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BAYESIAN INFERENCE OF MARKOVIAN QUEUEING MODEL WITH TWO HETEROGENEOUS SERVERS

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Abstract. In this paper we consider the Bayesian inference of Markovian queueing model with two heterogeneous servers with service rate μ_1 and μ_2 where $\mu_1 > \mu_2$. Assuming Mckays bivariate gamma prior distribution for service rates (μ_1, μ_2) and gamma prior distribution for arrival rate λ, closed form expressions for the Bayes estimates of the queue parameters under squared error loss function are obtained. Bayes estimates and bootstrap credible interval for queue parameters are computed for different set of simulated data. Also we apply Markov Chain Monte Carlo method using the same prior distribution and computed Bayes estimates and credible interval of queue parameters for different set of hyper parameters and compare the values with bootstrap estimates.

1. Introduction

Statistical inference plays an extremely vital role in any use of queueing as an aid to decision-making. The statistical inference problems can be generally divided into two types, viz. "parameter estimation" and "distribution selection". The pioneer work in the problem of statistical inference for queuing models dates back to 1957, when Clarke[10] proposed the problem of obtaining maximum likelihood estimates (MLE) for the parameters involved in the stationary M/M/1 queue. Bene[7] presented a similar exposition for M/M/∞. A general overview of the statistical inference for Markov processes can be found in Billingsley [8] and Basawa and Rao [6]. Dave and Shah [11] discussed the maximum likelihood estimates of the parameters of a stationary M/M/2 queue with heterogeneous serves. Jain and Templeton [17] obtained the confidence interval for an M/M/2 queue from the MLE. Wang et al. [27] derived the MLE and confidence intervals of an M/M/c queue.

The queue parameters λ and μ may not be deemed to be constants in real life situations, but they are random variables. Combining the prior information about these parameters with the current data on the queueing system, we can improve the estimates of the queue characteristics. This can be done by the Bayesian analysis. McGrath and Singpurwalla [20] and McGrath et al.[21] used the subjective Bayesian approach to the statistical inference in queues. Armero

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and Bayarri [2] developed the Bayesian prediction on M/M/1 queue. Armero and Conesa [4] studied Bayesian inference and prediction for Markovian queues with bulk arrivals. David et al.[12] described Bayesian inference and prediction for some M/G/1 queueing models. Choudhury and Borthakur [9] also discussed the Bayesian inference for an M/M/1 queue in detail using system size data. Mukharjee and Chowdhury [23] derived the Bayes estimates of traffic intensity and measures of effectiveness using beta distribution as prior. Jose and Manoharan [18] obtained the Bayes estimate of queue parameters using bivariate prior distribution for λ and μ with order restriction $\lambda < \mu$.

The Bayesian estimation of Markovian queueing system with heterogeneous servers has not been addressed in literture. In this paper we discuss Bayesian estimation of $M/M/2$ queueing model with two heterogeneous servers. That is we have a two-server queue under the assumption of Poisson arrivals with rate λ and exponentially distributed service times with different service rates μ_i (i= 1, 2) for each of the two servers. The service rates are such that $\mu_1 > \mu_2$. Thus, μ_1 corresponds to the faster server and μ_2 to the slower server. An arriving customer finding both servers free, chooses for his service the faster one. Mckay's bivariate gamma distribution [22] satisfying the order restriction $\mu_1 > \mu_2$, as the joint prior distribution for μ_1 and μ_2 and gamma distribution as the prior for λ.

This paper is organized as follows. In section 2, we obtain the posterior joint and marginal distribution of queue parameters. Bayes estimates of queue characteristics under squared error loss function and its numerical analysis are also obtained in section 2. In section 3 , we apply Markov Chain Monte Carlo method to obtain Bayes estimates of queue parameters.

In the derivation of posterior marginal distribution of $\lambda,\ \mu_1,\ \mu_2$ and the Bayes estimates of $\lambda,\ \mu_1$ and μ_2, we use various results on special functions such as confluent hyper geometric function and Gauss hyper geometric function. Confluent hyper geometric function is defined as

$$F(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{(b)_k k!} \tag{1.1}$$

and the Gauss hyper geometric function is defined as

$$G(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}\ , \tag{1.2}$$

where $(a)_k = a(a+1)(a+2)...(a+k-1)$ denotes the ascending factorial.

2. Bayesian Inference

Consider a stationary M/M/2 queue with heterogeneous servers. The mean arrival rate is λ and mean service rates are μ_1 and μ_2, where $\mu_1 > \mu_2$ and the traffic intensity is $\rho = \frac{\lambda}{\mu_1+\mu_2}$. The steady state condition for the queue is $\lambda < \mu_1 + \mu_2$. Here we assume that customers wait in a line in order of their arrival and those arriving when both servers are idle go to the faster one for service. Otherwise, they enter service in the order of their arrival as and when servers become free i.e. the queue is in front of the faster server. Assume that the queue begins operation with n_0 customers present and the queue is being observed for a fixed amount of time T, where T is sufficiently large enough to give adequate number of observations. Suppose that during T, there are n_a number of arrivals to the queue and n_d number of departures from the queue.

When the queue is empty, the servers of an M/M/2 queue is idle and the amount of time during which both the servers are idle is T_e. If there is one customer in the queue, only the faster server is busy. T_{fb} denotes the time during which only one customer in the queue. When there are more than one customers in the queue then both the servers are busy and the amount of time during which both the servers are busy is denoted by T_{sb}. If both the servers are busy, the queue is empty and n_c be the number of arrivals to an empty queue. When the faster server is busy, then the queue is said to be partially busy and the number of arrivals to a partially busy queue is denoted by n_{fu}.The number of departures from a partially busy queue is denoted as n_{fd}. The number of arrivals to a completely busy queue is n_{su} and the number of departures from a partially busy queue is n_{sd}. Then we have

$$T = T_e + T_{fb} + T_{sb} \tag{2.1}$$

$$n_a = n_e + n_{fu} + n_{su} \tag{2.2}$$

$$n_d = n_{sd} + n_{fd} \tag{2.3}$$

The corresponding likelihood function is made up of the following four basic components: The probability that there are initial n_0 customers in the system, $p(n_0)$, $n_0 \geq 2$. The probability density function of n_e arrivals occurring during time T_e is given by $p_1 = \lambda^{n_e} e^{-\lambda T e}$. The probability density function of n_{fu} arrivals occurring and n_{fd} departures occurring during T_{fb} is given by $p_2 = \lambda^{n_{fu}} e^{-\lambda T_{fb}} {\mu_1}^{n_{fd}} e^{-\mu_1 T_{fb}}$. The probability density function of n_{su} arrivals occurring and n_{sd} departures occurring during time T_{sb} is given by $p_3 = \lambda^{n_{su}} e^{-\lambda T_{sb}} (\mu_1 + \mu_2)^{n_{sd}} e^{-(\mu_1+\mu_2)T_{sb}}$.

Combining the respective components, the corresponding likelihood function becomes

$$L(\lambda, \mu_1, \mu_2) = \lambda^{n_a} e^{-\lambda T} e^{-\mu_1 T_{fb}} {\mu_1}^{n_{fd}} (\mu_1 + \mu_2)^{n_{sd}} e^{-(\mu_1+\mu_2)T_{sb}} p(n_0) \tag{2.4}$$

Under Bayesian setup the assumption $\mu_1 > \mu_2$ of M/M/2 queueing system with heterogeneous servers is incorporated by using Mckay's bivariate gamma distribution as the joint prior for μ_1 and μ_2, which is given by

$$\pi_1(\mu_1, \mu_2) = \frac{c^{\alpha+\gamma}}{\Gamma\alpha\Gamma\gamma} {\mu_2}^{\alpha-1} (\mu_1 - \mu_2)^{\gamma-1} e^{-c\mu_2}, 0 < \mu_2 < \mu_1 < \infty, \alpha > 0, \gamma > 0 \tag{2.5}$$

We use gamma distribution as the prior distribution for arrival rate λ with probability density function

$$\pi_2(\lambda) = \frac{m^p}{\Gamma p} e^{-m\lambda} \lambda^{p-1}, \lambda > 0, m > 0, p > 0. \tag{2.6}$$

Hence the joint prior distribution of λ, μ_1 and μ_2 is

$$\begin{aligned} \pi(\lambda, \mu_1, \mu_2) &= \pi_1(\mu_1, \mu_2)\pi_2(\lambda) \\ &= \frac{c^{\alpha+\gamma}}{\Gamma\alpha\Gamma\gamma} {\mu_2}^{\alpha-1} (\mu_1 - \mu_2)^{\gamma-1} e^{-c\mu_2} \frac{m^p}{\Gamma p} e^{-m\lambda} \lambda^{p-1} \end{aligned} \tag{2.7}$$

Using the likelihood function given in (2.4) and the joint prior distribution of (λ, μ_1, μ_2) given in (2.7), we can obtain the joint likelihood function of (λ, μ_1, μ_2) and x=$(T_e, T_{fb}, T_{sb}, n_e, n_{fu}, n_{su}, n_{fd}, n_{sd})$ as

$$\begin{aligned} h(x, \lambda, \mu_1, \mu_2) &= L(x, \lambda, \mu_1, \mu_2)\pi(\lambda, \mu_1, \mu_2) \\ &= \frac{c^{\alpha+\gamma}}{\Gamma\alpha\Gamma\gamma}\frac{m^p}{\Gamma p}\lambda^{n_a+p-1}e^{-\lambda(m+T)}e^{-\mu_1 T_{fb}}{\mu_1}^{n_{fd}}(\mu_1+\mu_2)^{n_{sd}} \\ &\times\ e^{-(\mu_1+\mu_2)T_{sb}}{\mu_2}^{\alpha-1}(\mu_1-\mu_2)^{\gamma-1}e^{-c\mu_2} \end{aligned} \tag{2.8}$$

Here the posterior joint distribution of (λ, μ_1, μ_2) is

$$\begin{aligned} \pi(\lambda, \mu_1, \mu_2/x) &= k^{'}\lambda^{n_a+p-1}e^{-\lambda(m+T)}e^{-\mu_1 T_{fb}}{\mu_1}^{n_{fd}}(\mu_1+\mu_2)^{n_{sd}} \\ &\times\ e^{-(\mu_1+\mu_2)T_{sb}}{\mu_2}^{\alpha-1}(\mu_1-\mu_2)^{\gamma-1}e^{-c\mu_2}, \end{aligned} \tag{2.9}$$

where the normalizing constant $k^{'}$ is such that

$$\int_{\lambda=0}^{\infty}\int_{\mu_1=0}^{\infty}\int_{\mu_2=0}^{\mu_1}\pi(\lambda, \mu_1, \mu_2/x)d\mu_2 d\mu_1 d\lambda = 1$$

Using the result (Equation(2.3.6.1), Volume 1 of Prudnikov et al.(1986))

$$\int_0^a x^{\alpha-1}(a-x)^{\beta-1}e^{-px}dx = \beta(\alpha,\beta)a^{\alpha+\beta-1}F(\alpha;\alpha+\beta;-ap), \alpha>0, \beta>0 \tag{2.10}$$

we get

$$\frac{1}{k^{'}} = \frac{\Gamma(n_a+p)}{(m+T)^{n_a+p}}\frac{m_\alpha!}{(T_{fb}+T_{sb})^{m_\alpha}}\sum_{x=0}^{n_{sd}}\begin{pmatrix} n_{sd} \\ x \end{pmatrix}\beta(x_\alpha,\gamma)G(x_\alpha, m_\alpha; x_\alpha+\gamma; -\xi_T), \tag{2.11}$$

where $m_\alpha = n_{fd}+n_{sd}+\alpha+\gamma$, $x_\alpha = x+\alpha$ and $\xi_T = \frac{c+T_{sb}}{T_{fb}+T_{sb}}$.

2.1. Posterior marginal distribution of λ, μ_1 and μ_2. Posterior marginal distribution of λ is

$$\begin{aligned}
\pi(\lambda/x) &= \int_{\mu_1=0}^{\infty}\int_{\mu_2=0}^{\mu_1}\pi(\lambda,\mu_1,\mu_2/x)d\mu_1 d\mu_2 \\
&= k^{'}\lambda^{n_a+p-1}e^{-\lambda(m+T)}\sum_{x=0}^{n_{sd}}\left(\begin{array}{c} n_{sd} \\ x \end{array}\right)\beta(x_\alpha,\gamma) \\
&\times \int_{\mu_1=0}^{\infty} e^{-\mu_1(T_{fb}+T_{sb})}{\mu_1}^{m_\alpha-1}F(x_\alpha;x_\alpha+\gamma;-\mu_1(c+T_{sb})) \\
&= k^{'}\lambda^{n_a+p-1}e^{-\lambda(m+T)}\sum_{x=0}^{n_{sd}}\left(\begin{array}{c} n_{sd} \\ x \end{array}\right)\beta(x_\alpha,\gamma) \\
&\times \sum_{k=0}^{\infty}\frac{(x_\alpha)_k}{(x_\alpha+\gamma)_k}-\frac{(c+T_{sb})^k}{k!}\frac{\Gamma(m_\alpha+k)}{(T_{fb}+T_{sb})^{m_\alpha+k}} \\
&= \frac{(m+T)^{n_a+p}}{\Gamma(n_a+p)}\lambda^{n_a+p-1}e^{-\lambda(m+T)} \qquad (2.12)
\end{aligned}$$

Posterior marginal distribution of μ_1 is

$$\begin{aligned}
\pi(\mu_1/x) &= \int_{\lambda=0}^{\infty}\int_{\mu_2=0}^{\mu_1}\pi(\lambda,\mu_1,\mu_2/x)d\lambda d\mu_2 \\
&= k^{'}e^{-\mu_1(T_{fb}+T_{sb})}{\mu_1}^{n_{fd}}\int_{\lambda=0}^{\infty}\lambda^{n_a+p-1}e^{-\lambda(m+T)}d\lambda \\
&\times \int_{\mu_2=0}^{\mu_1}(\mu_1+\mu_2)^{n_{sd}}{\mu_2}^{\alpha-1}(\mu_1-\mu_2)^{\gamma-1}e^{-\mu_2(c+T_{sb})}d\mu_2 \\
&= k^{''}\sum_{x=0}^{n_{sd}}\left(\begin{array}{c} n_{sd} \\ x \end{array}\right)\beta(x_\alpha,\gamma)e^{-\mu_1(T_{fb}+T_{sb})}{\mu_1}^{m_\alpha-1}F(x_\alpha;x_\alpha+\gamma;-\mu_1(c+T_{sb})) \\
&= k^{''}e^{-\mu_1(T_{fb}+T_{sb})}{\mu_1}^{m_\alpha-1}\sum_{x=0}^{n_{sd}}\left(\begin{array}{c} n_{sd} \\ x \end{array}\right)\beta(x_\alpha,\gamma)F(x_\alpha;x_\alpha+\gamma;-\mu_1(c+T_{sb})),
\end{aligned} \qquad (2.13)$$

where $k^{''}=\frac{(T_{sb}+T_{fb})^{m_\alpha}}{m_\alpha!}\frac{1}{\sum_{x=0}^{n_{sd}}\left(\begin{array}{c} n_{sd} \\ x \end{array}\right)\beta(x_\alpha,\gamma)G(x_\alpha,m_\alpha;x_\alpha+\gamma;-\frac{c+T_{sb}}{T_{fb}+T_{sb}})}$

Posterior marginal distribution of μ_2 is

$$\begin{aligned}
\pi(\mu_2/x) &= \int_{\lambda=0}^{\infty}\int_{\mu_1=\mu_2}^{\infty}\pi(\lambda,\mu_1,\mu_2/x)d\lambda d\mu_1\\
&= k^{'}e^{-\mu_2(c+T_{sb})}{\mu_2}^{\alpha-1}\int_{\lambda=0}^{\infty}\lambda^{n_a+p-1}e^{-\lambda(m+T)}\\
&\times \int_{\mu_1=\mu_2}^{\infty}e^{-\mu_1(T_{fb}+T_{sb})}\mu_1^{n_{fd}}(\mu_1+\mu_2)^{n_{sd}}(\mu_1-\mu_2)^{\gamma-1}d\mu_1 d\lambda\\
&= k_1^{'}\sum_{x=0}^{n_{sd}}\begin{pmatrix} n_{sd}\\ x\end{pmatrix}e^{-\mu_2(c+T_{sb})}{\mu_2}^{\alpha+x-1}\\
&\times \int_{\mu_1=\mu_2}^{\infty}e^{-\mu_1(T_{fb}+T_{sb})}(\mu_1-\mu_2)^{\gamma-1}{\mu_1}^{n_{fd}+n_{sd}-x}d\mu_1\\
&= k_1^{'}\sum_{x=0}^{m_{sd}}\begin{pmatrix} n_{sd}\\ x\end{pmatrix}\sum_{y=0}^{n_{fd}+n_{sd}-x}\begin{pmatrix} n_{fd}+n_{sd}-x\\ y\end{pmatrix}\frac{\Gamma\tau_{xy}}{(T_{fb}+T_{sb})^{\tau_{xy}}}\\
&\times \left\{e^{-\mu_2 c_\tau}{\mu_2}^{x+y+\alpha-1}\right\}, \qquad (2.14)
\end{aligned}$$

where $\tau_{xy}=n_{fd}+n_{sd}+\gamma-x-y$, $c_\tau=c+2T_{sb}+T_{fb}$ and

$$k_1^{'}=\frac{1}{\sum_{x=0}^{n_{sd}}\begin{pmatrix} n_{sd}\\ x\end{pmatrix}\sum_{y=0}^{n_{sd}+n_{fd}-x}\begin{pmatrix} n_{sd}+n_{fd}-x\\ y\end{pmatrix}\frac{\Gamma\tau_{xy}}{(T_{fb}+T_{sb})^{\tau_{xy}}}\frac{\Gamma(x+y+\alpha)}{{c_\tau}^{x+y+\alpha}}}$$

2.2. Bayes Estimator of queue characteristics under squared error loss function. The Bayes estimates of λ, μ_1 and μ_2 under squared error loss function are obtained as

$$\begin{aligned}
\lambda^* &= \int_0^\infty \lambda\pi(\lambda/x)d\lambda\\
&= \int_{\lambda=0}^{\infty}\frac{(m+T)^{n_a+p}}{\Gamma n_a+p}\lambda^{n_a+p}e^{-\lambda(m+T)}d\lambda=\left(\frac{n_a+p}{m+T}\right)
\end{aligned} \qquad (2.15)$$

$$\begin{aligned}
{\mu_1}^* &= \int_{\mu_1=0}^{\infty}\mu_1\pi(\mu_1/x)d\mu_1\\
&= k^{'}\frac{\Gamma(n_a+p)}{(m+T)^{n_a+p}}\sum_{x=0}^{n_{sd}}\begin{pmatrix} n_{sd}\\ x\end{pmatrix}\beta(x_\alpha,\gamma)\\
&\times \sum_{k=0}^{\infty}\frac{(x_\alpha)_k}{(x_\alpha+\gamma)_k}\frac{-(c+T_{sb})^k}{k!}\int_0^\infty e^{-\mu_1(T_{fb}+T_{sb})}{\mu_1}^{m_\alpha+k}d\mu_1\\
&= c_1\frac{\sum_{x=0}^{n_{sd}}\begin{pmatrix} n_{sd}\\ x\end{pmatrix}\beta(x_\alpha,\gamma)G(x_\alpha,m_\alpha+1;x_\alpha;-\frac{c+T_{sb}}{T_{fb}+T_{sb}})}{\sum_{x=0}^{n_{sd}}\begin{pmatrix} n_{sd}\\ x\end{pmatrix}\beta(x_\alpha,\gamma)G(x_\alpha,m_\alpha;x_\alpha+\gamma;-\frac{c+T_{sb}}{T_{fb}+T_{sb}})}, \qquad (2.16)
\end{aligned}$$

where $c_1=\left(\frac{m_\alpha+1}{T_{fb}+T_{sb}}\right)$

$$\begin{aligned}
{\mu_2}^* &= \int_{\mu_2=0}^{\infty} \mu_2 \pi(\mu_2/x) d\mu_2 \\
&= k_1^{'} \frac{\Gamma(n_a+p)}{(m+T)^{n_a+p}} \sum_{x=0}^{n_{sd}} \begin{pmatrix} n_{sd} \\ x \end{pmatrix} \sum_{y=0}^{n_{fd}+n_{sd}-x} \begin{pmatrix} n_{fd}+n_{sd}-x \\ y \end{pmatrix} \\
&\times \left\{ \frac{\Gamma(y+\gamma)}{(T_{fb}+T_{sb})^{y+\gamma}} \int_0^{\infty} e^{-\mu_2 c_\tau} {\mu_2}^{\tau_y} \right\} \\
&= \frac{\sum_{x=0}^{n_{sd}} \begin{pmatrix} n_{sd} \\ x \end{pmatrix} \sum_{y=0}^{n_{fd}+n_{sd}-x} \begin{pmatrix} n_{fd}+n_{sd}-x \\ y \end{pmatrix} \frac{\Gamma(y+\gamma)}{(T_{fb}+T_{sb})^{y+\gamma}} \frac{\Gamma(\tau_y+1)}{c_\tau^{\tau_y+1}}}{\sum_{x=0}^{n_{sd}} \begin{pmatrix} n_{sd} \\ x \end{pmatrix} \sum_{y=0}^{n_{fd}+n_{sd}-x} \begin{pmatrix} n_{fd}+n_{sd}-x \\ y \end{pmatrix} \frac{\Gamma(y+\gamma)}{(T_{fb}+T_{sb})^{y+\gamma}} \frac{\Gamma\tau_y}{c_\tau^{\tau_y}}}
\end{aligned} \tag{2.17}$$

where $\tau_y = n_{fd} + n_{sd} + \alpha - y$

2.3. Numerical Analysis. We simulate data on M/M/2 queueing system with two heterogeneous servers for different set of hyper parameters. Using the simulated data, we have computed Bayes estimates for λ, μ_1 and μ_2 using the expressions obtained in subsection 2.2. Confidence intervals for arrival and service rate parameters are also computed using bootstrap method. The algorithm for generating samples from Markovian queueing model with two heterogeneous servers and the computation of bootstrap sampling distribution and the credible interval of the Bayes estimates of the queue parameters are as follows.

Monte Carlo Simulation Method

Step 1: Fix the values of hyper parameters.

Step 2: Generate an observation (λ) from gamma distribution and observations (μ_1, μ_2) from Mckay's bivariate gamma distribution.

Step 3: Generate observations from exponential distribution with rates λ, μ_1 and μ_2 respectively.

Step 4: Repeat step 1-3 n times.

Step 5: Generate observations from M/M/2 queue with unequal service rates using the observations generated from exponential distribution with rate λ as interarrival time and the observation generated from exponential distribution with rates μ_1 and μ_2 as service times corresponding to server 1 and server 2 respectively.

Step 6: Compute Bayes estimates under squared error loss function using the formula given in the subsection 2.2.

Step 7: Repeat step 1-6 N times to generate N bootstrap samples,.

Step 8: The bootstrap estimate is computed as

$$\hat{\theta} = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N}$$

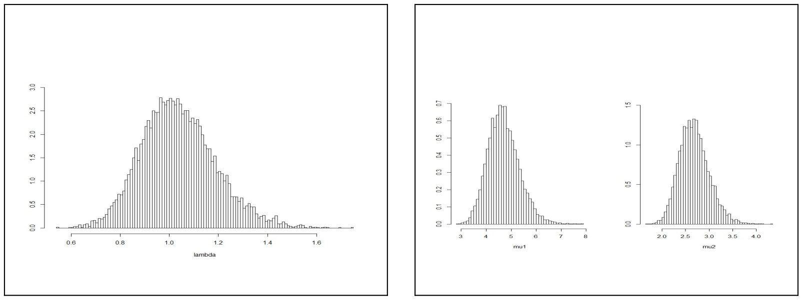

FIGURE 1. Sampling distribution of λ,μ_1 and $\mu_2$when m=4,p=5,c=3,α=15,γ=2

and the rmse corresponding to the estimator of parameter θ is

$$rmse = \sqrt{\sum_{i=1}^{N} \frac{(\hat{\theta}_i - \theta)^2}{N}}$$

Bayes estimates of the queue parameters along with the corresponding confidence intervals and root mean squared error(rmse) of the sampling distribution of the Bayes estimates of queue parameters for different set of hyper parameters and sample of size (n=25,50) are computed and are presented in tables. We fix m=4,p=5,c=3,α=15,γ=2 as hyper parameter values and draw the histograms corresponding to the sampling distribution of λ, μ_1 and μ_2. Bayes estimates, credible region and rmse of λ, μ_1 and μ_2 under Monte Carlo simulation method are given in table 1. The sampling distribution of λ, μ_1 and μ_2 are given in figure 1.

3. Markov Chain Monte Carlo Method

Markov Chain Monte Carlo(MCMC) techniques enables us to simulate random samples from a distribution by embedding it as a limiting distribution of a Markov chain. A detailed discussion of MCMC can be found in Gelfand and Smith[13], Gilks and Wild [14], Tierney [26] and Gilks et al.[14]. MCMC is essentially Monte Carlo integration using Markov chains. In frequentist as well as Bayesian methods, we need to integrate over high dimensional probability distributions to make inference about the modal parameters. The important algorithm used in MCMC method for sampling from arbitrary distribution are the Metropolis-Hastings sampler, the Gibbs sampler, the independence sampler, and the random walk. Gibbs sampling is one of the powerful tool in the MCMC method for drawing dependent samples from complex high dimensional probability densities. In Bayesian context, these distributions are usually posterior distribution of the modal parameters and samples produced by Gibbs sampler can be used directly for Bayesian inference. A brief discussion on MCMC technique including Gibbs sampler and the Metropolis-Hastings sampler is given in [25].

David et al.[12] described Bayesian inference and prediction for M/E_r/1 and M/H_k/1 queues. They have used Markov chain Monte Carlo methods for estimation procedure. Landauskas and Valakevieius [19] presents numerical results on modelling an M/G/1/∞ queuing system using Markov chain Monte Carlo method.

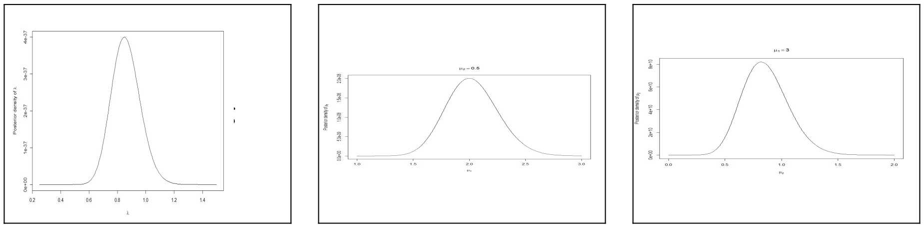

FIGURE 2. Posterior conditional densities of parameters λ, μ_1 and μ_2

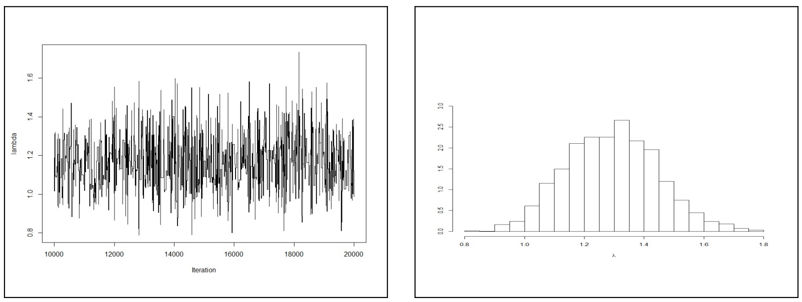

FIGURE 3. Trace plot and posterior distribution of λ(m=4,p=5,c=3,α=15,γ=2)

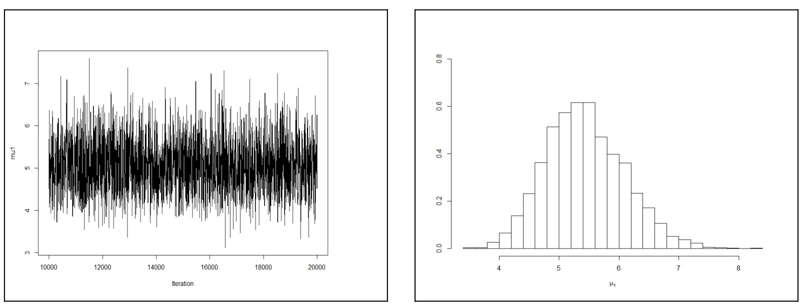

FIGURE 4. Trace plot and posterior distribution of μ_1(m=4,p=5,c=3,α=15,γ=2)

Wang et al.[28] described a Bayesian approach to parameter inference in queueing networks. They used a hybrid Monte Carlo Markov Chain method to perform inference and prediction.

In this section, we choose Metropolis-Hasting algorithm within Gibbs sampling procedure to generate random samples from the conditional densities of the parameters and utilize them to obtain the Bayes estimates and credible intervals of queue parameters.

We have the posterior joint distribution of λ, μ_1 and μ_2 (see equation(2.9))

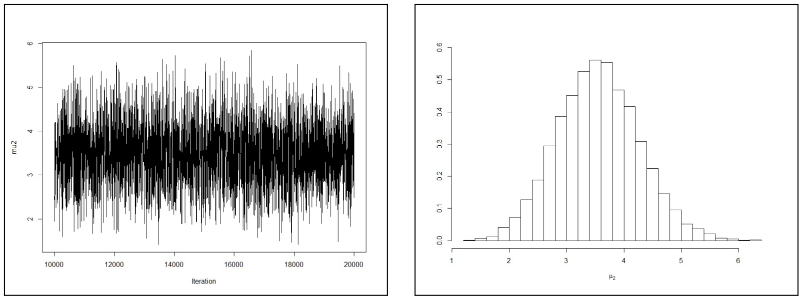

FIGURE 5. Trace plot and posterior distribution of μ_2(m=4,p=5,c=3,α=15,γ=2)

$$\begin{aligned}\pi(\lambda,\mu_1,\mu_2/x) &\propto \lambda^{n_a+p-1}e^{-\lambda(m+T)}e^{-\mu_1 T_{fb}}{\mu_1}^{n_{fd}}(\mu_1+\mu_2)^{n_{sd}}\\ &\times e^{-(\mu_1+\mu_2)T_{sb}}{\mu_2}^{\alpha-1}(\mu_1-\mu_2)^{\gamma-1}e^{-c\mu_2}\end{aligned}$$

The conditional posterior densities of λ,μ_1 and μ_2 are derived as

$$\pi(\lambda/x) \propto \lambda^{m_u+p-1}e^{-\lambda(m+T)} \tag{3.1}$$

$$\pi(\mu_1/\mu_2,x) \propto e^{-\mu_1(T_{fb}+T_{sb})}{\mu_1}^{m_{fd}}(\mu_1+\mu_2)^{m_{sd}}(\mu_1-\mu_2)^{\gamma-1} \tag{3.2}$$

$$\pi(\mu_2/\mu_1,x) \propto (\mu_1+\mu_2)^{m_{sd}}e^{-\mu_2(c+T_{sb})}{\mu_2}^{\alpha-1}(\mu_1-\mu_2)^{\gamma-1} \tag{3.3}$$

The graph of the conditional densities of the queue parameters is given in Figure 2. The conditional densities 3.2 and 3.3 are not in the form of known distributions and, hence, it is not possible to generate samples directly from these distributions by standard methods.

Here we use the Metropolis-Hasting(M-H) method with chi-square distribution as proposal distribution g(.) to generate random samples from the posterior conditional densities of λ,μ_1 and μ_2.

Algorithm for generating samples using M-H sampler is as follows.

Step 1: Set initial values (λ^0,${\mu_1}^0$,${\mu_2}^0$) at t=1

Step 2: Consider the proposal distribution as chi square g(λ)$\equiv \chi^2_{(\lambda^{t-1})}$,$\lambda >$ 0, generate λ^t from $\pi(\lambda/data)$ using M-H algorithm as follows.

a)Generate Y from the proposal distribution.

b)Generate U from Uniform(0,1).

c)If $U \leq \frac{\pi(Y)g(\lambda/Y)}{\pi(\lambda)g(Y/\lambda)}$ accept Y and set $\lambda^t = Y$; otherwise set λ^t=λ^{t-1}

d)Increment t.

Observe that in step (2c) the candidate point Y is accepted with probability $\alpha(\lambda_{t-1},Y) = min\left(1,\frac{\pi(Y)g(\lambda/Y)}{\pi(\lambda)g(Y/\lambda)}\right)$

Step 3: For a bivariate distribution, Generate (μ_1^t,μ_2^t) from the corresponding conditional distribution using M-H method with the proposal distribution $g(\mu_1)\equiv\chi^2_{({\mu_1}^{t-1})}$, $\mu_1 > 0$ and $g(\mu_2)\equiv\chi^2_{({\mu_2}^{t-1})}$, $\mu_2 > 0$.

a)Sets $(\mu_1,\mu_2) = \mu(t-1)$.

b)Generate μ_1^* from $\pi(\mu_1/\mu_2,data)$.

c)Update μ_1=$\mu_1^*(t)$.
d)Generate μ_2^* from $\pi(\mu_2/\mu_1, data)$.
e)Set $\mu(t) = (\mu_1^*(t), \mu_2^*(t))$.
Step 4: Repeat Steps 2-3, N times, and obtain λ^t, μ_1^t, μ_2^t for t=1,2,...,N.

Then by using the generated samples, the Bayes estimates of the parameters λ, μ_1 and μ_2 become $\lambda^* = \frac{\sum_{t=1}^N \lambda^t}{N}$, $\mu_1^* = \frac{\sum_{t=1}^N \mu_1^t}{N}$ and $\mu_2^* = \frac{\sum_{t=1}^N \mu_2^t}{N}$ respectively.

For computation of equal tailed credible region, N estimates are arranged in increasing order. Then from the ordered estimates, upper and lower credible limits are obtained. Bayes estimates, credible region and rmse of λ, μ_1 and μ_2 under Markov Chain Monte Carlo method for different set of hyper parameters are given in table 2. Trace plot and posterior distribution of λ, μ_1 and μ_2 are given in 3-5.

Table 1: Bayes estimates of queue characteristics under Monte-Carlo method

n	Hyper parameter	λ^*	μ_1^*	μ_2^*
25	m=4,p=5,c=3 $\alpha = 15$,$\gamma = 2$	1.088 (0.686,1.616) rmse=0.237	5.634 (3.807,8.317) rmse= 1.151	3.094 (2.206,4.203) rmse=0.516
	m=8,p=9,c=12 $\alpha = 10$,$\gamma = 8$	1.055 (0.580,1.448) rmse=0.180	1.311 (0.872,1.946) rmse=0.274	0.580 (0.401, 0.803) rmse=0.102
	m=1,p=2,c=10 $\alpha = 10$,$\gamma = 10$	1.214 (0.477,2.181) rmse=0.4304	1.759 (1.175, 2.603) rmse=0.365	0.722 (0.499,1.000) rmse=0.128
	m=2,p=3,c=10 $\alpha = 15$,$\gamma = 10$	1.120 (0.593,1.817) rmse=0.309	2.541 (1.732,3.685) rmse=0.505	1.050 (0.762,1.391) rmse=0.162
50	m=4,p=5,c=3 $\alpha = 15$,$\gamma = 2$	1.051 (0.744,1.431) rmse=0.175	4.909 (3.646,6.586) rmse=0.742	2.781 (6.586,6.586) rmse=0.368
	m=8,p=9,c=12 $\alpha = 10$,$\gamma = 8$	1.035 (0.786,1.341) rmse=0.141	1.009 (0.755,1.341) rmse=0.148	0.536 (0.405,0.697) rmse= 0.074
	m=1,p=2,c=10 $\alpha = 10$,$\gamma = 10$	1.126 (0.536,1.797) rmse=0.316	1.334 (1.004,1.755) rmse=0.192	0.677, (0.509,0.877) rmse=0.094
	m=2,p=3,c=10 $\alpha = 15$,$\gamma = 10$	1.071 (0.664,1.553) rmse=0.228	1.934 (1.457,2.550) rmse=0.278	0.964 (0.745, 1.227) rmse=0.123

Table 2: Bayes estimates under Markov Chain Monte-Carlo method

n	Hyper parameter	λ^*	μ_1^*	μ_2^*
25	m=4,p=5,c=3 $\alpha = 15, \gamma = 2$	1.089 (0.718,1.535) rmse=0.202	5.666 (3.964,7.766) rmse=0.977	3.686 (2.257,5.394) rmse=0.783
	m=8,p=9,c=12 $\alpha = 10, \gamma = 8$	1.259 (0.900,1.691) rmse=0.212	1.631 (1.105,2.267) rmse=0.289	0.461 (0.238,0.740) rmse=0.289
	m=1,p=2,c=10 $\alpha = 10, \gamma = 10$	0.782 (0.520, 1.111) rmse=0.155	1.963 (1.407,2.630) rmse=0.319	0.504 (0.249,0.828) rmse=0.146
	m=2,p=3,c=10 $\alpha = 15, \gamma = 10$	0.704 (0.465,0.994) rmse= 0.129	2.271 (1.665,2.997) rmse=0.345	0.855 (0.498,1.289) rmse=0.201
50	m=4,p=5, c=3 $\alpha = 15, \gamma = 2$	1.011 (0.775, 1.297) rmse=0.140	5.369 (4.126,6.845) rmse=0.696	3.839 (2.490,5.340) rmse=0.739
	m=8,p=9,c=12 $\alpha = 10, \gamma = 8$	0.972 (0.750,1.222) rmse=0.125	1.114 (0.847,1.413) rmse=0.143	0.377 (0.197,0.603) rmse=0.103
	m=1,p=2,c=10 $\alpha = 10, \gamma = 10$	0.705 (1.052,1.771) rmse=0.125	1.558 (1.153,2.011) rmse=0.205	0.482 (0.254,0.782) rmse=0.135
	m=2,p=3,c=10 $\alpha = 15, \gamma = 10$	1.006 (0.729,1.291) rmse=0.138	1.726 (1.336,2.174) rmse=0.209	0.714 (0.440, 1.051) rmse=0.156

4. Conclusion

The Bayes estimate of queue parameters, arrival rate and service rates of M/M/2 queueing model with heterogeneous servers under squared error loss function are obtained assuming the prior distributions as Mckay's bivariate gamma distribution for service rates and gamma distribution for arrival rate. Closed form expressions of the Bayes estimates are derived using the properties of some special function. We obtained bootstrap estimates and estimates using MCMC method along with credible interval and rmse. It is observed from the simulation study that the rmse associated with all the estimates decrease with increase in the sample size. Bayes estimate of μ_1, μ_2 and the corresponding rmse increase with increase in hyper parameter α and decrease in hyper parameter c and γ under both the method.

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