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ON LIMIT THEOREMS FOR BRANCHING PROCESSES WITH DEPENDENT IMMIGRATION

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Abstract. In this paper we consider branching processes with $m-$dependent and increasing immigration. We derive the asymptotic normality for fluctuations of such processes when the process is a supercritical. Moreover, under some mild conditions on immigration process we will prove convergence in L^2-sense of properly normalized supercritical branching process with $m-$dependent immigration.

1. Introduction

Let $\{\xi_{k,i}, k, i \geq 1\}$ and $\{\varepsilon_k, k \geq 1\}$ be two sequence of non-negative integer-valued random variables such that the two families $\{\xi_{k,i}, k, i \geq 1\}$ and $\{\varepsilon_k, k \geq 1\}$ are independent, $\{\xi_{k,i}, k, i \geq 1\}$ are independent and identically distributed (i.i.d.). We consider a sequence of branching processes with immigration $X_k, k \geq 0$ defined recursively as

$$X_0 = 0, \quad X_k = \sum_{i=1}^{X_{k-1}} \xi_{k,i} + \varepsilon_k, \quad k \geq 1. \tag{1.1}$$

We can interpret $\xi_{k,i}$ as the number of offsprings produced by the $i-$th individual belonging to the $(k-1)-$th generation and ε_k is the number of immigrants in the $k-$th generation. We can interpret X_k as the number of individuals in the $k-$th generation.

Assume that $a = \mathbb{E}\xi_{1,1} < \infty$. Process X_k is called subcritical, critical or supercritical depending on $a < 1$, $a = 1$ or $a > 1$, respectively.

The asymptotic behavior the distribution of X_n as $n \to \infty$ has been studied by many authors, see, e.g., the survey of Vatutin and Zubkov [32]. For the first time, Sevast'yanov [29] proved limit theorems for continuous-time Markov branching processes when $\{\varepsilon_n, n \geq 1\}$ is an independent and distributed by Poisson law. Then many research works appeared in which various generalizations of the immigration process were considered. For instance, in the critical case, Nagaev [20] considered a wide-sense stationary immigration process $\{\varepsilon_n, n \geq 1\}$, and proved weak convergence of distribution of $n^{-1}X_n$ as $n \to \infty$ to a gamma distribution. Asadullin and Nagaev [1] managed to weaken conditions of [20] up to the general condition that there exists a random variable ε, such that $n^{-1}\mathbb{E}\left|\sum_{k=1}^{n}(\varepsilon_k - \varepsilon)\right| \to 0$

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as $n \to \infty$. Badalbaev and Zubkov [5] for the sequence of special random processes (including branching processes with immigration) proved a limit theorem which contains results of [20] and [1] as a special case. Concerning functional limit theorems for (1.1), we refer to Wei and Winnicki [33], Sriram [30], Ispány et al.[9],[10], Khusanbaev [14], [16] and see references therein.

While most of previous literature was concerned with stationary immigration, the non-stationary (or time-dependent) and increasing immigration case were first studied by Rahimov [22]. In this class of processes, unlike in classical models, immigration rate may depend on time of immigration. Later, Rahimov [23] obtained for all cases central limit theorems (CLT's) for fluctuation of (1.1). A key reference is a monograph by Rahimov [24], where one can find a variety of results and references related to this process. We refer to [28], [13] and [15] to the rates of convergence in CLT for process (1.1) Later, Rahimov [25] proved functional limit theorems for a fluctuation of critical process defined by (1.1). For deterministic approximation for (1.1), we refer to Rahimov [26], Khusanbaev [11]-[12] and see references therein.

However, in all of previous works, the immigration process was assumed to be independent. Guo and Zhang [8] proved a functional limit theorem for fluctuations of the critical branching process (1.1) under the condition of $m-$ dependent immigration. Recently, Khusanbaev et al. [17] managed to weaken conditions of [8] to the case when immigration satisfies $\phi-$mixing condition and proved functional limit theorems for fluctuation of (1.1). The results of above cited papers showed that the limit behavior of fluctuations of process (1.1) clearly depends on the rate of convergence to infinity of the mean number of immigrants.

The aim of this paper is to establish CLT for fluctuations of supercritical process X_n defined by (1.1) in the case when the immigration process is $m-$dependent and the mean of immigrants tends to infinity. Moreover, we study conditions ensuring convergence in L^2-sense of properly normalized supercritical branching process with $m-$dependent and regularly varying immigration to some random variable.

The paper is organized as follows. In Section 2, we provide basic facts and definitions. Section 3 contains main results and their proofs.

2. Notations and definitions

We begin by introducing basic facts. First, let us recall the notation of m-dependence.

Definition 2.1. It is said that a sequence of random variables $\{\xi_n, n \geq 1\}$ satisfies the $m-$dependence condition for some $m \geq 0$ if the random vectors $(\xi_1, ..., \xi_k)$ and $(\xi_{k+m+1}, ...,)$ are independent for all $k \geq 1$ (see [6]).

Throughout the paper, we will assume that the immigration process is heterogeneous, i.e., random variables $\{\varepsilon_k, k \geq 1\}$ are non-identically distributed and $m-$dependent, and the particles entering the population act independently of the other particles and according to the same law as particles of the population. We denote by $Z_i^j(k)$, $k = i, i+1, ..., 1 \leq j \leq \varepsilon_i$, the branching Galton-Watson process generated by $j-$th particles arriving at the moment i with $Z_i^j(i) = 1$. According to our assumptions branching processes $Z_i^j(k)$, $k \geq i$; $i, j \geq 1$ are independent and $Z_i^j(k+i)$, $k = 0, 1, ...$ has the same distribution as $Z_1^1(k)$, $k = 1, 2, ...$

Set

$$p_k = \mathbb{P}(\xi_{1,1} = k), \quad B = \mathbb{E}(\xi_{1,1})(\xi_{1,1} - 1), \quad b^2 = \mathrm{Var}(\xi_{1,1}),$$

$$B_t = \sum_{k=2}^{\infty} k^{t-1}(k-1)p_k < \infty, \quad t > 1.$$

We also assume that $\alpha(n) = \mathbb{E}\varepsilon_n < \infty$ and $\beta(n) = \mathrm{Var}(\varepsilon_n) < \infty$ for all $n \in \mathbb{N}$.

Definition 2.2 ([7])**.** A measurable function $l : (0, \infty) \to (0, \infty)$ is slowly varying if for all $\lambda > 0$

$$\lim_{x \to \infty} \frac{l(\lambda x)}{l(x)} = 1.$$

Definition 2.3 ([7])**.** A measurable function $f : (0, \infty) \to (0, \infty)$ is called regularly varying at infinity if it can be represented in the form

$$f(x) = x^{\rho} l(x),$$

where ρ is called index of regular variation and $\rho \in (-\infty, \infty)$ and $l_\rho(x)$ is a slowly varying function.

Definition 2.4 ([7])**.** A sequence of positive numbers $\{x(n), n \geq 1\}$ is called a regularly varying sequence of index ρ if there is a sequence of positive terms $\{y(n), n \geq 1\}$ satisfying

$$x(n) \sim Cy(n), \quad n(1 - (y(n-1)/y(n))) \to \rho < \infty, \quad n \to \infty.$$

If a sequence $\{x(n), n \geq 1\}$ is regularly varying with index ρ, we will write $\{x(n), n \geq 1\} \in R_\rho$.

Under the above assumptions $A(a, n) = \mathbb{E}X_n$ and $B^2(a, n) = \mathrm{Var}(X_n)$ are finite for all $n \geq 1$. Denote

$$f_k(t) = \mathbb{E}e^{itZ_1^1(k)}, \quad \Psi_k(t) = \mathbb{E}e^{itX_k}, \quad k \geq 1.$$

Denote by $\Phi(x)$ distribution function of the standard normal law. In this paper, $a_n \sim b_n$ denotes $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$ and $a_n = o(b_n)$ denotes $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$, $a \wedge b = \min(a, b)$ and $\xrightarrow{P}$ denotes the convergence of random variables in probability. C denotes a generic constant which may be different from place to place.

3. Main results and their proofs

Our first result provides conditions for validity of CLT for supercritical branching processes X_n. Note that the immigration mean and variance is not assumed to be a regularly varying at infinity.

Theorem 3.1. *Assume $a > 1$, $B_{2+\delta} < \infty$ for some $0 < \delta < 1$ and let $\{\varepsilon_n, n \geq 1\}$ be a sequence of $m-$dependent random variables. If $\chi(a, n) := \sum\limits_{k=1}^{n} a^{-2k}\alpha(k) \to \infty$ and $\beta(n) = o(\chi(a, n))$ as $n \to \infty$ then*

$$\mathbb{P}\left(\frac{X_n - A(a, n)}{B(a, n)} < x\right) \to \Phi(x), \quad n \to \infty, \quad x \in \mathbb{R}.$$

Remark 3.2. Note that the condition $\chi(a,n) \to \infty$ holds if $\liminf \frac{\alpha(n+1)}{\alpha(n)} \geq a^2$. This means that immigration mean tends to infinity fast enough. Clearly, if $\alpha(n) \in R_\alpha$ then $\chi(a,n)$ does not converge to infinity.

Proof. Note that the process X_k can be represented as

$$X_k = \sum_{i=1}^{k} \sum_{j=1}^{\varepsilon_i} Z_i^j(k), \quad k \geq 1. \tag{3.1}$$

By (3.1) and taking into account independence of random variables $Z_i^j(k)$, it follows

$$\Psi_n(t) = \mathbb{E} \prod_{k=1}^{n} f_{n-k}^{\varepsilon_k}(t).$$

Indeed, $Z_i^j(k)$ are independent of $\varepsilon_n, n \geq 1$, since by assumption two sequences $\{\xi_{k,i}, k, i \geq 1\}$ and $\varepsilon_n, n \geq 1$ are independent, obtain

$$\Psi_n(t) = \mathbb{E}\exp(itX_n) = \mathbb{E}\exp(it\sum_{i=1}^{n}\sum_{j=1}^{\varepsilon_i} Z_i^j(n))$$

$$= \mathbb{E}[\mathbb{E}[\prod_{i=1}^{n}\exp(it\sum_{j=1}^{\varepsilon_i} Z_i^j(n)|\sigma(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)]] = \mathbb{E}[\prod_{i=1}^{n}[\mathbb{E}(itZ_1^1(n))]^{\varepsilon_i}].$$

Therefore

$$\mathbb{E}e^{it\frac{X_n - A(a,n)}{B(a,n)}} = e^{-it\frac{A(a,n)}{B(a,n)}}\mathbb{E}\prod_{k=1}^{n} f_{n-k}^{\varepsilon_k}\left(\frac{t}{B(a,n)}\right).$$

Since $\mathbb{E}\xi_{1,1}^{2+\delta} < \infty$, $0 < \delta < 1$ then we may expand $f_k(t)$ into the Taylor series (see Loev [19], 212 pp.):

$$f_k(t) = 1 + it\mathbb{E}Z_1^1(k) - \frac{t^2}{2}\mathbb{E}\big[Z_1^1(k)\big]^2 + 2^{1-\delta}\theta_k\frac{t^{2+\delta}}{(1+\delta)(2+\delta)}\mathbb{E}\big[Z_1^1(k)\big]^{2+\delta}, \ |\theta_k| \leq 1.$$

Using the well-known formulas (see [3])

$$\mathbb{E}Z_1^1(k) = a^k, \quad \mathbb{E}\big[Z_1^1(k)\big]^2 = \frac{a^{k-1}\left(a^{k-1}-1\right)}{a-1}b^2 + a^{2k},$$

and expansion $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + O\left((x-1)^3\right)$, $x \to 1$, we obtain

$$\ln\left[e^{-it\frac{A(a,n)}{B(a,n)}}\prod_{k=1}^{n} f_{n-k}^{\varepsilon_k}\left(\frac{t}{B(a,n)}\right)\right] =$$

$$= \frac{it}{B(a,n)}\sum_{k=1}^{n} a^{n-k}\left(\varepsilon_k - \alpha(k)\right) - \frac{t^2}{2B^2(a,n)}\sum_{k=1}^{n}\varepsilon_k\frac{a^{n-k}\left(a^{n-k}-1\right)}{a(a-1)}b^2 +$$

$$+O\left(\frac{t^{2+\delta}}{B^{2+\delta}(a,n)}\sum_{k=1}^{n}\varepsilon_k\mathbb{E}\big[Z_1^1(n-k)\big]^{2+\delta} + \frac{|t|^3}{B^3(a,n)}\sum_{k=1}^{n} a^{3(n-k)}\alpha(k)\right) =$$

$$= J_1(n) + J_2(n) + O\left(J_3(n)\right), \quad a.s. \tag{3.2}$$

We have to evaluate each term involved in the right-hand side of (3.2). Before we start, it should be pointed out the asymptotic behavior of $B^2(a,n)$ as $n \to \infty$. Observe that under conditions of Theorem 3.1, we have

$$B^2(a,n) \sim \Delta^2(a,n) \sim Ca^{2n}\chi(a,n), \quad n \to \infty. \tag{3.3}$$

It is obvious that $\mathbb{E}[J_1(n)] = 0$. By Cauchy-Bunyakovsky inequality and inequality $2xy \le x^2 + y^2$, $x \in \mathbb{R}$, Lemma 3.2 from [27] and also using the properties of regularly varying functions at infinity, we deduce

$$\mathbb{E}[J_1(n)]^2 \le \frac{t^2}{B^2(a,n)} \sum_{k=1}^{n} a^{2(n-k)}\beta(k) +$$

$$+\frac{2t^2}{B^2(a,n)} \sum_{k=1}^{n-1} \sum_{j=k+1}^{(k+m-1)\wedge n} a^{2n-j-k}\sqrt{\beta(k)}\sqrt{\beta(j)} \le$$

$$\le C\frac{a^{2n}t^2}{B^2(a,n)} \max_{1\le k\le n} \beta(k) +$$

$$+\frac{a^{2n}(m-1)t^2}{B^2(a,n)} \max_{1\le k\le n} \beta(k) \sum_{k=1}^{n} a^{-2k} + \frac{a^{2n}(m-1)t^2}{B^2(a,n)} \sum_{k=1}^{n} a^{-2k}\beta(k) \sim$$

$$\sim Cm\frac{t^2}{\chi(a,n)} \max_{1\le k\le n} \beta(k) \to 0, \quad n \to \infty.$$

Hence, we may claim that

$$J_1(n) \xrightarrow{P} 0, \quad n \to \infty. \tag{3.4}$$

Let us consider $J_2(n)$. From (3.3) it follows that

$$\mathbb{E}J_2(n) = -\frac{t^2}{2B^2(a,n)}\Delta^2(a,n) \to -\frac{t^2}{2}, \quad n \to \infty.$$

Now we estimate the variance of $J_2(n)$. Observe that

$$\mathrm{Var}J_2(n) \le \frac{(2m-1)t^4}{4B^4(a,n)} \sum_{k=1}^{n} \beta(k) \frac{a^{2(n-k)}\left(a^{n-k}-1\right)^2}{a^2(a-1)^2} b^4 \sim$$

$$\sim \frac{C(2m-1)b^4t^4}{4\left(a^{2n}\chi(a,n)\right)^2} a^{4n} \max_{1\le k\le n} \beta(k) \to 0, \quad n \to \infty.$$

Thus, by Chebyshev's inequality

$$J_2(n) \xrightarrow{P} -\frac{t^2}{2}, \quad n \to \infty. \tag{3.5}$$

Concerning $J_3(n)$, we get

$$\mathbb{E}|J_3(n)| \le \frac{t^{2+\delta}}{B^{2+\delta}(a,n)} \sum_{k=1}^{n} \alpha(k)\,\mathbb{E}\left[Z_1^1(n-k)\right]^{2+\delta} + \frac{|t|^3}{B^3(a,n)} \sum_{k=1}^{n} a^{3(n-k)}\alpha(k)$$

$$= J_{3,1}(n) + J_{3,2}(n). \tag{3.6}$$

Let us consider $J_{3,2}(n)$. From (3.3) and $\alpha(n) \sim \chi(a,n)$, $n \to \infty$, we have

$$J_{3,2}(n) \le \frac{|t|^3}{a^{2n}\chi(a,n)\sqrt{a^{2n}\chi(a,n)}}\alpha(n)\frac{a^{3n}-1}{a^3-1} \to 0, \quad n \to \infty. \tag{3.7}$$

Finally, it remains to consider $J_{3,1}(n)$. From Nagaev ([21], Theorem 1.7), we know that if $B_t < \infty$ and $a \ge 1$ then for any $t > 2$,

$$\mathbb{E}[Z_1^1(n)]^t < c_1(t)a^{nt}\left(\frac{B_t}{B}\right)^{\frac{t}{(t-2)}} + c_2(t) +$$

$$+c_3(t)B^t\left(\frac{a^n-1}{a-1}\right)^t + c_4(t)a^{nt}\frac{a^n-1}{a-1}\beta_t, \tag{3.8}$$

where

$$\beta_t = \mathbb{E}\xi_{1,1}^t, \quad c_1(t) = 1,02t\left(\frac{5et}{t-2}\right)^t, \quad c_2(t) = t(t+1)e^{1,5t-1}\Gamma(t),$$

$$c_3(t) = 2^{-t}e^{-t-1}t(t+1)(t+3,5)^t\Gamma(t), \quad c_4(t) = \frac{4}{5}\left(\frac{5}{2}\right)^{2t}\left(\frac{t}{t-2}\right)^t,$$

and $\Gamma(t)$ is a Euler's function.

From (3.8) and (3.3) it follows

$$J_{3,1}(n) \le \frac{\alpha(n)\,t^{2+\delta}}{B^{2+\delta}(a,n)}\sum_{k=1}^{n}\mathbb{E}\big[Z_1^1(n-k)\big]^{2+\delta} \le$$

$$\le \frac{Ct^{2+\delta}}{a^{(2+\delta)n}(\chi(a,n))^{\frac{\delta}{2}}}\frac{a^{(2+\delta)n}-1}{a^{2+\delta}-1} + \frac{nt^{2+\delta}}{a^{(2+\delta)n}(\chi(a,n))^{\frac{\delta}{2}}} +$$

$$\le \frac{Ct^{2+\delta}}{a^{(2+\delta)n}(\chi(a,n))^{\frac{\delta}{2}}}\frac{a^{(2+\delta)n}-1}{a^{2+\delta}-1} + \frac{Ct^{2+\delta}}{a^{(2+\delta)n}(\chi(a,n))^{\frac{\delta}{2}}}\frac{a^{(3+\delta)n}-1}{a^{3+\delta}-1} \to 0 \tag{3.9}$$

as $n \to \infty$. Then from (3.7), (3.9) and taking into account (3.6), it yields

$$J_3(n) \xrightarrow{P} 0, \quad n \to \infty. \tag{3.10}$$

Thus, combining relations (3.4)-(3.5) and (3.10), we obtain

$$\ln\left[e^{-it\frac{A(a,n)}{B(a,n)}}\prod_{k=1}^{n}f_{n-k}^{\varepsilon_k}\left(\frac{t}{B(a,n)}\right)\right] \xrightarrow{P} -\frac{t^2}{2}, \quad n \to \infty.$$

Hence, according to the Lebesgue's majorized convergence theorem, we may deduce

$$e^{-it\frac{A(a,n)}{B(a,n)}}\Psi_n\left(\frac{t}{B(a,n)}\right) \to e^{-\frac{t^2}{2}}, \quad n \to \infty,$$

which finishes the proof of Theorem 3.1. □

The last result deals with convergence of properly normalized supercritical branching processes with dependent immigration when condition $\chi(a,n) \to \infty$ is not valid. It should be noted that from Theorem 6.1 in [2] follows a.s. convergence of normalized supercritical branching processes with immigration. We will additionally prove the L_2-convergence of normalized X_n and make sure that the first two moments of the limit distribution exist.

Theorem 3.3. *Assume that $a > 1$, $b^2 > 0$. Let $\{\varepsilon_n, n \geq 1\}$ be a sequence of $m-$dependent random variables with $\alpha(n) \in R_\alpha$ and $\beta(n) \in R_\beta$ where $\alpha, \beta \geq 0$. Moreover, assume the following condition holds:*

$$\sum_{k=1}^{\infty} a^{-k}\alpha(k) < \infty, \quad \sum_{k=1}^{\infty} a^{-k}\beta(k) < \infty. \tag{3.11}$$

Then there exists a random variable V such that

$$V(n) := \frac{X_n}{a^n} \to V, \quad n \to \infty \tag{3.12}$$

where convergence means in L^2-sense, and with $\mathbb{E}V < \infty$, $\mathrm{Var}(V) < \infty$.

Proof. We will show that

$$\mathbb{E}[V(n+k) - V(k)]^2 \to 0, \quad k \to \infty \tag{3.13}$$

uniformly for $n \geq 0$. It is obvious that

$$\mathbb{E}[V(n+k) - V(k)]^2 = \mathrm{Var}(V(n+k)) + \mathrm{Var}(V(k)) - 2\,\mathrm{cov}(V(n+k), V(k)) \\ +(\mathbb{E}V(n+k) - \mathbb{E}V(k))^2. \tag{3.14}$$

First, we have

$$\mathrm{cov}(X_{n+k}, X_k) = a^n \mathrm{Var}(X_k) + \sum_{j=k+1}^{k+n} a^{k+n-j}\,\mathrm{cov}(X_k, \varepsilon_j)$$

$$= a^n \mathrm{Var}(X_k) + \sum_{j=k+1}^{k+n} \sum_{l=1}^{k} a^{2k+n-j-l}\,\mathrm{cov}(\varepsilon_l, \varepsilon_{k+i}). \tag{3.15}$$

Now using the identity

$$\mathrm{cov}(V(n+k), V(k)) = \frac{1}{a^{2k+n}}\,\mathrm{cov}(X_{n+k}, X_k), \tag{3.16}$$

and by substituting the expression of (3.15) in (3.16), we obtain the two terms of the following form:

$$\mathrm{cov}(V(n+k), V(k)) = \frac{1}{a^{2k}}\mathrm{Var}(X_k) + \sum_{j=k+1}^{k+n} \sum_{l=1}^{k} a^{-j-l}\,\mathrm{cov}(\varepsilon_l, \varepsilon_j). \tag{3.17}$$

We will show that under condition (3.11) the second term of the right hand side of (3.17) converges to zero as $n, k \to \infty$. Indeed,

$$\sum_{j=k+1}^{k+n} \sum_{l=1}^{k} a^{-j-l}\,\mathrm{cov}(\varepsilon_l, \varepsilon_j) \leq \sum_{j=k+1}^{k+n} \sum_{l=1}^{k} a^{-j-l}(\beta(l) + \beta(j))$$

$$\leq \sum_{j=k+1}^{k+n} a^{-j} \sum_{l=1}^{k} a^{-l}\beta(l) + \sum_{j=k+1}^{k+n} a^{-l}\beta(l) \sum_{l=1}^{k} a^{-l} \leq$$

$$\leq \frac{a^{-1}(1-a^{-n})}{1-a^{-1}} \frac{1}{a^k} \sum_{l=1}^{k} a^{-l}\beta(l) + \frac{a^{-1}(1-a^{-k})}{1-a^{-1}} \sum_{j=k+1}^{\infty} a^{-j}\beta(j) \to 0. \tag{3.18}$$

Now, we may write

$$|\mathrm{Var}\,(V\,(n+k)) - \mathrm{Var}\,(V\,(k))| \leq$$

$$\leq \sum_{j=k+1}^{\infty} a^{-2j}\,(C\alpha\,(j) + \beta\,(j)) + \sum_{i=k+1}^{k+n}\sum_{j=1}^{i-1} a^{-i-j}\,\mathrm{cov}\,(\varepsilon_j, \varepsilon_i). \tag{3.19}$$

Note that from (3.11) it follows that the first term of the right hand side of (3.19) tends to zero as $k \to \infty$. It remains to show that the second term converges to zero as $k \to \infty$. For this purpose, we may write

$$\sum_{i=k+1}^{k+n}\sum_{j=1}^{i-1} a^{-i-j}\,\mathrm{cov}\,(\varepsilon_j, \varepsilon_i) \leq \sum_{i=k+1}^{k+n}\sum_{j=1}^{i-1} a^{-i-j}(\beta(j) + \beta(i)) \leq$$

$$\leq \sum_{i=k+1}^{k+n} a^{-i}\sum_{j=1}^{i-1} a^{-j}\beta(j) + \sum_{i=k+1}^{k+n} a^{-i}\beta(i)\sum_{j=1}^{i-1} a^{-j} \to 0, \quad n, k \to \infty. \tag{3.20}$$

Therefore, from (3.11) and (3.19)-(3.20) it entails that uniformly for $n \geq 0$

$$\mathrm{Var}\,(V\,(n+k)) - \mathrm{Var}\,(V\,(k)) \to 0, \quad k \to \infty. \tag{3.21}$$

Again, by condition (3.11), we obtain

$$\mathbb{E}V\,(n+k) - \mathbb{E}V\,(k) \leq \sum_{j=k+1}^{\infty} a^{-j}\alpha\,(j) \to 0, \quad k \to \infty \tag{3.22}$$

uniformly for $n \geq 0$. Consequently, from (3.18),(3.21)-(3.22) it yields (3.13). Hence, the existence of a random variable V implies from (3.13) and well-known Cauchy criterion for average convergence of order $p \geq 1$ (see [31], Theorem 7, Chapter 2, p.333). Consequently, $V(n)$ converges in L^2-sense to V as $n \to \infty$. Now, the application of the inequality $(\sqrt{\mathbb{E}X^2} - \sqrt{\mathbb{E}Y^2})^2 \leq \mathbb{E}(X-Y)^2$ which can be easily verified, gives us

$$\left(\sqrt{\mathbb{E}(V\,(n))^2} - \sqrt{\mathbb{E}V^2}\right)^2 \leq \mathbb{E}(V\,(n) - V)^2. \tag{3.23}$$

By (3.23), we may conclude that

$$\mathbb{E}(V\,(n))^2 \to \mathbb{E}(V)^2, \quad n \to \infty.$$

On the other hand, by the inequality $(\mathbb{E}X)^2 \leq \mathrm{E}X^2$, we have

$$(\mathbb{E}\,(V\,(n) - V))^2 \leq \mathbb{E}(V\,(n) - V)^2 \to 0, \quad n \to \infty$$

which ends the proof of Theorem 3.3. □

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