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FITTING OF GOMPERTZ MODEL BETWEEN RAINFALL AND GROUND WATER LEVELS BY USING PARTIAL SUMS METHOD – A CASE STUDY

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Abstract. Present paper deals with the application of Time Series model to analyze and predict Rainfall (RF) and Ground water levels (GWLs) in Anantapuramu district based on the data collected from January 2007 to December 2016. Through with Gompertz model by using 'Partial Sums Method' for the purpose of analysis the district is divided into five zones or Revenue Divisions (RD) namely, 1. Anantapuramu RD 2. Penukonda RD 3. Kadiri RD 4. Kalyandurg RD 5. Dharmavaram RD. We have estimated the Gompertz model values by using the method of Partial Sums and compared among them by using the data. Further, validation of the fitted model identified the best suitable zone. i.e., least Mean Square Error (MSE) value of the zone and forecast on the Rainfall and Ground water levels of this district. We also calculate Critical Difference (C.D) test and conclusions are drawn based on the results obtained

1. INTRODUCTION

It was noted in the previous paper [13] that method of curve fitting is the best for estimating trend. The nature of the curve that is appropriate for the given data can be satisfactorily decided either by a theoretical understanding of the data or by observing the scatter diagram that is constructed for the given data.

The methods of fitting Straight Line, Second Degree Parabola, Exponential Curve and Power Curvesby least squares method was discussed earlier [13].

Linear, Parabolic, Exponential and Power Curve projections generally assume that growth or decline continues without limit. While these trends continue for some time they are not continue forever. There are a number of situations in which there is an asymptote to growth or decline. There are three types of Growth Curves or Models is there that is 1. Modified Exponential Model 2. Gompertz Model 3. Logistic Model.

Modified Exponential Model, in this method there are two types of methods used for the fitting of Modified Exponential Model:

1. Method of Three Selected Time Points [14].
2. Method of Partial Sums.

Gompertz Model, in this method there are two types of methods used for the fitting of

Gompertz Model:

1. Method of Three Selected Time Points [14].
2. Method of Partial Sums. In this paper by using method of partial sums.

Logistic Model, in this method there are four types of methods used for the fitting of Logistic Model:

1. Method of Three Selected Time Points [14].
2. Yule's Method
3. Hotelling's Method
4. Method of Successive Approximations

The data is collected on Average Rainfall and Average Ground Water Levels are given in the following Table-1.1 for a ready reference [13, 14, 15 and 16].

Table-1.1

Average Rainfall and Average Ground water levels data from 2007 to 2016

Year	Zone-I		Zone-II		Zone-III		Zone-IV		Zone-V	
	RF (in mm)	GWL	RF (in mm)	GWL	RF (in mm)	GWL	RF (in mm)	GWL	RF (in mm)	GWL
2007	65.60	10.57	58.20	22.58	67.20	14.23	52.00	14.97	60.50	17.03
2008	53.90	9.96	77.90	20.73	65.20	9.27	61.30	10.88	62.70	9.09
2009	45.40	12.17	50.60	17.53	46.30	11.08	57.10	9.58	38.70	10.24
2010	53.90	12.74	71.50	15.02	70.80	12.03	64.60	8.58	56.30	11.79
2011	39.50	12.69	42.30	15.20	48.90	11.48	31.80	8.93	36.60	12.84
2012	43.20	14.98	43.40	20.49	45.30	16.08	40.50	13.76	41.90	13.22
2013	35.00	15.94	52.30	23.03	47.10	18.69	34.80	16.98	38.10	14.30
2014	31.10	15.87	30.30	23.40	27.10	21.16	37.10	18.92	22.80	16.30
2015	44.10	14.90	62.60	26.88	66.30	25.80	46.00	19.26	54.30	17.66
2016	33.50	15.57	33.40	27.27	32.30	15.35	25.70	19.51	30.10	16.15

2. STATISTICAL ANALYSIS

Some of the Preliminary Statistical analysis is done for the data provided in the above table -1.1, such as yearly averages of Rainfall and Ground water levels are calculated and Karl-Pearson's Correlation Coefficient (r) is calculated between Average Rainfall(X) and Average Ground water levels (Y) Zonal wise[13, 14 and 15].

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To forecast **Rainfall** and **Ground Water Levels** through Gompertz model by using Method of Partial Sums for different zones we consider given as follows:

The Gompertz Model $\boldsymbol{y_t = k * a^{b^t}}$ .... (2.1)

$\log_e y_t = \log_e k + b^t \log_e a$ .... (2.2)

$Y_t = K + Ab^t$.............. (2.3)

$a = antilog(A),\ k = antilog(K)$.........(2.4)

Where $b = (\frac{s_3 - s_2}{s_2 - s_1})^{(\frac{1}{n})}$..... (2.5)

$A = \log_e a = \frac{(b-1)}{b} * \frac{(s_2 - s_1)^3}{(s_3 - 2s_2 + s_1)^2}$... (2.6)

$k = \log_e k = \frac{1}{n}[\frac{(s_1 s_3 - s_2^2)}{(s_3 - 2s_2 + s_1)}]$..... (2.7)

Here, $t$means the years its converted into time points, n is the number of values if added that is partial sums and$s_1, s_2, s_3$are partial sums their correspondence rainfall or ground water level values it's taken.

To fit the above Gompertz model and to estimate the values of the parameters 'a', 'b' and 'k' by solving the related normal equations and following trend curve is fitted for the data given in table 1.1 and fitted model is given below.

The fitted Gompertz model for Average RF and Average GWLs:

A: For Average Rainfall

Zone-I

Gompertz Curve $\log_e y_t = (5.01) + (-0.94) * (1.06)^t$

Zone-II

Gompertz Curve $\log_e y_t = (3.75) + (0.59) * (0.78)^t$

Zone-III

Gompertz Curve $\log_e y_t = (4.15) + (-0.04) * (1.32)^t$

Zone-IV

Gompertz Curve $\log_e y_t = (3.58) + (0.77) * (0.76)^t$

Zone-V

Gompertz Curve $\log_e y_t = (5.14) + (-1.11) * (1.05)^t$

B: For Average Ground water levels

Zone-I
Gompertz Curve $\log_e y_t = (3.08) + (-0.86) * (0.89)^t$

Zone-II
Gompertz Curve $\log_e y_t = (2.94) + (-0.11) * (-1.26)^t$

Zone-III
Gompertz Curve $\log_e y_t = (2.38) + (0.02) * (1.55)^t$

Zone-IV
Gompertz Curve $\log_e y_t = (2.43) + (-0.02) * (-1.64)^t$

Zone-V
Gompertz Curve $\log_e y_t = (2.42) + (0.02) * (1.46)^t$

Gompertz Curve $\hat{y}_t = \hat{e}(\log_e y_t)$ here substitutes the $\boldsymbol{log_e y_t}$values for required estimated Gompertz Curve values.

3. VALIDATION OF THE FITTED MODEL

Validation of the fitted model is necessary to check the suitability of the model for the given data this is done by consideringX = Years and Y = Average RF or Average GWL given in table-1.1 and estimated the Average RF (Y) or Average GWL (Y) denoted by$\hat{y}$. The estimated Average RF and Average GWLs are given in the following tables.

Table-3.1
Estimated Average RF$\hat{y}$for GompertzModel by using Partial Sums Method

Year	Zone-I		Zone-II		Zone-III		Zone-IV		Zone-V	
	Actual	Estimates	Actual	Estimates	Actual	Estimates	Actual	Estimates	Actual	Estimates
2007	65.60	55.15	58.20	67.36	67.20	60.34	52.00	64.72	60.50	52.98
2008	53.90	52.46	77.90	60.95	65.20	59.15	61.30	56.26	62.70	50.40

2009	45.40	48.91	50.60	56.26	46.30	57.97	57.10	50.40	38.70	46.99
2010	53.90	46.06	71.50	52.98	70.80	56.26	64.60	46.06	56.30	44.26
2011	39.50	42.52	42.30	50.40	48.90	54.05	31.80	43.38	36.60	41.26
2012	43.20	39.65	43.40	48.91	45.30	51.42	40.50	41.68	41.90	38.47
2013	35.00	36.60	52.30	47.47	47.10	47.94	34.80	40.45	38.10	35.52
2014	31.10	33.78	30.30	46.06	27.10	43.82	37.10	38.86	22.80	33.12
2015	44.10	30.57	62.60	45.15	66.30	38.86	46.00	38.09	54.30	30.57
2016	33.50	27.94	33.40	44.70	32.30	33.45	25.70	37.71	30.10	27.94

Table-3.2

Estimated Average GWL $\hat{y}$ for Gompertz Model by using Partial Sums Method

Year	Zone-I		Zone-II		Zone-III		Zone-IV		Zone-V	
	Actual	Estimates	Actual	Estimates	Actual	Estimates	Actual	Estimates	Actual	Estimates
2007	10.57	10.07	22.58	21.76	14.23	11.13	14.97	11.70	17.03	11.59
2008	9.96	11.02	20.73	15.96	9.27	11.36	10.88	10.80	9.09	11.70
2009	12.17	11.94	17.53	23.57	11.08	11.59	9.58	12.43	10.24	11.94
2010	12.74	12.68	15.02	14.30	12.03	12.18	8.58	9.87	11.79	12.30
2011	12.69	13.46	15.20	26.84	11.48	12.94	8.93	14.44	12.84	12.81
2012	14.98	14.15	20.49	12.18	16.08	14.30	13.76	7.69	13.22	13.60
2013	15.94	14.88	23.03	32.79	18.69	16.61	16.98	21.54	14.30	14.88
2014	15.87	15.49	23.40	9.39	21.16	21.12	18.92	3.97	16.30	16.95
2015	14.90	16.12	26.88	45.60	25.80	30.27	19.26	63.43	17.66	20.49
2016	15.57	16.61	27.27	6.23	15.35	53.52	19.51	0.68	16.15	27.11

In the above tables -3.1 and 3.2 for the validation of the model Mean Square Errors (MSE's) are calculated zone wise by considering

MSE = $\sum(y-\hat{y})^2$ .....(3.1)

Where y represents actual or observed values given in table-1.1 and $\hat{y}$ is the estimated valuesthrough fitted Gompertz model is given in tables- 3.1 and 3.2. Byusing Method of Partial Sums for fittedGompertz model respectively. MSE's were calculated and are given in the following table.

Table-3.3

MSE's for Average RF- Gompertz Model by using Partial Sums Method.

Type of the Model	Zone-I	Zone-II	Zone-III	Zone-IV	Zone-V
Gompertz Model by	430.49	1546.11	1529.77	953.14	1135.94

using Partial Sums Method					

Table-3.4

MSE's for Average GWLs- Gompertz Model by using Partial Sums Method.

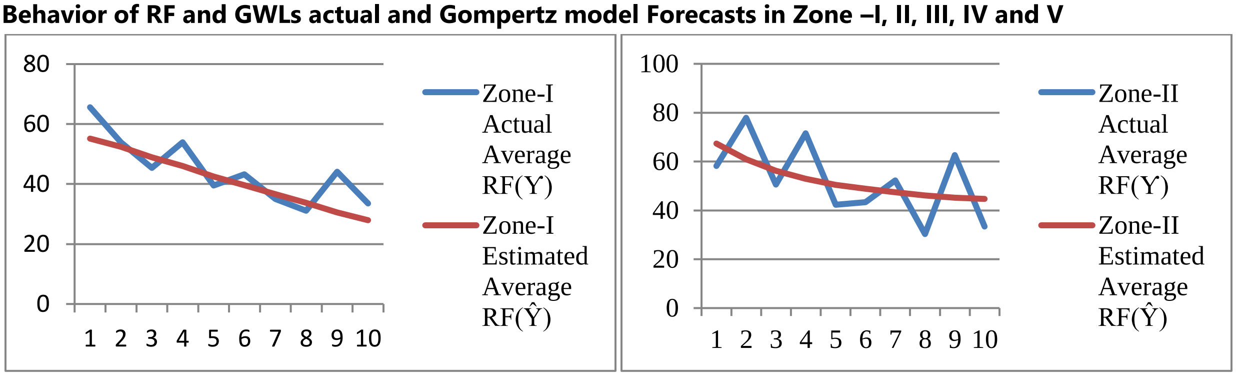
Type of the Model	Zone-I	Zone-II	Zone-III	Zone-IV	Zone-V
Gompertz Model by using Partial Sums Method	6.53	1349.63	1500.82	2637.53	168.58

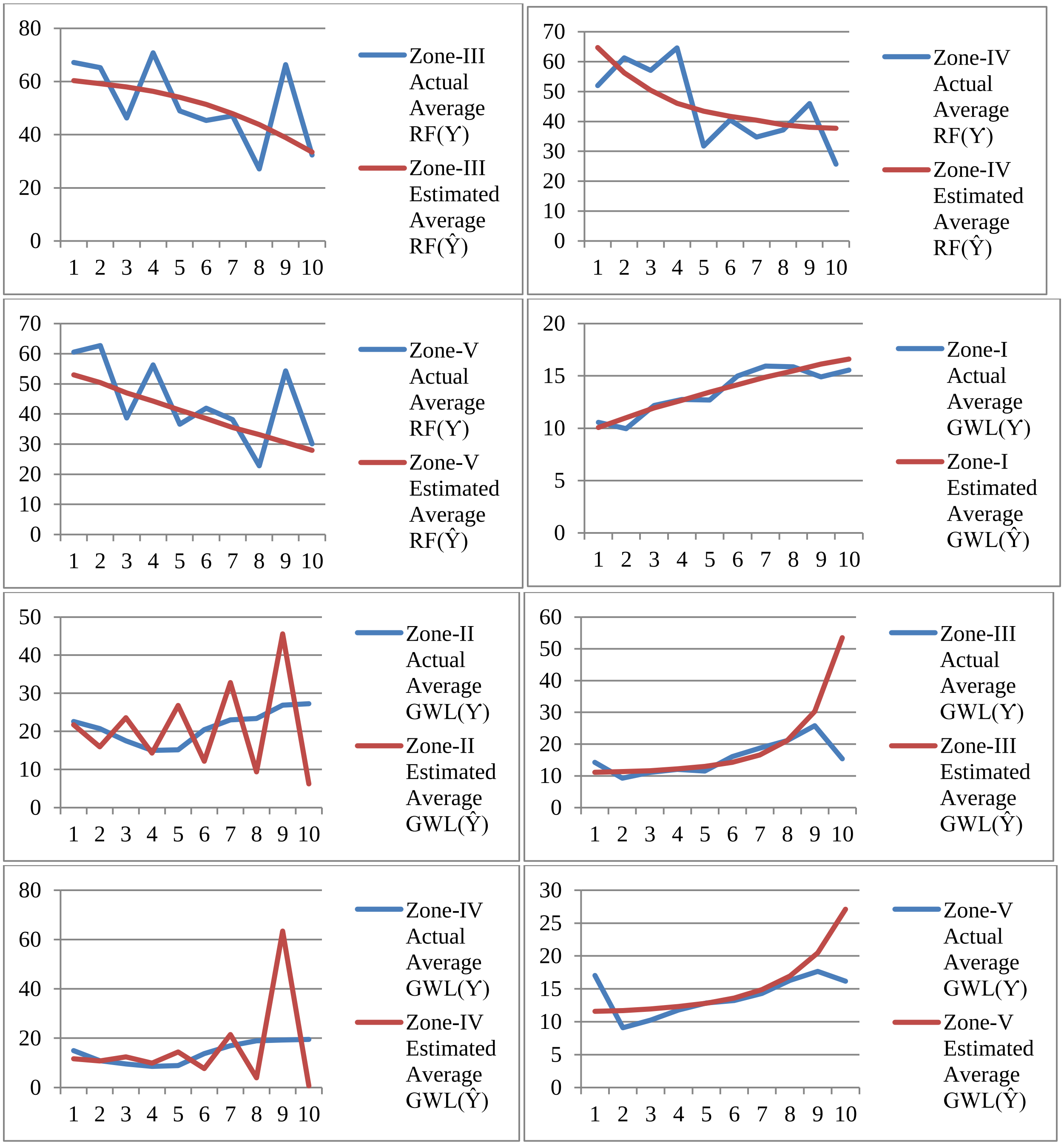
CONCLUSIONS:

By Comparing MSE's for Average RF and Average GWLs through Method of Partial Sums for Gompertz model under consideration, for RF of zone-I is least and GWLs for zone-I Gompertz model is the most suitable model because MSEs is least. Next to zone-I, zone-IV has least MSEs in RF and GWLs zone-V is least. Further, the behaviors of RF and GWL through this model i.e. Gompertz model in different zones are represented in the following Figure-3.1. Similarconclusionscan be drawn from the following graphs also.

Fig-3.1

Behavior of RF and GWLs actual and Gompertz model Forecasts in Zone –I, II, III, IV and V





Note: In the above graphs x-axis represents years in the last decade i.e. from 2007 to 2016.

On y-axis Average RF measured in Mille Meters or Average GWLs measured in Meters.

4. FURTHER STATISTICAL ANALYSIS

Now, we proceed to analyze the given estimates in tables-3.1 and 3.2 using ANOVA two-way classification by considering rows as different years and columns as different zones and the following Null Hypothesis are formed and tested.

H_{01}: There is no significant difference between different years of Average RF in Anantapuramu

District.

H_{02}: There is no significant difference between Average RF of different zones in Anantapuramu District.

H_{03}: There is no significant difference between different years of Average Ground Water Levels inAnantapuramu District.

H_{04}: There is no significant difference between Average Ground Water Levels of different zones in Anantapuramu District.

Table-4.1

ANOVA Two-way Table for RF

Source of variation	d.f	S.S	M.S.S	F-cal
Rows (years)	9	3184.16	353.7956	61.87883
Columns (Zones)	4	1107.353	276.8382	48.41899
Error	36	205.832	5.717555	
Total	49	4497.345		

By comparing F-calculated value of Rows (Years) with F-critical value at 5% level of significance we reject the H_{01} i.e. There is a significant difference between different years of Average RF in Anantapuramu District. Similarly by comparing F-calculated value of Columns (Zones) with F-critical value at 5% level of significance we reject the H_{02} i.e.There is a significant difference between Average RF of different zones in Anantapuramu District.

Table-4.2

ANOVA Two-way Table for GWLs

Source of variation	d.f	S.S	M.S.S	F-cal
Rows (years)	9	2287.27	254.1411	2.472185
Columns (Zones)	4	370.2581	92.56451	0.900431
Error	36	3700.808	102.8002	
Total	49	6358.336		

By comparing F-calculated value of Rows (Years) with F-critical value at 5% level of significance we rejectthe H_{03} i.e. There is a significant difference between different years of Average GWLs in Anantapuramu District. Similarly by comparing F-calculated value of Columns (Zones) with F-critical value at 5% level of significance we accept the H_{04}i.e.There is no significant difference between Average GWLs of different zones in Anantapuramu District.

Since F-cal value related to Rows (Years) in RF is high so there is a necessity for Critical Difference (C.D) Test for sub-grouping various years using the following formula.

C.D. = $\sqrt{2 \times Error M.S.S/m}$ × $t_{0.01}$ for error d.f. in tables (4.1) and (4.2) ... (4.1)

Where m represents number of replicates in each zone and as well as year.

5. CRITICAL DIFFERENCE (C.D) TEST: Average RF for Years

Table-5.1

Year wise Aggregate Average RF for Gompertz model estimates by using Method of Partial Sums

Year	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Average	60.11	55.844	52.106	49.124	46.322	44.026	41.596	39.128	36.648	34.348
Ranking	X	IX	VIII	VII	VI	V	IV	III	II	I

Table 5.2

If we can arranged Ascending Order

Year	2016	2015	2014	2013	2012	2011	2010	2009	2008	2007
Average	34.348	36.648	39.128	41.596	44.026	46.322	49.124	52.106	55.844	60.11

S.E = $\sqrt{2 \times Error\ M.S.S/m}$

= 1.51

1% l.o.f C.D = 2.58×1.51

= 3.90

2016 2015 2014 2013 2012 2011 2010 2009 2008 2007

Above notation indicates that 2016-2015, 2015-2014, 2014-2013, 2013-2012, 2012-2011, 2011-2010, 2010-2009, 2009-2008 years Average RF come under one category and 2007 year Average RF come under another category because there is no Significant Difference in average RF. These years are ranked based on their respective Average RF.

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