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CENTRAL LIMIT THEOREM FOR WEAKLY DEPENDENT RANDOM FIELDS WITH VALUES IN l_p $(1 \leq p \leq 2)$.

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ABSTRACT. In the paper the central limit theorem for l_p space-valued random fields satisfying some dependence condition is proved.

1. Introduction and main result

Central limit theorems for real-valued random fields were studied by many authors (for example, see [2] - [4], [6] - [8] and the references therein). The authors mainly considered random fields satisfying mixing conditions. In contrast, in the papers [4], [9], the authors considered random fields satisfying the conditions of weak dependence, which differ from the mixing conditions.

The central limit theorem for random fields with values in Banach spaces was investigated in the papers [10] - [12] (see also the bibliography therein). Our goal is to generalize the results of [13] to the case of the l_p space.

Introduce notation and definitions.

N and Z denote the set of natural numbers and integers, respectively.

For $n = (n_1, n_2, ..., n_d) \in N^d$, $n \to \infty$ means $n_q \to \infty$ for all $q = 1, ..., d$, $|n| = n_1, ..., n_d$.

For vectors from R^d, the relations $\leq, <, \pm, \wedge, \vee$ are fulfilled coordinatewise.

Definition 1.1. A family of σ -algebras $\left\{\mathcal{F}_i, i \in Z^d\right\}$ is called filtration if $\mathcal{F}_i \subset \mathcal{F}_j$ for all $i, j \in Z^d$ with $i \leq j$. This family is called commuting if, in addition, for all $k, l \in Z^d$ and all bounded $\mathcal{F}_l$ - measurable random variables Y, the equality $E\left(Y|F_k\right) = E\left(Y|F_{k\wedge l}\right)$ a.s. takes place.

For commuting filtration $\left\{\mathcal{F}_i, i \in Z^d\right\}$ the corresponding filtration $\mathcal{F}^{(q)} = \left\{\mathcal{F}_l^{(q)}, l \in Z\right\}$ defined by

$$\mathcal{F}_l^{(q)} = \bigvee_{i \in Z^d,\ i_q \leq l} \mathcal{F}_i, \quad l \in Z,\ q = 1, ..., d$$

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are commuting (see [13]). The projection operator corresponding to the commuting filtration $\left\{\mathcal{F}_i, i \in Z^d\right\}$ is defined by

$$P_j = \prod_{q=1}^{d} P_{j_q}^{(q)}, \ j \in Z^d \tag{1.1}$$

where $P_l^{(q)} : L^1(\mathcal{F}) \to L^1(\mathcal{F})$ and $P_l^{(q)} f = E_l^{(q)} f - E_{l-1}^{(q)} f, \ \ f \in L^1(\mathcal{F}), \ l \in Z, \ q = 1, ..., d$, $E_l^{(q)}(\cdot) = E\left(\cdot|\mathcal{F}_l^{(q)}\right)$ $L^1(\mathcal{F})$-is the $L^1(\Omega, \mathcal{F}, P)$ space.

Consider a family of independent identically distributed random variables $\left\{\varepsilon_i, i \in Z^d\right\}$ with common distribution μ.

Consider the probability space $(\Omega, \mathcal{F}, P) = \left(R^{Z^d}, B_R^{Z^d}, \mu^{Z^d}\right)$ and the completely commuting transformation (see definition in [13]) $\left\{T_i, i \in Z^d\right\}$ and filtration $\left\{\mathcal{F}_i, i \in Z^d\right\}$ generefed by $\left\{\varepsilon_i, i \in Z^d\right\}$ (see theorem 5.1 in [13]). By $P_0, \ 0 \in Z^d$ we denote the operator in (1.1) (see (2.3) from [13]).

Consider the random field

$$X_i = f \circ T_i\left(\left\{\varepsilon_k, k \in Z^d\right\}\right), \ i \in Z^d \tag{1.2}$$

$f : R^{Z^d} \to R$ is a measurable function, $f \circ T_i$- composition of mappings. Denote $\|X\|_2 = \left(EX^2\right)^{\frac{1}{2}}$.

In [13], the following theorem was proved.

Theorem 1.2. *Let the stationary random field $\left\{X_i, i \in Z^d\right\}$ defined in (1.2) satisfy the following conditions*

$$EX_{\mathbf{1}} = 0, \ EX_{\mathbf{1}}^2 < \infty,$$
$$\sum_{i \in Z^d} ||P_0 X_i||_2 < \infty.$$

Then, for $n \to \infty$, the following weak convergence holds:

$$\frac{1}{|n|^{1/2}} \sum_{\mathbf{1} \le i \le n} X_i \Rightarrow N\left(0, \sigma^2\right)$$

where

$$\sigma^2 = E\left|\sum_{i \in Z^d} P_0 X_i\right|^2.$$

Recall that l_p $(1 \le p \le 2)$ is the space of sequences $x = (x^{(1)}, x^{(2)}, ...)$ such that $\sum\limits_{i=1}^{\infty} |x^{(i)}|^p < \infty$ with a norm $\|x\| = \left(\sum\limits_{i=1}^{\infty} |x^{(i)}|^p\right)^{\frac{1}{p}}$.

Consider a random field $\left\{Y_i, i \in Z^d\right\}$ such that

$$Y_i = g \circ G_i\left(\left\{\varepsilon_j, \ i \in Z^d\right\}\right) \tag{1.3}$$

where $\left\{\varepsilon_j, j \in Z^d\right\}$ is a family of independent identically distributed random variables, G_i is the shift operator on R^{Z^d} defined as in (1.1) and $g : R^{Z^d} \to l_p$ is a measurable function.

Introduce the notation:

$\{e_j\}-$ standard basis of the space l_p, ie $e_j = (0, 0, ..., 0, 1, 0, 0, ...)$ where $j-$ th component is 1.

$$Y_i = \sum_{j=1}^{\infty} Y_i^{(j)} e_j,$$

$$S_n = \frac{1}{\sqrt{|n|}} \sum_{\mathbf{1} \le i \le n} Y_i, \quad n \in Z^d,$$

$$S_n = \sum_{i=1}^{\infty} S_n^{(i)} e_i,$$

$$t_{ij} = \lim_{n\to\infty} ES_n^{(i)} S_n^{(j)}, i, j = 1, 2, ...$$

$K = (t_{i,j})_{i,j\ge 1}$ is an infinite matrix.

The main result is the following theorem.

Theorem 1.3. *Let a $\{Y_i, i \in Z^2\}$, be a strictly stationary random field (1.3)) with values in $l_p(1 \le p \le 2)$ satisfying the following conditions*

$$EY_{\mathbf{1}} = 0, \ E||Y_{\mathbf{1}}||^2 < \infty, \tag{1.4}$$

for the projection operator defined in (1.1)

$$\sum_{i\in Z^d} \left|\left| P_0 Y_i^{(j)} \right|\right|_2 < \infty., \ j = 1, 2, ...$$

$$\sum_{i=1}^{\infty} t_{ii}^{\frac{p}{2}} < \infty, \quad \sigma_{ii} > 0, \quad i = 1, 2, ... \tag{1.5}$$

Then the following weak convergence holds

$$S_n \Rightarrow N(0, K) \quad \text{as} \quad n \to \infty$$

where $N(0, K)$ is a Gaussian random variable with values in l_p, with mean zero and covariance matrix $K = (t_{ij})$.

The above theorem generalizes the result from [13].

2. Proof of Theorem

Proof. Proof of Theorem 1.3. The proof is based on the following theorem from Billingsley[1].

Theorem 2.1. *Let B be a separable metric space. Assume that $Y_k, X_{k1}, X_{k2}, ...$ is a sequence of B-valued random variables. Assume that for each k, $X_{kn} \Rightarrow X_k$ as $n \to \infty$ and $X_k \Rightarrow X$ as $k \to \infty$. Moreover assume that for any $\varepsilon > 0$*

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\{||X_{kn} - Y_n|| > \varepsilon\} = 0. \tag{2.1}$$

Then the weak convergence $Y_n \Rightarrow X$ as $n \to \infty$ holds.

We will use the following representation

$$S_n = S_{nk} + \overline{S}_{nk}$$

where

$$S_{nk} = \sum_{i=1}^{k} S_n^{(i)} e_i, \quad \overline{S}_{nk} = \sum_{i=k+1}^{\infty} S_n^{(i)} e_i$$

In the first step we will prove that S_{nk} converges weakly ($as \;\; n \to \infty$) to some random variable $\eta_k \in l_p$. In the second step we will prove that $\eta_k \Rightarrow \eta$ where η is a Gaussian random variable in l_p. In the last step we will prove (2.1).

Weak convergence $S_{n1} \Rightarrow \eta_1$ follows from a central limit theorem for real-valued random fields i.e Theorem 1.2.

For fixed $k > 1$ convergence $S_{nk} \Rightarrow \eta_k$ follows from Theorem 1.2 and Cramer-Wald device.

Thus we have for each $k \geq 1$

$$S_{nk} \Rightarrow \eta_k$$

where $\eta_k = \left\{\eta_k^{(1)}, \eta_k^{(2)}, ..., \eta_k^{(k)}, 0, 0\right\}$ and $\left\{\eta_k^{(1)}, \eta_k^{(2)}, ..., \eta_k^{(k)}\right\}$ is a Gaussian k-dimensional vector with parameters $(0, K_k)$ and $K_k = (t_{ij})_{k \times k}$ is a covariance matrix.

Now we will prove the convergence

$$\eta_k \Rightarrow \eta = (\eta^{(1)}, \eta^{(2)}, ...) \quad as \;\; k \to \infty$$

where η is a mean zero Gaussian random variable with values in l_p and covariance matrix $K = (t_{ij})$. The existence of such a Gaussian random variable follows from (1.5).

Because of Gaussionaty the convergence $\eta_k \Rightarrow \eta$ follows from the convergence of characteristic functionals i.e. for

$$g_k(f) = exp\left\{-\frac{1}{2}\sum_{i,j=1}^{k} t_{ij} \cdot f_i f_j\right\},$$

$$g(f) = exp\left\{-\frac{1}{2}\sum_{i,j=1}^{\infty} t_{ij} \cdot f_i f_j\right\},$$

$$f = (f_1, f_2, ...) \in l_p^*,$$

we have

$$g_k(f) \to g(f)$$

for any $f \in l_p^*$.

Thus the convergence $\eta_k \Rightarrow \eta$ as $k \to \infty$ is proved.It remains to prove

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\left\{||S_n - S_{nk}|| > \varepsilon\right\} = 0$$

Using Chebishev's inequality and (1.5) we have

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\left\{||S_n - S_{nk}|| > \varepsilon\right\} \leq \lim_{k\to\infty} \limsup_{n\to\infty} \frac{E||S_{nk}||^p}{\varepsilon^p} \leq$$

$$\frac{1}{\varepsilon^p}\lim_{k\to\infty}\sum_{i=k+1}^{\infty}\limsup_{n\to\infty}E|S_n^{(i)}|^p \le \frac{1}{\varepsilon^p}\lim_{k\to\infty}\limsup_{n\to\infty}\left(E|S_n^{(i)}|^2\right)^{\frac{p}{2}} =$$

$$\frac{1}{\varepsilon^p}\lim_{k\to\infty}\sum_{i=k+1}^{\infty}t_{ii}^{\frac{p}{2}} = 0$$

The theorem is proved. □

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