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SOLUTION OF THE ILL-POSED CAUCHY PROBLEM FOR MATRIX FACTORIZATIONS OF THE HELMHOLTZ EQUATION ON THE PLANE

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Abstract. In this paper, the problem of continuation of the solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation in a two-dimensional bounded domain is studied. It is assumed that the solution to the problem exists and is continuously differentiable in a closed domain with exactly given Cauchy data. For this case, an explicit formula for the continuation of the solution is established, as well as a regularization formula for the case when, under the indicated conditions, instead of the Cauchy data, their continuous approximations with a given error in the uniform metric are given. A stability estimate for the solution of the Cauchy problem in the classical sense is obtained.

1. Introduction

The paper studies the construction of exact and approximate solutions to the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation. Such problems naturally arise in mathematical physics and in various fields of natural science (for example, in electro-geological exploration, in cardiology, in electrodynamics, etc.). In general, the theory of ill-posed problems for elliptic systems of equations has been sufficiently formed thanks to the works of A.N. Tikhonov, V.K. Ivanov, M.M. Lavrent'ev, N.N. Tarkhanov of many other famous mathematicians. Among them, the most important for applications are the so-called conditionally well-posed problems, characterized by stability in the presence of additional information about the nature of the problem data. One of the most effective ways to study such problems is to construct regularizing operators. For example, this can be the Carleman-type formulas (as in complex analysis) or iterative processes (the Kozlov-Maz'ya-Fomin algorithm, etc.).

The work is devoted to the main problem for partial differential equations, which is the Cauchy problem. There are classes of equations for which this problem behaves well - the so-called hyperbolic equations. The main attention is paid to the regularization formulas for solutions of the Cauchy problem. The question of the existence of a solution to the problem is not considered - it is assumed a priori. At the same time, it should be noted that any regularization formula leads to an approximate solution of the Cauchy problem for all data, even if there

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is no solution in the usual classical sense. Moreover, for explicit regularization formulas, one can indicate in what sense the approximate solution turns out to be optimal. In this sense, exact regularization formulas are very useful for real numerical calculations. There is good reason to hope that numerous practical applications of regularization formulas are still ahead.

This problem concerns ill-posed problems, i.e., it is unstable. It is known that the Cauchy problem for elliptic equations is unstable relatively small change in the data, i.e., incorrect (example Hadamard, see, for instance [7], p. 39). There is a sizable literature on the subject (see, e.g. [1]-[2], [4]-[5], [24] and [33]). N.N. Tarkhanov [31] has published a criterion for the solvability of a larger class of boundary value problems for elliptic systems. In unstable problems, the image of the operator is not is closed, therefore, the solvability condition can not be is written in terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on part of the boundary of the domain the solution is usually unique, the problem is solvable for everywhere dense a set of data, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than theory of solvability of Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg, A.M. Kytmanov, N.N. Tarkhanov (see, for instance [32]).

The uniqueness of the solution follows from Holmgren's general theorem (see [4]). The conditional stability of the problem follows from the work of A.N. Tikhonov (see [33]), if we restrict the class of possible solutions to a compactum.

We note that when solving applied problems, one should find the approximate values of $U(x)$ and $\frac{\partial U(x)}{\partial x_j},\ x \in G,\ j = 1, 2.$

In this paper we construct a family of vector-functions $U(x, f_\delta) = U_{\sigma(\delta)}(x)$ and $\frac{\partial U(x, f_\delta)}{\partial x_j} = \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$, $(j = 1, 2)$ depending on a parameter σ, and prove that under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$, at $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ converges in the usual sense to a solution $U(x)$ and its derivative $\frac{\partial U(x)}{\partial x_j},\ x \in G$ at a point $x \in G$.

Following A.N. Tikhonov (see [33]), a family of vector-valued functions $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem.

Formulas that allow finding a solution to an elliptic equation in the case when the Cauchy data are known only on a part of the boundary of the domain are called Carleman type formulas. In [5], Carleman established a formula giving a solution to the Cauchy - Riemann equations in a domain of a special form. Developing his idea, G.M. Goluzin and V.I. Krylov [6] derived a formula for determining the values of analytic functions from data known only on a portion of the boundary, already for arbitrary domains. A multidimensional analogue of Carleman's formula for analytic functions of several variables was constructed in (see [1]). A formula of the Carleman type, in which the fundamental solution of a differential operator with special properties (the Carleman function) is used, was obtained by M.M. Lavrent'ev (see, for instance [23]-[24]). Using this method, Sh. Ya. Yarmukhamedov (see, for instance [34]-[37]) constructed the Carleman functions for the Laplace and Helmholtz operators with $n(x, y) \equiv 1$ for spatial domains of a special form, when the part of the boundary of the domain where the data is unknown is a conical surface or a hyper surface $\{x_3 = 0\}$. In [32] an integral formula is proved for systems of equations of elliptic type of the first order, with constant coefficients in a bounded domain. Using the methodology of works [34]-[37], Ikehata [8] was considered the probe method and Carleman functions for the Laplace and Helmholtz equations in the three-dimensional domain. Using exponentially growing solutions, Ikehata [9] was obtained a formula for solving the Helmholtz equation with a variable coefficient for regions in space where the unknown data are located on a section of the hypersurface $\{x \cdot s = t\}$. Carleman type formulas for various elliptic equations and systems were also obtained in works [3], [6], [8]-[9], [10]-[21], [38], [25]-[37]. In work [3] it was considered the Cauchy problem for the Helmholtz equation in an arbitrary bounded plane domain with Cauchy data, known only on the region boundary. The solvability criterion the Cauchy problem for the Laplace equation in the space $\mathbb{R}^m$ it was considered by Shlapunov in work [27]. In work [22], was be continuation the problem for the Helmholtz equation is investigated and the results of numerical experiments are presented.

The construction of the Carleman matrix for elliptic systems was carried out by: Sh. Yarmukhamedov, N.N. Tarkhanov, A.A. Shlapunov, I.E. Niyozov, D.A. Juraev and others (see, for instance [34]-[37], [27]-[28], [25]-[26] and [10]-[21], [38]). The system considered in this paper was introduced by N.N. Tarkhanov. For this system, he studied correct boundary value problems and found an analogue of the Cauchy integral formula in a bounded domain (see, for instance [32]).

In many well-posed problems for systems of equations of elliptic type of the first order with constant coefficients that factorize the Helmholtz operator, it is not possible to calculate the values of the vector function on the entire boundary. Therefore, the problem of reconstructing the solution of systems of equations of first order elliptic type with constant coefficients, factorizing the Helmholtz operator (see, for instance [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21] and [38]), is one of the topical problems in the theory of differential equations.

For the last decades, interest in classical ill-posed problems of mathematical physics has remained. This direction in the study of the properties of solutions of the Cauchy problem for the Laplace equation was started in [22]-[23], [33]-[36] and subsequently developed in [3], [6], [29]-[32], [8]-[9], [25]-[28] and [10]-[21], [38].

Let $\mathbb{R}^2$ be the two-dimensional real Euclidean space,

$$x = (x_1, x_2) \in \mathbb{R}^2, \; y = (y_1, y_2) \in \mathbb{R}^2.$$

$G \subset \mathbb{R}^2$ is a bounded simply-connected domain with piecewise smooth boundary consisting of the plane T: $y_2 = 0$ and some smooth curve S lying in the half-space $y_2 > 0$, i.e., $\partial G = S \bigcup T$.

We introduce the following notation:

$$r = |y - x|, \alpha = |y_1 - x_1|, w = i\sqrt{u^2 + \alpha^2} + y_2, \; u \geq 0,$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)^T, \ \frac{\partial}{\partial x} \to \xi^T, \ \xi^T = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \text{ be a transposed vector } \xi,$$

$$U(x) = (U_1(x), \ldots, U_n(x))^T, \ u^0 = (1, \ldots, 1) \in \mathbb{R}^n, \ n = 2^m, \ m = 2,$$

$$E(z) = \left\| \begin{matrix} z_1 \ldots 0 \\ \ldots\ldots. \\ 0 \ldots z_n \end{matrix} \right\| - \text{diagonal matrix}, z = (z_1, \ldots, z_n) \in \mathbb{R}^n.$$

Let $D(\xi^T)$, $(n\times n)-$dimensional matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$D^*(\xi^T)D(\xi^T) = E((|\xi|^2 + \lambda^2)u^0),$$

where $D^*(\xi^T)$ is the Hermitian conjugate matrix $D(\xi^T)$, λ is a real number.

We consider in the region G a system of differential equations

$$D\left(\frac{\partial}{\partial x}\right)U(x) = 0, \tag{1.1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of first-order differential operators.

We denote by $A(G)-$the class of vector functions in the domain G continuous on $\overline{G} = G\bigcup \partial G$ and satisfying system (1.1).

2. Construction of the Carleman matrix and the Cauchy problem

Formulation of the problem. Suppose $U(y) \in A(G)$ and

$$U(y)|_S = f(y), \ y \in S. \tag{2.1}$$

Here, $f(y)$ a given continuous vector-function on S. It is required to restore the vector function $U(y)$ in the domain G, based on it's values $f(y)$ on S.

If $U(y) \in A(G)$, then the following integral formula of Cauchy type is valid

$$U(x) = \int\limits_{\partial G} N(y, x; \lambda)U(y)ds_y, \quad x \in G, \tag{2.2}$$

where

$$N(y, x; \lambda) = \left(E\left(\varphi_2(\lambda r)u^0\right) D^*\left(\frac{\partial}{\partial y}\right)\right) D(t^T).$$

Here $t = (t_1, t_2)-$is the unit exterior normal, drawn at a point y, the curve ∂G, $\varphi_2(\lambda r)-$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$, where $\varphi_2(\lambda r)$ defined by the following formula:

$$\varphi_2(\lambda r) = -\frac{i}{4}H_0^{(1)}(\lambda r). \tag{2.3}$$

We denote by $K(w)$ is an entire function taking real values for real w, $(w = u + iv,\ \ u, v-$real numbers) and satisfying the following conditions:

$$K(u) \neq 0,\ \sup_{v \geq 1} \left| v^p K^{(p)}(w) \right| = B(u, p) < \infty,$$
$$-\infty < u < \infty,\ p = 0, 1, 2. \tag{2.4}$$

We define the function $\Phi(y, x; \lambda)$ at $y \neq x$ by the following equality

$$\Phi(y, x; \lambda) = -\frac{1}{2\pi K(x_2)} \int_0^\infty \operatorname{Im} \left[\frac{K(w)}{w - x_2} \right] \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du, \tag{2.5}$$

where $I_0(\lambda u) = J_0(i\lambda u)-$is the Bessel function of the first kind of zero order [4].

In the formula (2.5), choosing

$$K(w) = \exp(\sigma w),\ K(x_2) = \exp(\sigma x_2),\ \sigma > 0, \tag{2.6}$$

we get

$$\Phi_\sigma(y, x; \lambda) = -\frac{e^{-\sigma x_2}}{2\pi} \int_0^\infty \operatorname{Im} \left[\frac{\exp(\sigma w)}{w - x_2} \right] \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du. \tag{2.7}$$

The formula (2.2) is true if instead $\varphi_2(\lambda r)$ of substituting the function

$$\Phi_\sigma(y, x; \lambda) = \varphi_2(\lambda r) + g_\sigma(y, x; \lambda), \tag{2.8}$$

where $g_\sigma(y, x)-$ is the regular solution of the Helmholtz equation with respect to the variable y, including the point $y = x$.

Then the integral formula has the form:

$$U(x) = \int_{\partial G} N_\sigma(y, x; \lambda) U(y) ds_y,\ \ x \in G, \tag{2.9}$$

where

$$N_\sigma(y, x; \lambda) = \left(E \left(\Phi_\sigma(y, x; \lambda) u^0 \right) D^* \left(\frac{\partial}{\partial y} \right) \right) D(t^T).$$

3. The continuation formula and regularization according to M.M. Lavrent'ev's

Theorem 3.1. *Let $U(y) \in A(G)$ it satisfy the inequality*

$$|U(y)| \leq M,\ y \in T. \tag{3.1}$$

If

$$U_\sigma(x) = \int_S N_\sigma(y, x; \lambda) U(y) ds_y,\ \ x \in G, \tag{3.2}$$

then the following estimates are true

$$|U(x) - U_\sigma(x)| \le C(\lambda, x)\sigma M e^{-\sigma x_2}, \ \sigma > 1, \ x \in G, \tag{3.3}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_\sigma(x)}{\partial x_j}\right| \le C(\lambda, x)\sigma M e^{-\sigma x_2}, \ \sigma > 1, \ x \in G, \ j = 1, 2. \tag{3.4}$$

Here and below functions bounded on compact subsets of the domain G, we denote by $C(\lambda, x)$.

Proof. Let us first estimate inequality (3.3). Using the integral formula (2.9) and the equality (3.2), we obtain

$$U(x) = \int\limits_S N_\sigma(y, x; \lambda)U(y)ds_y + \int\limits_T N_\sigma(y, x; \lambda)U(y)ds_y =$$

$$= U_\sigma(x) + \int\limits_T N_\sigma(y, x; \lambda)U(y)ds_y, \ x \in G.$$

Taking into account the inequality (3.1), we estimate the following

$$\begin{gathered} |U(x) - U_\sigma(x)| \le \left|\int\limits_T N_\sigma(y, x; \lambda)U(y)ds_y\right| \le \\ \le \int\limits_T |N_\sigma(y, x; \lambda)| \, |U(y)| \, ds_y \le M \int\limits_T |N_\sigma(y, x; \lambda)| \, ds_y, \ \ x \in G. \end{gathered} \tag{3.5}$$

To do this, we estimate the integrals $\int\limits_T |\Phi_\sigma(y, x; \lambda)| \, ds_y$, $\int\limits_T \left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_1}\right| ds_y$, and $\int\limits_T \left|\frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_2}\right| ds_y$ on the part T of the plane $y_2 = 0$.

Separating the imaginary part of (2.7), we obtain

$$\begin{gathered} \Phi_\sigma(y, x; \lambda) = \frac{e^{\sigma(y_2 - x_2)}}{2\pi} \left[\int\limits_0^\infty \frac{\cos\sigma\sqrt{u^2 + \alpha^2}}{u^2 + r^2} \ u I_0(\lambda u) \, du - \right. \\ \left. - \int\limits_0^\infty \frac{(y_2 - x_2)\sin\sigma\sqrt{u^2 + \alpha^2}}{u^2 + r^2} \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du\right], \ x_2 > 0. \end{gathered} \tag{3.6}$$

Given (3.6) and the inequality

$$I_0(\lambda u) \le \sqrt{\frac{2}{\lambda \pi u}}, \tag{3.7}$$

we have

$$\int\limits_T |\Phi_\sigma(y, x; \lambda)|\, ds_y \le C(\lambda, x)\sigma e^{-\sigma x_2},\ \sigma > 1,\ x \in G, \tag{3.8}$$

To estimate the second integral, we use the equality

$$\begin{gathered} \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_1} = \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s} \frac{\partial s}{\partial y_1} = 2(y_1 - x_1) \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial s}, \\ s = \alpha^2. \end{gathered} \tag{3.9}$$

Given equality (3.6), inequality (3.7) and equality (3.9), we obtain

$$\int\limits_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_1} \right| ds_y \le C(\lambda, x)\sigma e^{-\sigma x_2},\ \sigma > 1,\ x \in G, \tag{3.10}$$

Now, we estimate the integral $\int\limits_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_2} \right| ds_y$.

Taking into account equality (3.6) and inequality (3.7), we obtain

$$\int\limits_T \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial y_2} \right| ds_y \le C(\lambda, x)\sigma e^{-\sigma x_2},\ \sigma > 1,\ x \in G, \tag{3.11}$$

From inequalities (3.7), (3.10) and (3.11), bearing in mind (3.5), we obtain an estimate (3.3).

Now let us prove inequality (3.4). To do this, we take the derivatives from equalities (2.9) and (3.2) with respect to x_j, $(j = 1, 2)$ then we obtain the following:

$$\begin{gathered} \frac{\partial U(x)}{\partial x_j} = \int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y + \int\limits_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y, \\ \frac{\partial U_\sigma(x)}{\partial x_j} = \int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y,\ \ x \in G,\ j = 1, 2. \end{gathered} \tag{3.12}$$

Taking into account the (3.12) and inequality (3.1), we estimate the following

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{\partial_\sigma U(x)}{\partial x_j}\right|\le\left|\int_T\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right|\le$$

$$\le\int_T\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right||U(y)|\,ds_y\le M\int_T\left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y, \tag{3.13}$$

$$x\in G,\ j=1,2.$$

To do this, we estimate the integrals $\int_T\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_1}\right|ds_y$ and $\int_T\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_2}\right|ds_y$ on the part T of the plane $y_2=0$.

To estimate the first integrals, we use the equality

$$\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_1}=\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial s}\frac{\partial s}{\partial x_1}=-2(y_1-x_1)\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial s},$$

$$s=\alpha^2. \tag{3.14}$$

Given equality (3.6), inequality (3.7) and equality (3.14), we obtain

$$\int_T\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_1}\right|ds_y\le C(\lambda,x)\sigma e^{-\sigma x_2},\ \sigma>1,\ x\in G. \tag{3.15}$$

Now, we estimate the integral $\int_T\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_2}\right|ds_y$.

Taking into account equality (3.6) and inequality (3.7), we obtain

$$\int_T\left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_2}\right|ds_y\le C(\lambda,x)\sigma e^{-\sigma x_2},\ \sigma>1,\ x\in G. \tag{3.16}$$

From inequalities (3.15) and (3.16), bearing in mind (3.13), we obtain an estimate (3.4).

Theorem 3.1 is proved. □

Corollary 3.2. *For each $x\in G$, the equalities are true*

$$\lim_{\sigma\to\infty}U_\sigma(x)=U(x),\ \lim_{\sigma\to\infty}\frac{\partial U_\sigma(x)}{\partial x_j}=\frac{\partial U(x)}{\partial x_j},\ j=1,2.$$

We denote by $\overline{G}_\varepsilon$ the set

$$\overline{G}_\varepsilon=\left\{(x_1,x_2)\in G,\ a>x_2\ge\varepsilon,\ a=\max_T\psi(x_1),\ 0<\varepsilon<a\right\}.$$

It is easy to see that the set $\overline{G}_\varepsilon \subset G$ is compact.

Corollary 3.3. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_\sigma(x)\}$ and $\left\{\frac{\partial U_\sigma(x)}{\partial x_j}\right\}$ converge uniformly for $\sigma \to \infty$, i.e.:*

$$U_\sigma(x) \rightrightarrows U(x),\ \frac{\partial U_\sigma(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j},\ j = 1, 2.$$

It should be noted that the set $E_\varepsilon = G \backslash \overline{G}_\varepsilon$ serves as a boundary layer for this problem, as in the theory of singular perturbations, where there is no uniform convergence.

4. Estimation of the stability of the solution to the Cauchy problem

Suppose that the curve S is given by the equation

$$y_2 = \psi(y_1),\ y_1 \in \mathbb{R},$$

where $\psi(y_1)$ is a single-valued function satisfying the Lyapunov conditions.

We put

$$a = \max_T \psi(y_1),\ b = \max_T \sqrt{1 + \psi'^2(y_1)}.$$

Theorem 4.1. *Let $U(y) \in A(G)$ satisfy condition (3.10), and on a smooth surface S the inequality*

$$|U(y)| \leq \delta,\ 0 < \delta < Me^{-\sigma a}. \tag{4.1}$$

Then the following estimates are true

$$|U(x)| \leq C(\lambda, x)\sigma M^{1-\frac{x_2}{a}}\delta^{\frac{x_2}{a}},\ \sigma > 1,\ x \in G. \tag{4.2}$$

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \leq C(\lambda, x)\sigma M^{1-\frac{x_2}{a}}\delta^{\frac{x_2}{a}},\ \sigma > 1,\ x \in G,\quad j = 1, 2. \tag{4.3}$$

Proof. Let us first estimate inequality (4.2). Using the integral formula (2.9), we have

$$U(x) = \int_S N_\sigma(y, x; \lambda)U(y)ds_y + \int_T N_\sigma(y, x; \lambda))U(y)ds_y,\ x \in G. \tag{4.4}$$

We estimate the following

$$|U(x)| \leq \left|\int_S N_\sigma(y, x; \lambda)U(y)ds_y\right| + \left|\int_T N_\sigma(y, x; \lambda)U(y)ds_y\right|,\ x \in G. \tag{4.5}$$

Given inequality (4.1), we estimate the first integral of inequality (4.5).

$$\left|\int_S N_\sigma(y,x;\lambda)U(y)ds_y\right| \le \int_S |N_\sigma(y,x;\lambda)|\,|U(y)|\,ds_y \le$$
$$\le \delta \int_S |N_\sigma(y,x;\lambda)|\,ds_y,\ x\in G. \tag{4.6}$$

To do this, we estimate the integrals $\int_S |\Phi_\sigma(y,x;\lambda)|\,ds_y$, $\int_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_1}\right| ds_y$, and $\int_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_2}\right| ds_y$ on a smooth surface S.

Given equality (3.6) and the inequality (3.7), we have

$$\int_S |\Phi_\sigma(y,x;\lambda)|\,ds_y \le C(\lambda,x)\sigma e^{\sigma(a-x_2)},\ \sigma>1,\ x\in G. \tag{4.7}$$

To estimate the second integral, using equalities (3.6) and (3.9) as well as inequality (3.7), we obtain

$$\int_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_1}\right| ds_y \le C(\lambda,x)\sigma e^{\sigma(a-x_2)},\ \sigma>1,\ x\in G. \tag{4.8}$$

To estimate the integral $\int_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_2}\right| ds_y$, using equality (3.6) and inequality (3.7), we obtain

$$\int_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial y_2}\right| ds_y \le C(\lambda,x)\sigma e^{\sigma(a-x_2)},\ \sigma>1,\ x\in G. \tag{4.9}$$

From (4.7) - (4.9), bearing in mind (4.6), we obtain

$$\left|\int_S N_\sigma(y,x;\lambda)U(y)ds_y\right| \le C(\lambda,x)\sigma\delta\, e^{\sigma(a-x_2)},\ \sigma>1,\ x\in G. \tag{4.10}$$

The following is known

$$\left|\int_T N_\sigma(y,x;\lambda)U(y)ds_y\right| \le C(\lambda,x)\sigma M e^{-\sigma x_2},\ \sigma>1,\ x\in G. \tag{4.11}$$

Now taking into account (4.10) - (4.11), bearing in mind (4.5), we have

$$|U(x)| \le \frac{C(\lambda,x)\sigma}{2}(\delta\, e^{\sigma a}+M)e^{-\sigma x_2},\ \sigma>1,\ x\in G. \tag{4.12}$$

Choosing σ from the equality

$$\sigma = \frac{1}{a} \ln \frac{M}{\delta}, \tag{4.13}$$

we obtain an estimate (4.2).

Now let us prove inequality (4.3). To do this, we find the partial derivative from the integral formula (2.9) with respect to the variable $x_j,\ j = 1, 2$:

$$\begin{aligned} \frac{\partial U(x)}{\partial x_j} &= \int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y + \int\limits_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y = \\ &= \frac{\partial U_\sigma(x)}{\partial x_j} + \int\limits_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y,\ x \in G,\ j = 1, 2. \end{aligned} \tag{4.14}$$

Here

$$\frac{\partial U_\sigma(x)}{\partial x_j} = \int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y. \tag{4.15}$$

We estimate the following

$$\begin{aligned} \left| \frac{\partial U(x)}{\partial x_j} \right| &\leq \left| \int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y \right| + \left| \int\limits_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y \right| \leq \\ &\leq \left| \frac{\partial U_\sigma(x)}{\partial x_j} \right| + \left| \int\limits_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y \right|,\ x \in G,\ j = 1, 2. \end{aligned} \tag{4.16}$$

Given inequality (4.1), we estimate the first integral of inequality (4.16).

$$\begin{aligned} \left| \int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} U(y) ds_y \right| &\leq \int\limits_S \left| \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} \right| |U(y)|\, ds_y \leq \\ &\leq \delta \int\limits_S \left| \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j} \right| ds_y,\ x \in G,\ j = 1, 2. \end{aligned} \tag{4.17}$$

To do this, we estimate the integrals $\int\limits_S \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_1} \right| ds_y$, and $\int\limits_S \left| \frac{\partial \Phi_\sigma(y, x; \lambda)}{\partial x_2} \right| ds_y$ on a smooth curve S.

Given equality (3.6), inequality (3.7) and equality (3.14), we obtain

$$\int_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_1}\right| ds_y \leq C(\lambda,x)\sigma e^{\sigma(a-x_2)},\ \sigma>1,\ x\in G, \tag{4.18}$$

Now, we estimate the integral $\int_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_2}\right| ds_y$.

Taking into account equality (3.6) and inequality (3.7), we obtain

$$\int_S \left|\frac{\partial\Phi_\sigma(y,x;\lambda)}{\partial x_2}\right| ds_y \leq C(\lambda,x)\sigma e^{\sigma(a-x_2)},\ \sigma>1,\ x\in G, \tag{4.19}$$

From (4.18) - (4.19), bearing in mind (4.17), we obtain

$$\left|\int_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y\right| \leq C(\lambda,x)\sigma\delta e^{-\sigma x_2},\ \sigma>1,\ x\in G, \quad j=1,2. \tag{4.20}$$

The following is known

$$\left|\int_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j} U(y) ds_y\right| \leq C(\lambda,x)\sigma M e^{-\sigma x_2},\ \sigma>1,\ x\in G, \quad j=1,2. \tag{4.21}$$

Now taking into account (4.20) - (4.21), bearing in mind (4.16), we have

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \leq \frac{C(\lambda,x)\sigma}{2}(\delta e^{\sigma a}+M)e^{-\sigma x_2},\ \sigma>1,\ x\in G, \quad j=1,2. \tag{4.22}$$

Choosing σ from the equality (4.13) we obtain an estimate (4.3).
Theorem 4.1 is proved. □

Let $U(y)\in A(G)$ and instead $U(y)$ on S with its approximation $f_\delta(y)$, respectively, with an error $0<\delta<Me^{-\sigma a}$,

$$\max_S |U(y)-f_\delta(y)| \leq \delta. \tag{4.23}$$

We put

$$U_{\sigma(\delta)}(x) = \int_S N_\sigma(y,x;\lambda) f_\delta(y) ds_y,\ x\in G. \tag{4.24}$$

Theorem 4.2. *Let $U(y) \in A(G)$ on the part of the plane $y_2 = 0$ satisfy condition (3.1)*

Then the following estimates is true

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \leq C(\lambda, x)\sigma M^{1-\frac{x_2}{a}}\delta^{\frac{x_2}{a}}, \ \sigma > 1, \ x \in G. \tag{4.25}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \leq C(\lambda, x)\sigma M^{1-\frac{x_2}{a}}\delta^{\frac{x_2}{a}}, \ \sigma > 1, \ x \in G, \quad j = 1, 2. \tag{4.26}$$

Proof. From the integral formulas (2.9) and (4.24), we have

$$U(x) - U_{\sigma(\delta)}(x) = \int\limits_{\partial G} N_\sigma(y, x; \lambda)U(y)ds_y -$$

$$-\int\limits_S N_\sigma(y, x; \lambda)f_\delta(y)ds_y = \int\limits_S N_\sigma(y, x; \lambda)U(y)ds_y +$$

$$+\int\limits_T N_\sigma(y, x; \lambda)U(y)ds_y - \int_S N_\sigma(y, x; \lambda)f_\delta(y)ds_y =$$

$$= \int\limits_S N_\sigma(y, x; \lambda)\left\{U(y) - f_\delta(y)\right\}ds_y + \int\limits_T N_\sigma(y, x; \lambda)U(y)ds_y.$$

and

$$\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} = \int\limits_{\partial G} \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j}U(y)ds_y -$$

$$-\int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j}f_\delta(y)ds_y = \int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j}U(y)ds_y +$$

$$+\int\limits_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j}U(y)ds_y - \int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j}f_\delta(y)ds_y =$$

$$= \int\limits_S \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j}\left\{U(y) - f_\delta(y)\right\}ds_y + \int\limits_T \frac{\partial N_\sigma(y, x; \lambda)}{\partial x_j}U(y)ds_y,$$

$$j = 1, 2.$$

Using conditions (3.1) and (4.23), we estimate the following:

$$\left|U(x)-U_{\sigma(\delta)}(x)\right| = \left|\int\limits_S N_\sigma(y,x;\lambda)\left\{U(y)-f_\delta(y)\right\}ds_y\right| +$$

$$+\left|\int\limits_T N_\sigma(y,x;\lambda)U(y)ds_y\right| \le \int\limits_S |N_\sigma(y,x;\lambda)|\left|\{U(y)-f_\delta(y)\}\right|ds_y +$$

$$+\int\limits_T |N_\sigma(y,x;\lambda)|\,|U(y)|\,ds_y \le \delta\int_S |N_\sigma(y,x;\lambda)|\,ds_y +$$

$$+M\int\limits_T |N_\sigma(y,x;\lambda)|\,ds_y.$$

and

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| = \left|\int\limits_S \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\left\{U(y)-f_\delta(y)\right\}ds_y\right| +$$

$$+\left|\int\limits_T \frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}U(y)ds_y\right| \le \int\limits_S \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|\left|\{U(y)-f_\delta(y)\}\right|ds_y +$$

$$+\int\limits_T \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right||U(y)|\,ds_y \le \delta\int_S \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y +$$

$$+M\int\limits_T \left|\frac{\partial N_\sigma(y,x;\lambda)}{\partial x_j}\right|ds_y,\ j=1,2.$$

Now, repeating the proof of Theorems 4.1 and 4.2, we obtain

$$\left|U(x)-U_{\sigma(\delta)}(x)\right| \le \frac{C(\lambda,x)\sigma}{2}(\delta e^{\sigma a}+M)e^{-\sigma x_2}.$$

$$\left|\frac{\partial U(x)}{\partial x_j}-\frac{U_{\sigma(\delta)}(x)}{\partial x_j}\right| \le \frac{C(\lambda,x)\sigma}{2}(\delta e^{\sigma a}+M)e^{-\sigma x_2},\ j=1,2.$$

From here, choosing σ from equality (4.13), we obtain an estimates (4.25) and (4.26).

Theorem 4.2 is proved. □

Corollary 4.3. *For each $x\in G$, the equalities are true*

$$\lim_{\delta\to 0} U_{\sigma(\delta)}(x)=U(x),\ \lim_{\delta\to 0}\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}=\frac{\partial U(x)}{\partial x_j},\ j=1,2.$$

Corollary 4.4. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_{\sigma(\delta)}(x)\}$ and $\left\{\dfrac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right\}$ converge uniformly for $\delta \to 0$, i.e.:*

$$U_{\sigma(\delta)}(x) \rightrightarrows U(x),\ \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j},\ j = 1, 2.$$

5. Conclusion

The article obtained the following results:

Using the Carleman function, a formula is obtained for the continuation of the solution of linear elliptic systems of the first order with constant coefficients in a spatial bounded domain $\mathbb{R}^2$. The resulting formula is an analogue of the classical formula of B. Riemann, W. Voltaire and J. Hadamard, which they constructed to solve the Cauchy problem in the theory of hyperbolic equations. An estimate of the stability of the solution of the Cauchy problem in the classical sense for matrix factorizations of the Helmholtz equation is given. The problem it is considered when, instead of the exact data of the Cauchy problem, their approximations with a given deviation in the uniform metric are given and under the assumption that the solution of the Cauchy problem is bounded on part T, of the boundary of the domain G, an explicit regularization formula is obtained.

Thus, functionals $U_{\sigma(\delta)}(x)$ and $\dfrac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ determines the regularization of the solution of problem (1.1), (2.1).

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