $p > 2$, at $k \ge 1$, and C_α^1 at $k = 0$), then the operators T, T_n maps continuously the Banach space $C_\alpha^k(\overline{G})$, $k \ge 0$, $0 < \alpha < 1$, in $C_\alpha^{k+1}(\overline{G})$ ($W_p^k(\overline{G})$, $p > 2$, in $W_p^{k+1}(\overline{G})$).*

If $G = D$, for operators T^* and T_n^* the same statements are true.

When $f(\zeta) \in L_p(\overline{G})$ the function $Tf(\zeta) \in C_\beta(E)$, $\beta = \dfrac{p-2}{p}$, is holomorphic outside $\overline{G}$ and is zero at infinity.

Based on these lemmas, we will establish now some properties (both new and known) of two–dimensional singular integral operators.

Lemma 2.3. *Let $\mathcal{L} = \partial G \in C^1_\alpha$, $0 < \alpha < 1$, $q_1(\zeta)$ and $q_2(\zeta)$ are bounded measurable functions, defined in G and satisfying inequality (1.6).*

Then the singular integral equation

$$\omega(\zeta) + q_1(\zeta)S\omega(\zeta) + q_2(\zeta)\overline{S\omega(\zeta)} = h(\zeta) \in L_p(\overline{G}),\ p > 2, \tag{2.1}$$

where $S = \Pi$ or $S = \Pi_n$, has a unique solution $\omega(\zeta) \in L_s(\overline{G})$, $2 < s \le p$, and s is generally speaking close enough to two.

If $q_1(\zeta), q_2(\zeta) \in C(\overline{G})$, then the operator on the left–hand side of (2.1) is a real linear isomorphism of the Banach space $L_p(\overline{G})$, i. e. $\omega(\zeta) \in L_p(\overline{G})$.

Proof. Preliminarily we note that the operator under consideration in both cases by Lemma 2.1 maps continuously the Banach space $L_s(\overline{G})$, $\forall s > 2$, into itself.

The existence of a unique solution $\omega(\zeta) \in L_s(\overline{G})$, $2 < s \le p$, at $S = \Pi$ actually proved in [1]. Let us recall some details here.

By Lemma 2.1 and (1.6) for some $s : 2 < s \le p$ we have $q_0\|\Pi\|_{L_s(\overline{G})} < 1$. So, the operator

$$\omega(\zeta) \to q_1(\zeta)S\omega(\zeta) + q_2(\zeta)\overline{S\omega(\zeta)}$$

on the left–hand side of (2.1) is a contractive mapping in $L_s(\overline{G})$. From here we have the first assertion of Lemma 2.3 at $S = \Pi$.

Let now $S = \Pi_n$. Let us show that the homogeneous equation has only a zero solution in this case. Let $\omega(\zeta) \in L_s(\overline{G})$, $s > 2$, be a solution to the homogeneous equation. We continue the coefficients $q_1(\zeta)$ and $q_2(\zeta)$ outside G with zeros. Then, by Lemma 2.2 the function $w(\zeta) = T_n\omega(\zeta) \in W^1_s(\overline{G})$ is a continuous solution to the equation

$$\partial_{\bar\zeta}w + q_1(\zeta)\partial_\zeta w + q_2(\zeta)\partial_{\bar\zeta}\overline{w} = 0$$

on the whole plane. This solution has not less than n zeros in $\overline{G}$ and a pole order not higher $n-1$ at infinity. Since the principle of argument is valid for the solution $w(\zeta)$ [1, p. 478], from here we have $w(\zeta) = T_n\omega(\zeta) \equiv 0$. Differentiating this equality by $\bar\zeta$, we receive $\omega(\zeta) \equiv 0$.

We rewrite (2.1) as

$$Q_1\omega(\zeta) + K_1\omega(\zeta) = h(\zeta), \tag{2.2}$$

where $Q_1\omega(\zeta) = \omega(\zeta) + q_1(\zeta)\Pi\omega(\zeta) + q_2(\zeta)\overline{\Pi\omega(\zeta)}$ is on the proved linear isomorphism of Banach space $L_s(\overline{G})$, and $K_1\omega(\zeta)$ is a line combination of polynomials that is not difficult to calculate, by differentiating by ζ $P_n\omega(\zeta)$ from (1.2).

Let's show that the operator $K_1 : L_s(\overline{G}) \to L_s(\overline{G})$ is compact. By Lemma 2.1 the operator K_1 maps any closed ball $\|w\|_{L_s(\overline{G})} \le \text{const}$ in the closed set which is contained in a compact subset of $L_s(\overline{G})$. This subset is homeomorphic to the Descartes product of n^2 closed disks $|\zeta| \le \text{const}$. Thus, the image of every closed ball is compact and the operator K_1 is compact.

So, $Q_1 + K_1$ is the sum of isomorphism and compact operator. From here we have that $Q_1 + K_1$ is isomorphism (see [9, §14]).

The first part of the lemma is proven. Let's prove the second part.

For $S = \Pi$ the relevant statement is proved in [7]. Using it, for $S = \Pi_n$ the proof repeats verbatim the previous reasoning, only with a replacement $L_s(\overline{G})$ with $L_p(\overline{G})$. $\square$

Lemma 2.4. *If $\partial G = \mathcal{L} \in C^{k+1}_\alpha$, $(W^{k+2-1/p})$, $q_1(\zeta), q_2(\zeta) \in C^k_\alpha(\overline{G})$ $(W^{k+1}_p(\overline{G}))$, $k \geq 0$, $0 < \alpha < 1$ $(p > 2)$, then the singular integral operator*

$$Q\omega(\zeta) \equiv \omega(\zeta) + q_1(\zeta)S\omega(\zeta) + q_2(\zeta)\overline{S\omega(\zeta)} \tag{2.3}$$

where $S = \Pi$ or $S = \Pi_n$, is a real linear isomorphism of the Banach space $C^k_\alpha(\overline{G})$ $(W^{k+1}_p(\overline{G}))$.

Proof. For $S = \Pi$ the assertion was proved in [7]. The proof for the case $S = \Pi_n$ is carried out by verbatim repetition of the relevant reasoning from the proof of Lemma 2.3, but with the replacement $L_s(\overline{G})$ with $C^k_\alpha(\overline{G})$ $(W^{k+1}_p(\overline{G}))$. $\square$

Lemma 2.5. *Let $q_1(z)$ and $q_2(z)$ are bounded measurable functions defined in D and satisfying* (1.6).

Then the singular integral equation

$$\omega(z) + q_1(z)S\omega(z) + q_2(z)\overline{S\omega(z)} = h(z) \in L_p(\overline{D}),\ p > 2, \tag{2.4}$$

where $S = \Pi^*$ or $S = \Pi^*_n$, has the unique solution $\omega(z) \in L_s(\overline{D})$, $2 < s \leq p$, and s is generally speaking close enough to two.

If $q_1(z), q_2(z) \in C(\overline{D})$, the operator on the left–hand side of (2.3) *is a real linear isomorphism of the Banach space $L_p(\overline{D})$, i. e. $\omega(z) \in L_p(\overline{D})$.*

Proof. Preliminarily note that according to Lemma 2.1 the operator on the left–hand side of (2.4) maps continuously the Banach space $L_s(\overline{D})$ into itself for $\forall s > 2$.

The first assertion of the lemma was proved for $S = \Pi^*$ in [1], and for $S = \Pi^*_n$ in [2].

Just as above, to prove the second statement of the lemma by virtue of Banach's theorem, it is enough to show that the solution $\omega(z) \in L_s(\overline{D})$ to Equation (2.4) actually belongs to $L_p(\overline{D})$. We will prove it for $S = \Pi^*$; for $S = \Pi^*_n$ the reasoning is repeated almost verbatim.

Denote $w(z) = T^*\omega(z)$. The function $w(z)$ is a solution to the equation

$$\partial_{\bar z} w + q_1(z)\partial_z w + q_2(z)\partial_{\bar z}\overline{w} = h(z),$$

and at $z \in \Gamma$ $\operatorname{Re} w(z) = 0$.

By virtue of Pompeiu's formula with the operator T^*, the function $w(z)$ is a solution to the equation (at $A = B \equiv 0$)

$$\Omega^* w(z) = iC + T^*h(z) \in W^1_p(\overline{D}),$$

where C is a real constant.

From here, by Theorem 1.8, $w(z) \in W^1_p(\overline{D})$ (the proof of Theorem 1.8 does not based on Lemma 2.5, see below). Differentiating by $\bar z$ the equality $w(z) = T^*\omega(z)$, we get $\omega(z) \in L_p(\overline{D})$. $\square$

Just as Lemma 2.3 derives Lemma 2.4, the following statement is derived from Lemma 2.5.

Lemma 2.6. *If $q_1(z), q_2(z) \in C^k_\alpha(\overline{D})$ $(W^{k+1}_p(\overline{D}))$, $k \geq 0$, $0 < \alpha < 1$ $(p > 2)$, then the singular integral operator*

$$\omega(z) + q_1(z)S\omega(z) + q_2(z)\overline{S\omega(z)}$$

where $S = \Pi^*$ or $S = \Pi^*_n$, is a real linear isomorphism of the Banach space $C^k_\alpha(\overline{D})$ $(W^{k+1}_p(\overline{D}))$.

3. Proofs of Theorems

3.1. Proofs of Theorems 1.6 and 1.7. We will prove Theorem 1.6 in detail. At first, we note that by Lemma 2.2 the linear operator Ω_n maps continuously the Banach space $W^1_p(\overline{G})$ into itself.

Let's show that the kernel of the operator Ω_n is zero. Let's continue the coefficients of Equation (1.5) outside $\overline{G}$ with zeros and consider a homogeneous equation

$$\Omega_n(w) = 0.$$

The solution $w(\zeta) \not\equiv 0$ to this equation by Lemma 2.2, is continuous on the whole complex plane E and holomorphic outside $\overline{G}$. Also, $w(\zeta)$ is a solution to Equation (1.5) with continued coefficients on the whole plane E. At the same time, it has at least n zeros in $\overline{G}$ and the pole of order not higher than $n-1$ at infinity. Because for the solution $w(\zeta)$ the principle of argument is valid [1, p. 478], hence $w(\zeta) \equiv 0$.

Further, by virtue of Banach's theorem, it is enough to show the unambiguous solvability of the equation

$$\Omega_n(w) \equiv \Omega(w) + P_n(w) = F,$$

in the class $W^1_p(\overline{G})$ for $\forall F(\zeta) \in W^1_p(\overline{G})$.

Let's show that the operator $P_n : W^1_p(\overline{G}) \to A^1_p(\overline{G}) \subset W^1_p(\overline{G})$ is compact. By Lemma 2.2 the operator P_n maps any closed ball $\|w\|_{W^1_p(\overline{G})} \leq \text{const}$ in the closed set which is contained in a compact subset of $A^1_p(\overline{G})$. This subset is homeomorphic to the Descartes product of n closed disks $|\zeta| \leq \text{const}$. Thus, the image of every closed ball is compact and the operator P_n is compact.

So, taking into account Theorem 1.4 we get that Ω_n is the sum of isomorphism Ω and compact operator P_n. It follows that Ω_n is isomorphism (see [9, §14]).

Theorem 1.6 is proven.

The proof of Theorem 1.7 is similar to previous proof. We have only to replace functional spaces in the above reasoning and replace the reference to Theorem 1.4 with a reference to Theorem 1.5.

3.2. Proofs of Theorems 1.8 and 1.9. We will prove Theorem 1.8 in detail. At first, we note that by Lemma 2.2 the linear operator Ω^* maps continuously the Banach space $W^1_p(\overline{G})$ into itself. Also, Ω^* has zero kernel [10, Ch. 4, §9].

Thus, by virtue of Banach's theorem, it is enough to show the unambiguous solvability of the equation

$$\Omega^*(w) = F(z) \tag{3.1}$$

in $W^1_p(\overline{D})$ at $\forall F(z) \in W^1_p(\overline{D})$.

By verbatim repetition of the reasoning from the proof of lemma 4 of [2] (with the replacement of the operator T_n^* to T^*), we get that the equation (3.1) has a unique solution $w(z) \in W_s^1(\overline{D})$, where $2 < s \leq p$ and s is close enough to two.

Subtracting from both parts of Equation (3.1) the number

$$-\frac{1}{\pi}\iint\limits_{D}\left[\frac{f(t)}{t-z_1}+\frac{z_1\overline{f(t)}}{1-\bar{t}z_1}\right]dxdy,\ t = x+iy,$$

where $f(t) = q_1\partial_t w + q_2\partial_{\bar{t}}\overline{w} + Aw + B\overline{w}$, z_1 is arbitrary point on Γ, we get that $w(z)$ is a solution to the equation

$$\Omega_0^*(w) = F(z) - \text{const} \in W_p^1(\overline{D}).$$

By Theorem 1.2 from here we get $w(z) \in W_p^1(\overline{D})$.

Theorem 1.8 is proven.

The proof of Theorem 1.9 is similar, it should only replace the functional spaces in the above reasoning and replace the reference to Theorem 1.2 with a reference to Theorem 1.3.

3.3. Proofs of Theorems 1.10 – 1.13. Because the function $w(z)$ is a solution to Equation (1.5), by Pomreiu's formula it satisfies to equations

$$\Omega(w) = \Phi(\zeta) \quad \text{è} \quad \Omega_n(w) = \Phi_n(\zeta),$$

where $\Phi(\zeta)$, $\Phi_n(\zeta) \in A_p^1(\overline{G})$. From here, by Theorem 1.4 and Theorem 1.6 we have:

$$\|w\|_{W_p^1(\overline{G})} \leq \text{const}\|\Phi\|_{W_p^1(\overline{G})} \ \text{ è } \ \|w\|_{W_p^1(\overline{G})} \leq \text{const}\|\Phi_n\|_{W_p^1(\overline{G})},$$

and, by virtue of the continuity of the operators Ω and Ω_n in $W_p^1(\overline{G})$,

$$\|\Phi\|_{W_p^1(\overline{G})} \leq \text{const}\|w\|_{W_p^1(\overline{G})} \ \text{ è } \ \|\Phi_n\|_{W_p^1(\overline{G})} \leq \text{const}\|w\|_{W_p^1(\overline{G})},$$

where constants do not depend on w.

Comparing all these inequalities, we get (1.25).

Theorem 1.11 is proved similarly, only using Theorems 1.5 and 1.7.

Theorems 1.12 and 1.13 are also proved similarly, using Theorems 1.2, 1.3, 1.8 and 1.9. Note only that the third inequality in (1.27), (1.28) and (1.29) is a simple consequence of the previous two.

First kind representations (1.7), (1.10) were proved in [1]. Representations (1.11) – (1.12) are proved similarly. Let's prove the representation (1.13), which at the same time demonstrates the idea of proving previous first kind representations.

For a solution $w(\zeta) \in W_p^1(\overline{G})$ to Equation (1.5) by Lemma 2.3 there exists the unique solution to Equation (1.14) $\omega(\zeta) \in W_s^1(\overline{G})$. Denote

$$\widetilde{f}(\zeta) = w(\zeta)e^{-T\omega(\zeta)} \in W_s^1(\overline{G}).$$

It remains to be shown that the function $\widetilde{f}(\zeta)$ satisfies to Equation (1.15).

The ratios are obvious:

$$\widetilde{f}_{\bar\zeta} = w_{\bar\zeta} e^{-T\omega} + w e^{-T\omega}(-\omega),$$
$$\widetilde{f}_{\zeta} = w_{\zeta} e^{-T\omega} + w e^{-T\omega}(-\Pi\omega),$$
$$\overline{\widetilde{f}}_{\bar\zeta} = \overline{w}_{\bar\zeta} e^{-\overline{T\omega}} + \overline{w} e^{-\overline{T\omega}}(-\overline{\Pi\omega}).$$

From here we have:

$$\widetilde{f}_{\bar\zeta} + q_1 \cdot \widetilde{f}_{\zeta} + q_2 \cdot \overline{\widetilde{f}}_{\bar\zeta} \cdot e^{\overline{T\omega} - T\omega} = e^{-T\omega}(w_{\bar\zeta} + q_1 w_{\zeta} + q_2 \overline{w}_{\bar\zeta}) -$$
$$- w e^{-T\omega}\left(\omega + q_1 \Pi\omega + q_2 \frac{\overline{w}}{w}\overline{\Pi\omega}\right) =$$
$$= - w e^{-T\omega}\left(\omega + q_1 \Pi\omega + q_2 \frac{\overline{w}}{w}\overline{\Pi\omega} + A + B\frac{\overline{w}}{w}\right) = 0.$$

3.4. Proofs of Theorems 1.14 – 1.17. We will prove Theorem 1.14 in detail. Consider the first double inequality from (1.30). By Lemma 2.3 the solution $\omega = \omega(\zeta)$ to Equation (1.9) belongs to $L_s(\overline{G})$, where $2 < s \le p$, and

$$\|\omega\|_{W_s^1(\overline{G})} \le \text{const},$$

where constant do not depend on w.

Next, by Lemma 2.2

$$\|T\omega\|_{W_s^1(\overline{G})} \le \text{const}\|\omega\|_{L_s(\overline{G})} \le \text{const},$$

where constants do not depend on w.

From here and from (1.7) we have $f(\zeta) \in W_s^1(\overline{G})$, and taking into account Theorem 1.4 we get

$$\|f\|_{W_s^1(\overline{G})} \le \text{const}\|w\|_{W_s^1(\overline{G})} \le \text{const}\|\Phi\|_{W_s^1(\overline{G})}, \tag{3.2}$$

where $\Omega(w) = \Phi$ and constants do not depend on w.

Because the operator Ω is continuous in $W_s^1(\overline{G})$,

$$\|\Phi\|_{W_s^1(\overline{G})} \le \text{const}\|w\|_{W_s^1(\overline{G})}, \tag{3.3}$$

where constant do not depend on w.

From (1.7) we have

$$\|w\|_{W_s^1(\overline{G})} \le \text{const}\|f\|_{W_s^1(\overline{G})}, \tag{3.4}$$

where constant do not depend on w.

By matching (3.2) – (3.4), we get the first double inequality from (1.30).

The second and third inequalities are proved similarly, only in addition to Theorem 1.4 you need to use Theorem 1.6.

Let now $q_1(\zeta) \in C(\overline{G})$, $q_2(\zeta) = B(\zeta) \equiv 0$. Then, by Lemma 2.3 the singular integral operators in (1.9) and (1.14) are isomorphisms of the Banach space $L_p(\overline{G})$ and in all reasoning for all three pairs of inequalities, the space $W_s^1(\overline{G})$ can be replaced with $W_p^1(\overline{G})$.

Theorem 1.14 is proven.

Theorems 1.15 – 1.17 are proved similarly, with references to the corresponding theorems and lemmas of this work.

References

[1] Bojarski, B.: Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients, *Mat Sb.* **43** (1957) 451–503. [in Russian] English translation in Rep. Univ. Jyväskylä Dept. Math. Stat., 2009; 118.

[2] Klimentov, S.B.: On a Method of Constructing the Solutions of Boundary-Value Problems of the Theory of Bendings of Surfaces of Positive Curvature, *Journal of Mathematical Sciences*, *51* N2 (1990) 2230–2248.

[3] Klimentov, S.B.: On Isomorphism of Some Functional Spaces under Action of Integro–Differential Operators, *Ufa Mathematical Journal* **11** N1 (2019) 42–62.

[4] Klimentov, S.B.: Representations of the Second Kind for the Solutions to the First Order General Linear Elliptic System in the Simply Connected Plane Domain, *Global and Stochastic Analysis* **6** N1 (2019) 7–21.

[5] Klimentov, S.B.: Representations of the "Second Kind" for the Hardy Classes of Solutions to the Beltrami Equation, *Siberian Mathematical Journal* **55** N2 (2014) 262–275.

[6] Klimentov, S.B.: Another version of Kellogg's theorem, *Complex Variables and Elliptic Equations* **60** (2015) 1647–1657.

[7] Klimentov, S.B.: On Isomorphism of Some Functional Spaces under Action of Two-dimensional Singular Integral Operators, *Operator Theory and Differential Equations* Birkhäuser, Trends in Mathematics, Springer Nature Switzerland AG. Part of Springer Nature (2020) 69–77.

[8] Klimentov, S.B.: Second Kind Representations of Sobolev Space Solutions to a First Order General Elliptic Linear System in a Simply Connected Plane Domain, *Sibirskii Matematicheskii Zhurnal*, **62** N3 (2021) 542–558.

[9] Krein, S.G.: *Linear Equations in Banach Spaces*, Birkhäuser, Boston, 1982.

[10] Vekua, I.N.: *Generalized Analytic Functions*, Pergamon Press, Oxford/London/New York/Paris, 1962.

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