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LOCATING THE POSITIONAL VALUES OF THE ANY FRACTIONS IN THE (FAREY AND STERN-BROCOT) SEQUENCE, MATRIX

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Abstract

In this paper we discuss about Farey sequence, Farey Matrix, Stern–Brocot sequence and itsMatrix. The Matrix corresponding to the Stern-Brocot sequence is constructed so that it contains a square zero matrix. The principal aim of this paper is about finding the location of any fraction in the Farey and Stern-Brocot Sequence and its Matrix.

Key words and phrases: Stern - Brocot Sequence, Farey sequence, Fibonacci sequence.

Notations: $\boldsymbol{S_n}$ -Stern - Brocot Sequence of index n

$\boldsymbol{F_n}$ **-Farey sequence of index** $\boldsymbol{n}$

$\boldsymbol{MS_n}$ −Stern-Brocot Matrix

$\boldsymbol{MF_n}$ – Farey Matrix

$\boldsymbol{PS_n}$ −Position of an element in $\boldsymbol{S_n}$

$\boldsymbol{PF_n}$ −Position of an element in $\boldsymbol{F_n}$

$\boldsymbol{P[MS_n]}$ −Position of an element in $\boldsymbol{MS_n}$

$\boldsymbol{P[MF_n]}$ −Position of an element in $\boldsymbol{MF_n}$

AMS (MOS) Subject Classifications: 11B57.

1. Introduction

The Stern–Brocot tree has received much attention recently due to its deep connections with physical chemistry [4]. The Stern - Brocot tree was discovered by Moritz Stern [1] in 1858 and AchilleBrocot [2] in 1861.It was originally used by Brocot to design gear systems with a gear ratio close to some desired value by finding a ratio of smooth numbers near that value. Since smooth numbers factor into small primes,several small gears could be connected in sequence to generate an effective ratio of the product of their teeth, thus creating a relative small gear train while minimizing its error[3][11][7][9].

The Stern – Brocot tree begins with the numbers $\frac{0}{1}$and $\frac{1}{0}$. The proceeding levels of the tree are formed by including the Mediant fraction $\frac{a+c}{b+d}$ between every pair of neighbours values $\frac{a}{b}$ and$\frac{c}{d}$ and the procedure is repeated to infinity. Retracing the tree upward gives a series of progressively worse rational approximations with decreasing denominators[3].

Definition 1.(Stern–Brocot Sequence).

Define I_1^n=$\frac{p_1^n}{q_1^n}$ and I_n^n=$\frac{p_n^n}{q_n^n}$where $p_1^n = 0, q_1^n = 1, p_n^n = 1, q_n^n = 0.$

Initially the sequence was $I_n = \{I_1^n, I_n^n\} = \left\{\frac{0}{1}, \frac{1}{0}\right\}$

Also set I_2^n=$\frac{p_2^n}{q_2^n} = \frac{(proceeding+suceeding)\ values\ of p_2^n}{(proceeding+suceeding)\ values\ of q_2^n}$= $\frac{(0+1)}{(1+0)} = \frac{1}{1}$.

In general, I_m^n=$\frac{p_m^n}{q_m^n} = \frac{(proceeding+suceeding)\ values\ of p_m^n}{(proceeding+suceeding)\ values\ of q_m^n}$

The sequence $I_n$is defined by setting I_1^n and I_n^n as extreme values

$$I_n = \{I_1^n, I_2^n, I_3^n \ldots \ldots \ldots \ldots . I_n^n\}$$

Definition 2. (Parents)[5]

We call $I_k^{n-1}$and I_{k+1}^{n-1}, the left and right parents respectively of I_{2k}^n

Defintion 3 (Levels of the Stern- Brocot Tree)[5] [11][7][9]

Let I_0 be level 0 of the Stern- Brocot Tree. For $n > 0$, level n of the Stern-Brocot tree is defined as med I_{n-1} where

$$\mathbf{Med}\boldsymbol{I_{n-1}} = \langle(\boldsymbol{I_{n-1,1}} \oplus \boldsymbol{I_{n-1,2}}), (\boldsymbol{I_{n-1,2}} \oplus \boldsymbol{I_{n-1,3}}) \ldots \ldots \ldots . (\boldsymbol{I_{n-1,2^{n-1}}} \oplus \boldsymbol{I_{n-1,2^{n-1}+1}})\rangle$$

and⊕ is the child's addition operator where by $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$

THROREM:1

If the order of Stern-Brocot Sequence is $'n'$ it can be represented as a Matrix of order $m^2$with the zero square matrix where $m$is given by

1. When n is odd the order $m = 2^k$
2. When n is even the order m is immediate to 2^{n-1}

Proof:

Consider Stern-Brocot sequence S_n of order n in $[0,1]$.The number of elements in this sequence using mediant property is 2^{n-1}, leaving out the middle most element $\frac{1}{2}$. These elements can be arranged in a matrix.

When n is odd, let $n - 1 = 2^{k+1}$ then the number of elements in$S_{2k+1} = 2^{2k} = (2^k)^2$

Therefore, taking the order of the matrix as 2^k all the positions in the matrix of order $m = 2^k$ Will be occupied in the matrix.

When n is even, $n-1$ is odd and hence the 2^{n-1} elements of S_n cannot be a square number. Here a square number immediate to 2^{n-1} can be chosen as the order of the matrix.

Formation of the Matrix:

Zero is put in the cell $(1,1)$ and one is put in the cell (m,m) leaving out zero and one, the elements left to $\frac{1}{2}$ are arranged row wise in the upper triangular position of the matrix and the remaining elements are arranged along the diagonal from the top position.

Similarly starting from column one the elements to the right of $\frac{1}{2}$ are arranged column wise, the remaining are put in the diagonal from the bottom.

When n is odd the elements are arranged as in the previous case and in the left out cells zero is put. It is observed that a near zero matrix is obtained in the lower triangular portion.

Illustration:1

Construction of the Stern –BrocotMatrix MS_n,where$n = 8$ (even)

$$MS_n = \begin{bmatrix}
\frac{0}{1} & \frac{1}{8} & \frac{1}{7} & \frac{2}{13} & \frac{1}{6} & \frac{3}{17} & \frac{2}{11} & \frac{3}{16} & \frac{1}{5} & \frac{4}{19} & \frac{3}{14} & \frac{5}{23} \\
\frac{7}{8} & \frac{0}{c} & \frac{2}{9} & \frac{5}{12} & \frac{3}{13} & \frac{4}{17} & \frac{1}{4} & \frac{5}{19} & \frac{4}{15} & \frac{7}{26} & \frac{3}{11} & \frac{8}{29} \\
\frac{6}{7} & \frac{7}{9} & \frac{0}{c} & \frac{5}{18} & \frac{7}{25} & \frac{2}{9} & \frac{7}{24} & \frac{5}{17} & \frac{8}{27} & \frac{3}{10} & \frac{7}{23} & \frac{4}{13} \\
\frac{11}{13} & \frac{17}{22} & \frac{13}{18} & \frac{0}{c} & \frac{5}{16} & \frac{1}{3} & \frac{6}{17} & \frac{5}{14} & \frac{9}{25} & \frac{4}{11} & \frac{11}{30} & \frac{7}{19} \\
\frac{5}{6} & \frac{10}{13} & \frac{18}{25} & \frac{11}{16} & \frac{0}{c} & \frac{10}{27} & \frac{3}{8} & \frac{11}{29} & \frac{8}{21} & \frac{13}{34} & \frac{5}{13} & \frac{12}{31} \\
\frac{14}{17} & \frac{13}{17} & \frac{5}{7} & \frac{2}{3} & \frac{17}{27} & \frac{0}{c} & \frac{7}{18} & \frac{9}{23} & \frac{2}{5} & \frac{9}{22} & \frac{7}{17} & \frac{12}{29} \\
\frac{9}{11} & \frac{3}{4} & \frac{17}{24} & \frac{11}{17} & \frac{5}{8} & \frac{11}{18} & \frac{0}{c} & \frac{5}{12} & \frac{13}{31} & \frac{8}{19} & \frac{11}{26} & \frac{4}{9} \\
\frac{13}{16} & \frac{14}{19} & \frac{12}{17} & \frac{9}{14} & \frac{18}{29} & \frac{14}{23} & \frac{7}{12} & \frac{0}{c} & \frac{10}{23} & \frac{7}{16} & \frac{11}{25} & \frac{4}{9} \\
\frac{4}{5} & \frac{11}{15} & \frac{19}{27} & \frac{16}{25} & \frac{13}{21} & \frac{3}{5} & \frac{18}{31} & \frac{13}{23} & \frac{0}{c} & \frac{9}{20} & \frac{5}{11} & \frac{6}{13} \\
\frac{15}{19} & \frac{19}{26} & \frac{7}{10} & \frac{7}{11} & \frac{21}{34} & \frac{13}{22} & \frac{11}{19} & \frac{9}{16} & \frac{11}{20} & \frac{0}{c} & \frac{0}{c} & \frac{0}{c} \\
\frac{11}{14} & \frac{8}{11} & \frac{16}{23} & \frac{19}{30} & \frac{8}{13} & \frac{10}{17} & \frac{15}{26} & \frac{14}{25} & \frac{6}{11} & \frac{0}{c} & \frac{0}{c} & \frac{0}{c} \\
\frac{18}{23} & \frac{21}{29} & \frac{9}{13} & \frac{12}{19} & \frac{19}{31} & \frac{17}{29} & \frac{4}{7} & \frac{5}{9} & \frac{7}{13} & \frac{0}{c} & \frac{0}{c} & \frac{1}{1}
\end{bmatrix}$$

Note: To overcome technical difficulties choose $0 = \frac{0}{c}$

Illustration:2

Construction of the Stern –BrocotMatrix MS_n,where$n = 7$(odd)

$$MS_n = \begin{bmatrix}
\frac{0}{1} & \frac{1}{7} & \frac{1}{6} & \frac{2}{11} & \frac{1}{5} & \frac{3}{14} & \frac{2}{9} & \frac{3}{13} \\
\frac{6}{7} & \frac{7}{16} & \frac{1}{4} & \frac{4}{15} & \frac{3}{11} & \frac{5}{18} & \frac{2}{7} & \frac{5}{17} \\
\frac{5}{6} & \frac{3}{4} & \frac{4}{9} & \frac{3}{10} & \frac{4}{13} & \frac{1}{3} & \frac{5}{14} & \frac{4}{11} \\
\frac{9}{11} & \frac{11}{15} & \frac{7}{10} & \frac{5}{11} & \frac{7}{19} & \frac{3}{8} & \frac{8}{21} & \frac{5}{13} \\
\frac{4}{5} & \frac{8}{11} & \frac{9}{13} & \frac{12}{19} & \frac{6}{11} & \frac{7}{18} & \frac{2}{5} & \frac{7}{17} \\
\frac{11}{14} & \frac{13}{18} & \frac{2}{3} & \frac{5}{8} & \frac{11}{18} & \frac{5}{9} & \frac{5}{12} & \frac{8}{19} \\
\frac{7}{9} & \frac{5}{7} & \frac{9}{14} & \frac{13}{21} & \frac{3}{5} & \frac{7}{12} & \frac{9}{16} & \frac{3}{7} \\
\frac{10}{13} & \frac{12}{17} & \frac{7}{11} & \frac{8}{13} & \frac{10}{17} & \frac{11}{19} & \frac{4}{7} & \frac{1}{1}
\end{bmatrix}$$

2. **Farey Sequence[13][14] [10]**

The Farey sequence F_n for any positive integers 'n' is the set of irreducible rational numbers $\frac{a}{b}$with $0 \le a \le b \le n$ and (a,b)=1 arranged in increasing order. The first few are

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$

$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$
$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

Theorem:2

The Farey sequence consists of fractals elements lying between 0 and 1 here the denominators of the elements does not exceed the index of the sequence as in Stern-Brocot sequence this sequence of elements may also be represented as the matrix with the lower triangular zero matrix.

Proof:

Consider MF_n is a minimum order $(m \times m)$ square matrix taken according to the total number of elements of the Farey sequence. Arrangement of the rows and columns follows the procedure as in construction of Stern-Brocot Matrix. The elements to the left of $\frac{1}{2}$ are placed row wise in the upper triangular position of the matrix in sequential order and the elements to the right of $\frac{1}{2}$ are placed column wise in the lower triangular position of the matrix in the sequential reverse order by omitting the diagonals of the matrix and the value $\frac{1}{2}$.

After entering all the left and right positional elements in upper and lower triangular matrices then if any values left out in the sequence was entered in the diagonal by entering left remaining elements of $\frac{1}{2}$to the upper half of the diagonals in sequential order and also entering right remaining elements of $\frac{1}{2}$ to the lower half of the diagonals in sequential reverse order.

<u>**Matrix is represented in to two forms**</u>

Case:1

If the index of the matrix MF_n is odd, then the total number of elements in the matrix is equal to the total number of elements in the Farey sequence.

Case:2

If the index of the matrix MF_n is even, the order of the matrix must be greater than the total number of elements of the Farey sequence.

Then the left out entries in the Farey matrix are filled by zeroes and some zeroes form a lower triangular matrix on the lower right hand corner of the Farey Matrix whose order is lesser than the order of m.

Theorem:3

The Stern-Brocot sequence of elements generated using mediant property can be classified through denominators. The formula exhibited here displays the sequence of points in the Stern-Brocot sequence in the successive iteration with a particular denominator can be defined as

i) if $m = 1$

$$D_m[A_p] = \bigcup_{i=0}^{m-1} \bigcup_{k=0}^{n-1} \left\{\frac{mk+i}{m}\right\}, n = 1,2,3 \dots$$

ii) if $2 \le m \le 50$

$$D_m[A_p] = \bigcup_{i=0}^{m-1} \bigcup_{k=0}^{n-1} \left\{\frac{mk+i}{m}\right\}, n = 1,2,3 \dots$$

whereD_m denotes denominator, A_p denotes iteration of the Stern-Brocot sequence for D_m

Proof:

The generalised denominator pattern can be defined as

If $m = 1$ then

$$D_m[A_p] = \bigcup_{i=0}^{m-1} \bigcup_{k=0}^{n-1} \left\{\frac{mk+i}{m}\right\}, n = 1,2,3 \dots$$

Where $'m'$stands as denominator in the A_p iteration and p represents the index of the iteration.

If $m = 2,3, \dots 50$ then

$$D_m[A_p] = \bigcup_{i=0}^{m-1} \bigcup_{k=0}^{n-1} \left\{\frac{mk+i}{m}\right\}, n = 1,2,3 \dots$$

Where $'m'$stands as corresponding denominator in the A_p iteration and p represents the index of the iteration.

Define $p = n + r$ where r is a positive integer varies for each denominator terms. Also for each and every denominator the starting value of $'n'$ in the union should be 1 and for its succeeding denominator iterations the next consecutive values of $'n'$ will be carried out.

The general formula may be separated into four cases

Case:1

For all values of denominator,the value of i in the union is $i \neq 1$ $when$ $k = n - (m - r)$ to $(n - 1)$where$m$is a denominator and r is a fixed value varies for each and every denominator and $n = 1,2,3 \dots\dots$

Case:2

If the denominator value is an odd number or prime number, $i \neq 2, M$ where M stands for median values of the corresponding denominator.

Case:3

For all other not equal values of $'i'$ in the union except the values of Case 1 and Case 2

Clearly those values must be factors of $(m+1)$ or $(m-1)$ where m is a denominator of the particular iterations. In some cases, if it is not a factor then its product of pairs or unique number will satisfies any one the formula

$$[(constant)\times m-1]\ or\ [(constant)\times m+1]\ or\ [constant\times m]$$

Case:4The other not equal values of $'i'$ except from above cases then $'i'$ is not equal to its conjugate.

Illustration:3

By the above theorem denominators $D_m(A_p)$is obtained from $m=1\ to\ 50$ by the following procedure

Solution:

The formula for generation of the denominator of the Stern-Brocot Sequence is

Case:1 Denominator $m=10$(even)

$D_{10}(A_p)=D_{10}(A_{n+6})=\bigcup_{i=1}^{D-1}\bigcup_{k=0}^{n-1}\left\{\frac{10k+i}{10}\right\}$when$k=(n-4)\ to\ (n-1), i$ is not equal to 1and $n=1,2,3$ .....

Case:2 Denominator$m=13$(prime)

$$D_{13}(A_p)=D_{13}(A_{n+6})=\bigcup_{i=1}^{D-1}\bigcup_{k=0}^{n-1}\left\{\frac{13k+i}{13}\right\}, n=1,2,3\ldots..$$

where $i\neq 1$ when $k=n-7\ to\ n-1$.

$i\neq 2, M$when$k=\ n-2\ to\ n-1$.

$i\neq 3,4$when$k=\ n-1$.

Case:3 Denominator $m=13$(odd)

$$D_{21}(A_p)=D_{21}(A_{n+7})=\bigcup_{i=1}^{D-1}\bigcup_{k=0}^{n-1}\left\{\frac{21k+i}{21}\right\}, n=1,2,3\ldots..$$

where $i\neq 1$ when $k=n-14\ to\ n-1$.

$i\neq 2, M$when$k=\ n-5\ to\ n-1$.

$i\neq 4,5$when$k=n-2\ to\ n-1$.

Theorem:4

The Stern-Brocot sequence of elements generated using median property can be classified through numerators. The formula exhibited here displays the sequence of points in the Stern-Brocot sequence in the successive iteration with a particular numerators can be defined as

i) if $a=1, N_a[B_q]=\bigcup_{i=0}^{a-1}\bigcup_{k=1}^{n}\left\{\frac{a}{ak+i}\right\}, n=1,2,3\ldots$

ii) if$a=2,3,\ldots 50$ then $N_a[B_q]=\bigcup_{i=1}^{a-1}\bigcup_{k=1}^{n}\left\{\frac{a}{ak+i}\right\}, n=1,2,3\ldots$where N_a denotes numerator, B_q denotes iteration of the Stern-Brocot sequence for N_a,M denotes the Mediant..

Proof:

The generalised numerator pattern can be defined as

If $a=1$then $N_a[B_q]=\bigcup_{i=0}^{a-1}\bigcup_{k=1}^{n}\left\{\frac{a}{ak+i}\right\}, n=1,2,3\ldots$where $'a'$stands as numerator in the B_q iteration and q represents the index of the iteration.

If $a=2,3,\ldots 50$ then $N_a[B_q]=\bigcup_{i=1}^{a-1}\bigcup_{k=1}^{n}\left\{\frac{a}{ak+i}\right\}, n=1,2,3\ldots$ where $'a'$stands as corresponding numerator in the B_q iteration and q represents the index of the iteration.

Define $q=n+u$ where u is a positive integer varies for each numerator terms from. Also for each and every numerator the starting value of $'n'$ in the union should be 1 and for its succeeding numerator iterations the next consecutive values of $'n'$ will be carried out.

The general formula was separated into four cases

Case:1

For all values of numerator,the value of i in the union is

$i\neq 1\ when\ k=n-(a-u)\ to\ (n-1)$where$a$is a numerator, u is a value varies for each and every numerator and $n=1,2,3$ ....

Case:2

If the numerator value is an odd number or prime number,$i\neq 2, M$ where M stands for median values of the corresponding numerator.

Case:3

For all other not equal values of $'i'$ in the union except the values of Case 1 and Case 2

Clearly those values must be factors of $(a+1)$ or $(a-1)$ where m is a numerator of the particular iterations. In some cases, if it is not a factor then its product of pairs or unique number will satisfies any one the formula

$$[(constant)\times a-1]\ or\ [(constant)\times a+1]\ or\ [constant\times a]$$

Case:4

The other not equal values of $'i'$ except from above cases then $'i'$ is not equal to its conjugate.

Illustration:4

Bythe above theorem Numerators $N_a(B_q)$is obtained from $m=1\ to\ 50$ by the following procedure

Solution:
The formula for generation of the denominator of the Stern-Brocot Sequence is
Case:1 Numerator$a$ =10(even)

$$N_{10}(B_q) = N_{10}(B_{n+7}) = \bigcup_{i=1}^{a-1} \bigcup_{k=1}^{n} \left\{\frac{10}{10k+i}\right\}, n = 1,2,3 \ldots\ldots \text{ when} k = (n-3) \text{ } to \text{ } n, i \neq 1$$

Case:2 Numerator a =19(Prime)

$$N_{19}(B_q) = N_{19}(B_{n+8}) = \bigcup_{i=1}^{a-1} \bigcup_{k=1}^{n} \left\{\frac{19}{19k+i}\right\}, n = 1,2,3 \ldots.$$

where $i \neq 1$ when $k = n - 11 \text{ } to \text{ } n.$
$i \neq 2, M$when$k = n - 3 \text{ } to \text{ } n.$
$i \neq 3,6$when$k = n - 1 \text{ } to \text{ } n.$
$i \neq 4,5$when$k = n.$

Case:3 Numerator a =21 (odd)

$$N_{21}(B_p) = N_{21}(A_{n+7}) = \bigcup_{i=1}^{a-1} \bigcup_{k=1}^{n} \left\{\frac{21}{21k+i}\right\}, n = 1,2,3 \ldots..$$

where $i \neq 1$ when $k = n - 13 \text{ } to \text{ } n.$
$i \neq 2, M$when$k = n - 4 \text{ } to \text{ } n.$
$i \neq 4,5$when$k = n - 1 \text{ } to \text{ } n.$

Theorem 5:

Every element $\frac{p}{q}$ in the Stern-Brocot Sequence its positional representation in the sequence and in the corresponding Stern-BrocotMatrix are given respectively by

$$PS_{n+k} = 2^k * (PS_n) - \sum_{i=0}^{k} 2^i$$

$$P[MS_n] = PS_n + \left[\frac{m(m+1)}{2} - 1\right]$$

Proof:

Let $\frac{p}{q}$ be any element in the Stern-Brocot sequence and PS_n denotes the current position of an element in the sequence.By theorem 3 the generalised denominator pattern verifies the existence of the element $\frac{p}{q}$ in S_n.

Case:1

If the index value of $D(A_p)$ is equal to index of S_n, $\frac{p}{q}$ represents origin position of the sequence S_n .The positional value of $\frac{p}{q}$ in the sequence is

$$\boldsymbol{PS_n = No\ of\ preceeding\ fraction\ of\ \frac{p}{q}\ in\ the\ origin\ sequence} + 1$$

It follows the condition that, preceding numerator and succeeding numerator additional value must be equal to $'p'$ and preceding denominator and denominator additional value must be equal to $'q'$.

Case:2

If the index value of $D(A_p)$ is less than the index of S_n, $\frac{p}{q}$ represents already existing position of the sequence S_n, ie., $p < n$.The positional value of $\frac{p}{q}$ in the sequence is

$\boldsymbol{PS_{n+k} = 2^k * (PS_n) - \sum_{i=0}^{k} 2^i}$where$n$ is the position of a term chosen and $n+k$ is the position of the term in a required sequence.

Matrix Position:

If the positional value of $\boldsymbol{S_n}$is equal or less than n the position of $\boldsymbol{MS_n}$remains same with the position of $\boldsymbol{S_n}$,say $S_1 = a_{11}$,$S_2 = a_{12}$,$S_3 = a_{13}$,……$S_n = a_{1n}$.If the positional value of $\boldsymbol{S_n}$is greater than n and less than $2n$,$P[MS_n] = PS_n + 2$.If the positional value of $\boldsymbol{S_n}$is greater than $2n$ and less than $3n$,$P[MS_n] = PS_n + (2+3)$.In general for mn ,if the positional value of $\boldsymbol{S_n}$is greater than mn and less than $(m-1)n$ then the $\boldsymbol{MS_n}$becomes

$$P[MS_n] = PS_n + (2 + 3 + 4 + 5 \ldots. +\text{m})$$
$$P[MS_n] = PS_n + (1 + 2 + 3 + 4 + 5 \ldots. +\text{m} - 1)$$
$$P[MS_n] = PS_n + \left[\frac{m(m+1)}{2} - 1\right]$$

3.2Position of an element in Farey sequence

Represent the denominators of Farey sequence in terms of order pairs write $\left\{\frac{0}{1},\frac{1}{1}\right\}$ as $(1,1)$, say F_1.Define$(a,b) = (b,a)$ and omit the repetitions of order pairs. if (a,b)exist then omit (b,a)here in these order pairs (a,b)whose sum is equal to or less than the index value of corresponding F_n the order pairs was represented as $(a,b) = (a,a+b)(a+b,b)$.Otherwise order pairs remains same as (a,b).

The Farey denominator ordered pairs was defined as

$F_1 = (1,1)$

$F_2 = (1,2)$

$F_3 = (1,3)(3,2)$

$F_4 = (1,4)(4,3)(3,2)$

To find the origin position of any fraction $\left(\frac{s}{t}\right)$in Farey sequence

Origin position of any fraction $\left(\frac{s}{t}\right)$ = Number of preceding order pairs of denominators of $\left(\frac{s}{t}\right)$

Follows the condition that,Preceding Numerator and succeeding Numerator additional value should be equal to s and Preceding Denominator and succeeding Denominator additional value should be equal to t.

To find the corresponding position of consecutive fractions $\left(\frac{s}{t}\right)$

The formula to find the corresponding elements position is

Current position of Farey fraction$\left(\frac{s}{t}\right)$ = The position of Farey fraction $\left(\frac{s}{t}\right)$ in previous iteration
+
Number of new pairs whose proceeding denominators
Whose sum is equal to n in $(n-1)$th iteration

3.3 Definition: RootFraction

If a fraction of a Farey sequence whose denominator value is equal to index value of the Farey sequence then the corresponding fraction is said to be root fraction of a Farey sequence.

3.4 Definition:Existing fraction

If a fraction of a Farey sequence which is not an root fraction then it is said to be existing fraction

Theorem 6:

Every element $\frac{s}{t}$of a Farey sequence the positional representation in the corresponding Farey Matrix are given respectively by

$$P[MF_n] = PF_n + \left[\frac{m(m+1)}{2} - 1\right]$$

Proof:

Let $\frac{s}{t}$ be any element in the Farey sequence and PF_n denotes the current position of an element in the sequence.By the definition 3.3 and 3.4 verify $\frac{s}{t}$is origin fraction or existing fractions of Farey sequence

Then matrix position is

$$P[MF_n] = PF_n + \left[\frac{m(m+1)}{2} - 1\right]$$

Where MF_n a minimum number order of square matrix $(n \times n)$ whose total number of elements is always equal to or greater than the total number of elements of Stern-Brocot sequence.If the positional value of $S_n$is equal to or less than n the position of MS_nremains same with the position of S_n. ie., $S_1 = a_{11}$,$S_2 = a_{12}$,$S_3 = a_{13,\dots\dots}S_n = a_{1n}$

If the positional value of S_n is greater than n and less than $2n$ then the MS_nbecomes

$$P[MF_n] = PF_n + 2$$

If the positional value of $\boldsymbol{S_n}$is greater than $2n$ and less than $3n$ then the $\boldsymbol{MS_n}$becomes

$$P[MF_n] = PF_n + (2+3)$$

In general for mn ,if the positional value of $S_n$is greater than mn and less than $(m-1)n$ then the MF_nbecomes

$$P[MF_n] = PF_n + (2+3+4+5\ldots.+\text{m})$$
$$P[MF_n] = PF_n + (1+2+3+4+5\ldots.+\text{m}-1)$$

$$P[MF_n] = PF_n + \left[\frac{m(m+1)}{2} - 1\right]$$

Illustration:5

To find the position value of the fraction $\frac{2}{7}$ in Stern Brocot sequence and Stern-Brocot Matrix of level $n = 10$.

Solution:

Let $\frac{p}{q} = \frac{2}{7}$ is$D_7(A_{n+5}) = \cup_{i=0}^{6} \cup_{k=0}^{n-1} \left\{\frac{7k+i}{7}\right\}, n = 1,2,3 \ldots..$

put$n = 1$,$D_7(A_6) = \left\{\frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}\right\}$

By making use of denominator formula we found that $\frac{2}{7}$ is in $D_7(A_6)$ provided whose proceeding Numerator and Successive Numerator additional value should be equal to $p = 2$

in the Stern – BrocotSequence.Other wise move to the next $D_7(A_{n+5})$ iteration with the same procedure,here$\frac{p}{q} = \frac{2}{7}$ is an origin fraction in $D_7(A_6)$.Consider the half of the portion of the Stern-Brocot sequence from level 1 as S_1=$\left\{\frac{0}{1}, \frac{1}{1}\right\}$.Here $\frac{p}{q} = \frac{2}{7}$ is an origin and 4th position in S_5.

Clearly, $\frac{p}{q} = \frac{2}{7}$ is an existing fraction inS_{10}

$PS_{n+k} = 2^k PS_n - \sum_{i=0}^{k-1} 2^i$,$PS_{10}$ =97

In the Stern-Brocot sequence of 10th iteration the positional value of $\left(\frac{2}{7}\right)$ is in 97th place.

To find the positional value of $\left(\frac{2}{7}\right)$ in stern–brocot matrix $(n = 10)$

Consider the formula for Matrix

$$P[MS_n] = S_n + \left[\frac{m(m+1)}{2} - 1\right]$$

The position value is111th is$a_{5,19}$ in Stern-Brocot Matrix.

$\frac{2}{7}$is in $a_{5,19}$ position in Stern –Brocot Matrix of $n = 10$.

Illustraction:6

To find the positional value of the fraction $\frac{3}{8}$ in Farey sequence Farey Matrix of $n = 10$.

Solution:

To check whether the fraction $\frac{3}{8}$is an root fraction or Existing fraction.

Since Denominator of the fraction $\frac{3}{8}$is less than the index value of the Farey sequence $n = 10$ $ie.$, $[8 < 10]$. Clearly, $\frac{3}{8}$ is an Existing fraction and the position value of the fraction $\frac{3}{8}$is given by 13.

To find the positional value of $\frac{3}{8}$ in the Farey Matrix

$$MF_n = 13 + \left[\frac{m(m+1)}{2} - 1\right] = 18$$

Therefore the positional value of the fraction $\frac{3}{8}$ in the Farey matrix is$a_{3,6}$.

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