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# ATTRACTORS FOR WEAK SOLUTION OF THE INITIAL-BOUNDARY VALUE PROBLEM FOR ONE CLASS OF VISCOELASTIC FLUIDS WITH MEMORY

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Abstract. This article is devoted to the study of the existence of the trajectory and global attractors for weak solution of the initial-boundary value problem for one class of viscoelastic fluids with memory. The initial-boundary value problem for a model of viscoelastic fluid motion which is under consideration some bounded measurable function. This function characterizes memory of particles of fluid. The main result of this work is the proof of the existence theorem of the trajectory and global attractors for the weak solution of this initial-boundary value problem. For the proof of this theorems the approximating-topological method is used. This method was introduced by V.G. Zvyagin and was developed in his papers and papers of his colleagues.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain with a smooth boundary $\Gamma$. Consider the initial boundary problem

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \mu_1 \operatorname{Div} \int_0^t \mathcal{L}(t,s)\mathcal{E}(v)(s, Z_\delta(s;t,x))\, ds - \mu_0 \operatorname{Div} \mathcal{E}(v)$$

$$= -\operatorname{grad} p + f, \quad (t,x) \in [0,+\infty) \times \Omega \tag{1.1}$$

$$\operatorname{div} v = 0, \quad (t,x) \in [0,+\infty) \times \Omega \tag{1.2}$$

$$v\Big|_\Gamma = 0,\ v(0,x) = v^0(x)(x \in \Omega),\ \int_\Omega p\, dx = 0. \tag{1.3}$$

Here $v = (v_1, \dots, v_n)$ is the vector of velocity of the particle, $p$ is the preassure of the fluid, $f$ is the vector of body force, $\mu_0 > 0$, $\mu_1 \geq 0$ are some constants, $\mathcal{E} = (\mathcal{E}_{ij})$ is the strain rate tensor, $\operatorname{Div} A$ is the divergence of the $(n \times n)$-matrix $A = (A_{ij})$, $\mathcal{L}(t,s)$ is a bounded measurable function, characterizing memory of particles of

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the fluid. We assume that $|\mathcal{L}(t,s)| \le e^{-(2\mu_1/\mu_0)(t-s)}$ $(s < t,\ s,t \in [0,+\infty))$. The results with a certain function $\mathcal{L}(t,s)$ can be found in [1, 2].

We look at the trajectory defined by the equation

$$z(\tau;t,x) = x + \int_{\tau}^{t} S_\delta v(s, z(s;t,x))ds, \tau \in [0,T], (t,x) \in (0,T)\times\Omega \qquad (1.4)$$

In (1.4) we use the regularization operator $S_\delta : H \to C^1(\Omega)\cap V$, where $\delta > 0$. It has the following properties: $S_\delta(v) \to v$ in $H$ as $\delta \to 0$ and the map

$$S_\delta : L_2(0,T;H) \to L_2(0,T;C^1(\Omega)\cap V)$$

generated by this operator is continuous. We must introduce a regularization operator since for problems in hydrodynamics the velocity vector $v$ belongs to $L_2(0,T;V)$, so a trajectory can only be defined for a regularized velocity field (smoother than the original velocity field). A certain construction of an operator of this type can be found in [3].

Thus for each $v \in L_2(0,T;V)$ equation (1.4) has a unique solution $Z_\delta(v)$. From (1.4) we obtain $z(\tau;t,x) = Z_\delta(\tau;t,x)$. Let $Z_\delta(s;t,x)$ be a regularization of the trajectory.

It is established in [4] that initial-boundary value problem (1.1)-(1.3) is solvable in the weak sense. A description of the large-time behaviour of solutions is usually considered to be next in importance. This description (so-called limiting regimes) also characterizes the process as a whole. These questions are answered using the theory of attractors of the corresponding systems (see, e.g. [5]).

The aim of this paper is to investigate trajectory and global attractors for problem (1.1)-(1.3).

## 2. Statement of the problem and the main result

Let $E$ and $E_0$ be Banach spaces. Suppose that $E$ is continuously embedded in $E_0$ and that $E$ is reflexive.

**Definition 2.1.** Consider a nonempty set $\mathcal{H}^+ \subset C(\mathbb{R}_+;E_0)\cap L_\infty(\mathbb{R}_+;E)$. We shall refer to this set as the trajectory space and to its elements as trajectories.

**Definition 2.2.** A set $P \subset C(\mathbb{R}_+;E_0)\cap L_\infty(\mathbb{R}_+;E)$ is called the attracting set (for $\mathcal{H}^+$), if for any nonempty set $B \subset \mathcal{H}^+$, bounded with respect to the norm of $L_\infty(\mathbb{R}_+;E)$, we have $\lim\limits_{h\to 0}\sup\limits_{u\in B}\inf\limits_{v\in P}\|\mathrm{T}(h)u - v\|_{C([0,M];E_0)} = 0\ (\forall M > 0)$.

**Definition 2.3.** A set $P \subset C(\mathbb{R}_+;E_0)\cap L_\infty(\mathbb{R}_+;E)$ is called absorbing (for $\mathcal{H}^+$), if for any set $B \subset \mathcal{H}^+$, bounded with respect to the norm of $L_\infty(\mathbb{R}_+;E)$, there exists $h \ge 0$ such that $\mathrm{T}(t)B \subset P$ whenever $t \ge h$.

It follows from these definitions that each absorbing set is attracting.

**Definition 2.4.** A set $\mathcal{U} \subset C(\mathbb{R}_+;E_0)\cap L_\infty(\mathbb{R}_+;E)$ is called a trajectory attractor of the trajectory space $\mathcal{H}^+$, if the following conditions hold:

(i) $\mathcal{U}$ is compact in $C(\mathbb{R}_+;E_0)$ and bounded in $L_\infty(\mathbb{R}_+;E)$;
(ii) the equality $\mathrm{T}(t)\mathcal{U} = \mathcal{U}$ holds for all $t \ge 0$;

(iii) the set $\mathcal{U}$ is attracting in the sense of Definition 2.2.

**Definition 2.5.** The minimal trajectory attractor of the trajectory space $\mathcal{H}^+$ is the least with respect to inclusion trajectory attractor, i.e., the trajectory attractor that is contained in any trajectory attractor.

**Definition 2.6.** A nonempty set $\mathcal{A} \subset E$ is called the global attractor (in $E_0$) of the trajectory space $\mathcal{H}^+$, if the following conditions hold:

(i) $\mathcal{A}$ is compact in $E_0$ and bounded in $E$;
(ii) for any set $B \subset \mathcal{H}^+$ bounded in $L_\infty(\mathbb{R}_+; E)$ the attraction condition holds: $\sup\limits_{u\in B} \inf\limits_{y\in\mathcal{A}} \|u(t) - y\|_{E_0} \to 0\,(t \to \infty)$;
(iii) $\mathcal{A}$ is contained in any nonempty set that satisfies (i) and (ii).

*Remark* 2.7. Obviously, minimal trajectory attractor and global attractor are unique if they exist.

To describe the corresponding trajectory space we need the concept of a weak solution of this problem. To introduce it we translate (1.1)-(1.3) in operator form.

Let $\mathfrak{D}(\Omega)^n$ be the space of $C^\infty$-functions from $\Omega$ into $\mathbb{R}^n$ which have compact support in $\Omega$. We set $\mathcal{V} = \{v = (v_1, \dots, v_n) \colon v_i \in \mathfrak{D}(\Omega)^n,\ i = 1, \dots, n;\ \operatorname{div} v = 0\}$. Let $H$ be the closure of $\mathcal{V}$ in the norm of $L_2(\Omega)^n$ and let $V$ be the closure of $\mathcal{V}$ in the norm of $W_2^1(\Omega)^n$. The space $V$ is a Hilbert space with the inner product $(v, u)_V = \sum\limits_{i,j=1}^{n} \int\limits_\Omega \mathcal{E}_{ij}(u) \cdot \mathcal{E}_{ij}(v)\, dx$ and the corresponding norm $\|v\|_V$.

We denote the dual spaces to $H$ and $V$ by $H^*$ and $V^*$, respectively. By Riesz's theorem $H$ can be identified canonically with $H^*$. Bearing in mind this identification we have the chain of embeddings

$$V \subset H \equiv H^* \subset V^*, \tag{2.1}$$

where both embeddings are dense and compact.

We set $CG[0, T] = C([0, T] \times [0, T], C^1D(\overline{\Omega}))$, where $C^1D(\overline{\Omega})$ is the class of continuous bijective maps $z \colon \overline{\Omega} \to \overline{\Omega}$ which coincide with the identity map on $\Gamma$ and have continuous first order partial derivatives such that $\det\left(\frac{\partial z}{\partial x}\right) = 1$ at each point in $\overline{\Omega}$ (the topology in $C^1D(\overline{\Omega})$ is induced from $C(\overline{\Omega})^n$).

Let $\langle \varphi, v\rangle$ denote the action on the vector $v$ of the linear functional $\varphi$ in the dual space. Introduce the maps:

1) a linear operator $A \colon V \to V^*$, $\langle A(u), h\rangle = \mu_0(\mathcal{E}(u), \mathcal{E}(h))_{L_2(\Omega)^{n^2}}$, $u, h \in V$;

2) a map $K \colon V \to V^*$, $\langle K(u), h\rangle = \sum\limits_{i,j=1}^{n} \left(u_i u_j, \frac{\partial h_i}{\partial x_j}\right)_{L_2(\Omega)}$, $u, h \in V$;

($V \subset L_4(\Omega)^n$ then $u_i u_j \in L_2(\Omega)$ and $\frac{\partial h_i}{\partial x_j} \in L_2(\Omega)$ therefore the inner product is well-posed);

3) if $v \in L_2(0, T; V)$ and $z \in CG[0, T]$, then for each fixed $t \in (0, T)$ we introduce the functional on $V$

$$\langle C(v, z)(t), h\rangle = \mu_1 \left( \int\limits_0^t \mathcal{L}(t, s)\mathcal{E}(v)(s, z(s; t, x))\, ds, \mathcal{E}(h) \right)_{L_2(\Omega)^{n^2}}.$$

Thus we have defined an operator $C$ acting between the spaces $L_2(0,T;V) \times CG[0,T] \to L_2(0,T;V^*)$ (see [6]) and $L_2^{\mathrm{loc}}(\mathbb{R}_+;V) \times CG \to L_2^{\mathrm{loc}}(\mathbb{R}_+;V^*)$, where $L_2^{\mathrm{loc}}(\mathbb{R}_+;V) = \{v : v|_{[0,T]} \in L_2(0,T;V)\ \forall T > 0\}$ and $CG = C(\mathbb{R}_+ \times \mathbb{R}_+, C^1 D(\overline{\Omega}))$.

**Definition 2.8.** The weak solution of problem (1.1)-(1.3) on $[0,T]$ with $f \in V^*$, $v^0 \in H$ is a function $v \in L_2(0,T;V)$ such that $v' \in L_1(0,T;V^*)$, if it satisfies the identity

$$v' + A(v) - K(v) + C(v, Z_\delta(v)) = f, \tag{2.2}$$

$$v(0) = v^0. \tag{2.3}$$

The weak solution $v$ belongs to the space $W_1(0,T) = \{v\colon v \in L_2(0,T;V),\ v' \in L_1(0,T;V^*)\}$ with the norm $\|v\|_{W_1(0,T)} = \|v\|_{L_2(0,T;V)} + \|v'\|_{L_1(0,T;V^*)}$. Since $W_1(0,T) \subset C([0,T],V^*)$ $(n = 2,3)$ and $v^0 \in H$ with (2.1), so condition (2.3) is well-posed.

The fact that problem (1.1)-(1.3) has weak solutions is shown in [4] in an even more general (nonautonomous) case.

Let $V_\theta$ $(\theta \in (0,1))$, be the closure of the set $\mathcal{V}$ in the norm of $H^\theta(\Omega)^n$ and let $V_\theta^*$ be its dual space.

We take $E = H$, $E_0 = V_\theta^*$ as the Banach spaces required in the definition of a trajectory space.

**Definition 2.9.** As the trajectory space of the problem (1.1)-(1.3), we take the set of functions $v \in L_\infty(\mathbb{R}_+;H) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+;V)$ with derivative $v' \in L_1^{\mathrm{loc}}(\mathbb{R}_+;V^*)$ such that the restriction of $v$ to any segment $[0,T]$ is a weak solution of the problem (1.1)-(1.3) and for any $t \geq 0$ the inequalities

$$\operatorname*{vrai\,max}_{s\in[t,t+1]} \|v(s)\|_H \leq C_1 \left(1 + \|v\|^2_{L_\infty(\mathbb{R}_+;H)} e^{-2\gamma t}\right)^{1/2}, \tag{2.4}$$

$$\int_0^t e^{-2\gamma(t-s)} \|v(s)\|_V^2\, ds \leq C_1^2 \left(1 + \|v\|^2_{L_\infty(\mathbb{R}_+;H)} e^{-2\gamma t}\right) \tag{2.5}$$

hold with some constants $C_1 > 0$ and $0 < \gamma < 2\mu_1/\mu_0$ independent of $v$.

Note that the introduced space is contained in the class $L_\infty(\mathbb{R}_+;H) \cap C(\mathbb{R}_+;V_\theta^*)$ and it is non-empty.

The main results of this paper are the following two existence theorems for attractors.

**Theorem 2.10.** *Let $f$ do not depend on $t$ and $f \in V^*$. Then there exists a minimal trajectory attractor $\mathcal{U}$ for the trajectory space $\mathcal{H}^+$ of (1.1)-(1.3). The attractor is bounded in $L_\infty(\mathbb{R}_+;H)$ and compact in $C(\mathbb{R}_+;V_\theta^*)$, and in the topology of $C(\mathbb{R}_+;V_\theta^*)$ it attracts sets of trajectories bounded in the norm of $L_\infty(\mathbb{R}_+;H)$.*

**Theorem 2.11.** *Let $f$ do not depend on $t$ and $f \in V^*$. Then there exists a global attractor $\mathcal{A}$ for the trajectory space $\mathcal{H}^+$ of (1.1)-(1.3). The attractor is bounded in $H$ and compact in $V_\theta^*$, and in the topology of $V_\theta^*$ it attracts sets of trajectories bounded in norm $L_\infty(\mathbb{R}_+;H)$.*

## 3. The trajectory space of problem (1.1)-(1.3)

In this paper we use the topological approximation method developed in [3] for analysing equations. Bearing this method in mind, we introduce the approximation equations which will be used in problem (1.1)-(1.3). To do this we modify equation (2.2), to fit all the terms in the space $L_2(0,T;V^*)$. We consider the operator

$$K_\varepsilon\colon V \to V^*,\ \langle K_\varepsilon(u), h\rangle = \sum_{i,j=1}^{n} \left(\frac{u_i u_j}{1+\varepsilon|u|^2}, \frac{\partial h_i}{\partial x_j}\right)_{L_2(\Omega)} \qquad (\varepsilon > 0).$$

Look at the approximation problem

$$v' + A(v) - K_\varepsilon(v) + C(v, Z_\delta(v)) = f \quad (\varepsilon > 0) \tag{3.1}$$

$$v(0) = v^0, \tag{3.2}$$

in the space

$$W(0,T) = \{v\colon v \in L_2(0,T;V),\ v' \in L_2(0,T;V^*)\}. \tag{3.3}$$

Assume that $W(0,T)$ is endowed with the norm

$$\|v\|_{W(0,T)} = \|v\|_{L_2(0,T;V)} + \|v'\|_{L_2(0,T;V^*)} \quad (v \in W(0,T)).$$

The space $W(0,T)$ is a Banach space and we know that $W(0,T) \subset C([0,T],H)$ (see [7], Ch. III, Lemma 1.2), thus the initial condition (3.2) makes sense for $v^0 \in H$.

**Lemma 3.1.** *Let $v$ be a solution of (3.1)–(3.2) on an interval $[0,T]$, $T > 1$. Then*

$$\operatorname*{vrai\,max}_{s\in[t,t+1]} \|v(s)\|_H \le C_1 \left(1 + \|v^0\|_H^2 e^{-2\gamma t}\right)^{1/2}, \quad t \in [0, T-1]; \tag{3.4}$$

$$\int_0^t e^{-2\gamma(t-s)} \|v(s)\|_V^2\, ds \le C_1^2 \left(1 + \|v^0\|_H^2 e^{-2\gamma t}\right), \quad t \in [0,T], \tag{3.5}$$

*where the constants $0 < \gamma < 2\mu_1/\mu_0$ and $C_1 > 0$ depend on $\mu_0$, $\mu_1$, $\|f\|_{V^*}$, but are independent of $v^0$ and $v$.*

**Lemma 3.2.** *Let $v$ is a weak solution of the problem (2.2), (2.3) at any interval $[0,T]$, ($T > 1$) and $v \in L_\infty(0,T;H)$. Then $v' \in L_{4/3}(0,T;V^*)$ and the inequality*

$$\|v'\|_{L_{4/3}(t,t+1;V^*)} \le \mathcal{M}(I_1(t), I_2(t)), \tag{3.6}$$

*holds for $0 \le t \le T-1$, where*

$$I_1(t) = \operatorname*{vrai\,max}_{s\in[t,t+1]} \|v(s)\|_H,\ I_2(t) = \max_{\tau\in[t,t+1]} \int_0^\tau e^{-2\gamma(\tau-s)} \|v(s)\|_V^2\, ds,$$

*and $\mathcal{M}$ is some continuous function of two non-negative real arguments such that $\mathcal{M}(I_1, I_2)$ is non-decreasing with respect to $I_1$ for a fixed $I_2$ and non-decreasing with respect to $I_2$ for a fixed $I_1$.*

Proofs of lemmas 3.1 and 3.2 have a similar structure to the proofs of lemmas 2.4 and 2.5 from [1] and takes a lot of space, so it is not given here.

**Theorem 3.3.** *For each $v_0 \in H$ there exists a trajectory $v \in \mathcal{H}^+$ such that $v(0) = v_0$.*

Proof of this theorem can be found in [1], theorem 3.1.

We set $C_2 = \mathcal{M}(2C_1, 4C_1^2)$, where $C_1$ is as in inequalities (2.4) and (2.5) and $\mathcal{M}$ is the function in Lemma 3.2.

**Definition 3.4.** The set $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is called the trajectory semi-attractor (of the trajectory space $\mathcal{H}^+$) if it satisfies the following conditions:

(i) $P$ is compact in $C(\mathbb{R}_+; E_0)$ and bounded in $L_\infty(\mathbb{R}_+; E)$;

(ii) $\mathrm{T}(t)P \subset P$ for all $t \geq 0$;

(iii) $P$ is an attracting set in the sense of Definition 2.2.

Consider the set

$$P = \Big\{ v \in L_\infty(\mathbb{R}_+; H) \cap C(\mathbb{R}_+; V_\theta^*) : \forall t \geq 0 \ \|v\|_{L_\infty(t,t+1;H)} + \|v'\|_{L_{4/3}(t,t+1;V^*)} \leq 2C_1 + C_2 \Big\}.$$

**Lemma 3.5.** *The set $P$ is a semi-attractor of the trajectory space $\mathcal{H}^+$.*

*Proof.* We shall verify that $P$ satisfies the conditions of Definition 3.4.

(i) By the definition of $P$, for each function $v \in P$ and any $t \geq 0$,

$$\operatorname*{vrai\,max}_{s \in [t,t+1]} \|v(s)\|_H \leq 2C_1 + C_2,$$

so that $\|v\|_{L_\infty(\mathbb{R}_+,H)} \leq 2C_1 + C_2$, which means that $P$ is bounded in $L_\infty(\mathbb{R}_+, H)$.

Let $P_t$ be the set of restrictions of functions in $P$ to the interval $[t, t+1]$. It immediately follows from the definition of $P$ that for each $v \in P_t$

$$\|v\|_{L_\infty(t,t+1;H)} \leq 2C_1 + C_2, \ \|v'\|_{L_{4/3}(t,t+1;V^*)} \leq 2C_1 + C_2,$$

and the set $P_t$ is relatively compact in $C([t, t+1]; V_\theta^*)$ by [8, Corollary 4].

Since the restrictions of functions in $P$ to an arbitrary interval $[t, t+1]$ form a relatively compact subset of $C([t, t+1]; V_\theta^*)$, the set of restrictions of functions in $P$ to an interval of the form $[0, T]$ is a relatively compact subset of $C([0, T]; V_\theta^*)$, and as already mentioned, this ensures that $P$ is relatively compact in $C(\mathbb{R}_+, V_\theta^*)$.

To prove that $P$ is compact, it remains to verify that it is closed in $C(\mathbb{R}_+, V_\theta^*)$. Assume that a sequence $\{v_m\} \subset P$ converges to a function $C(\mathbb{R}_+, V_\theta^*)$ in $v_*$. We must show that $v_* \in P$. Note that the sequence $\{v_m\}$ is bounded in $L_\infty(\mathbb{R}_+, H)$, so it converges to $v_*$ weak-$*$ in $L_\infty(\mathbb{R}_+, H)$. Hence $v_* \in L_\infty(\mathbb{R}_+, H)$.

Now we only need to prove that

$$\|v_*\|_{L_\infty(t,t+1;H)} + \|v_*'\|_{L_{4/3}(t,t+1;V^*)} \leq 2C_1 + C_2.$$

for any $t \geq 0$. The derivatives $\{v_m'\}$ form a bounded sequence in the norm of $L_{4/3}(t, t+1; V^*)$ and it converges weakly in this space. By the properties of weak convergence

$$\|v_*\|_{L_\infty(t,t+1;H)} + \|v_*'\|_{L_{4/3}(t,t+1;V^*)} \leq \varliminf_{m\to\infty} \|v_m\|_{L_\infty(t,t+1;H)} + \varliminf_{m\to\infty} \|v_m'\|_{L_{4/3}(t,t+1;V^*)}$$

$$\leq \varliminf_{m\to\infty} \left(\|v_m\|_{L_\infty(t,t+1;H)} + \|v'_m\|_{L_{4/3}(t,t+1;V^*)}\right) \leq 2C_1 + C_2,$$

as required.

Thus $v^* \in P$, which demonstrates that $P$ is closed and therefore compact in the space $C(\mathbb{R}_+, V^*_\theta)$.

(ii) If $v \in P$, $h \geq 0$, then $\mathrm{T}(h)v \in L_\infty(\mathbb{R}_+; H) \cap C(\mathbb{R}_+; V^*_\theta)$ and furthermore

$$\|\mathrm{T}(h)v\|_{L_\infty(t,t+1;H)} + \|\mathrm{T}(h)v'\|_{L_{4/3}(t,t+1;V^*)}$$

$$= \|v\|_{L_\infty(t+h,t+h+1;H)} + \|v'\|_{L_{4/3}(t+h,t+h+1;V^*)} \leq 2C_1 + C_2,$$

that is, $\mathrm{T}(h)v \in P$. Thus we have proved that $P$ is translationally invariant.

(iii) We claim that $P$ is an absorbing set. Let $B \subset \mathcal{H}^+$ be a set bounded in the norm of $L_\infty(\mathbb{R}_+, H)$; say, $\|v\|_{L_\infty(\mathbb{R}_+,H)} \leq R$ for $v \in B$. We take $t_0$ such that $1 + R^2 e^{-2\gamma t_0} \leq 4$. Let $h \geq t_0$. For $v \in B$, using (2.4) we obtain

$$\|\mathrm{T}(h)v\|_{L_\infty(t,t+1;H)} = \|v\|_{L_\infty(t+h,t+h+1;H)}$$

$$\leq C_1 \left(1 + \|v\|^2_{L_\infty(\mathbb{R}_+,H)} e^{-2\gamma(t+h)}\right)^{1/2} \leq C_1 \left(1 + R^2 e^{-2\gamma t_0}\right)^{1/2} \leq 2C_1. \tag{3.7}$$

We find an estimate for the derivative with the use of Lemma 3.2:

$$\|\mathrm{T}(h)v'\|_{L_{4/3}(t,t+1;V^*)} = \|v'\|_{L_{4/3}(t+h,t+h+1;V^*)} \leq \mathcal{M}(I_1(t+h), I_2(t+h)). \tag{3.8}$$

We estimate $I_1(t+h)$ and $I_1(t+h)$ by means of (2.4) and (2.5):

$$I_1(t+h) = \operatorname*{vrai\,max}_{s\in[t+h,t+h+1]} \|v(s)\|_H \leq C_1 \left(1 + R^2 e^{-2\gamma(t+h)}\right)^{1/2} \leq 2C_1;$$

$$I_2(t+h) = \max_{\tau\in[t+h,t+h+1]} \int_0^\tau e^{-2\gamma(\tau-s)} \|v(s)\|^2_V \, ds$$

$$\leq C_1^2 \max_{\tau\in[t+h,t+h+1]} \left(1 + R^2 e^{-2\gamma\tau}\right) = C_1^2 \left(1 + R^2 e^{-2\gamma(t+h)}\right)$$

$$\leq C_1^2 \left(1 + R^2 e^{-2\gamma t_0}\right) = 4C_1^2.$$

Since $\mathcal{M}$ is monotone, it follows from (3.8) that

$$\|\mathrm{T}(h)v'\|_{L_{4/3}(t,t+1;V^*)} \leq \mathcal{M}(2C_1, 4C_1^2) = C_2.$$

By this inequality and (3.7),

$$\|\mathrm{T}(h)v\|_{L_\infty(t,t+1;H)} + \|\mathrm{T}(h)v'\|_{L_{4/3}(t,t+1;V^*)} \leq 2C_1 + C_2,$$

so that $\mathrm{T}(h)v \in P$ for $h \geq t_0$. Thus we have proved that $P$ is an attracting set and therefore it is absorbing. □

Now we present the proof of our main theorems.

**Proof of Theorem 2.10.** By Theorem 4.2.1 in [3] a trajectory space has a minimal trajectory attractor if it has a trajectory semi-attractor. That a semi-attractor exists for $\mathcal{H}^+$ is proved in Lemma 3.5, so this trajectory space also has a minimal trajectory attractor.

*Remark* 3.6. The minimal trajectory attractor of the trajectory space $\mathcal{H}^+$ lies in the semi-attractor $P$ constructed above. This follows from [3, Theorem 4.2.1].

**Proof of Theorem 2.11.** By [3, Theorem 4.2.2] a global attractor of a trajectory space exists if this trajectory space has a minimal trajectory attractor. Hence Theorem 2.11 is a consequence of Theorem 2.10.

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