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POROELASTICITY AND ITS APPLICATION IN BONE ELEMENTS

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Abstract: Poroelasticity is a well-developed theory for the interaction of fluid and solid states of a fluid-saturated porous medium. The theory originally was developed for soil mechanics especially for consolidation problems. In the theory of biomechanics in which the various bone elements of living being are considered as fluid-filled porous elastic materials. The application of poroelasticity to bone differs from its application to soft tissues in two important ways. First, the deformations of bone are small while those of soft tissues are generally large. Second, the bulk modulus of the mineralized bone matrix is about six times stiffer than that of the fluid in the pores while the bulk moduli of the soft tissue matrix and the pore water are almost the same. In these works, it was suggested that the theory of poroelasticity should be applied to the stress and strain analysis of bones.
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1. Introduction

The theory of wave propagation in a porous elastic solid, saturated with a viscous compressible fluid was established by Biot [1]. The theory was, however, restricted to the low frequency range only where the effect of dissipation due to the relative motion of the fluid in the pores with respect to the solid frame was included. Subsequently, the theory was also generalized by the author [2] to the wider frequency range. On the basis of this theory, Biot predicted the existence of three types of attenuated body waves – two dilatation ($P_I -, P_{II} -$ wave) and one rotational wave (S - wave) in porous media. If the dissipation be ignored, these waves show unattenuated behaviour.

It is worthwhile to note that the viscous forces predominate over the inertia forces in the low frequency range while in the high frequency range, the inertia forces predominate over the viscous ones. This transition between the two frequency ranges is governed by a characteristic frequency. Another transition frequency lower than the characteristic frequency exists below which the flow in the pores is of Poiseuille type and above which Poiseuille flow breaks down and the viscosity must be regarded as a complex function of non-dimensional frequency parameter. In particular, if the frequency parameter is made tend to infinity, the viscosity tends to zero and consequently the dynamical equations for the non-dissipative porous case are obtained. On the other hand, when the frequency parameter tends to zero, the characteristic pore size tends to zero and as a consequence the dynamical equations for the classical case result.

Biot's dynamical theory of elasticity of fluid- filled porous elastic solid is extensively used in studying the seismological problems such as propagation of body and surface waves in crustal layers of the earth, fluid-filled rocks, ocean bottom sediments etc. Besides seismology, the theory finds applications in the field of biomechanics in which the various bone elements of living being are considered as fluid-filled porous elastic materials. It is also applicable in the fields of geophysics, acoustic engineering, foundation engineering and so on.

The theory of dynamical poroelasticity was extensively applied by Deresiewicz and his co-workers [3-14] to study the effect of boundaries on the wave propagation in liquid-filled porous elastic solid. Paul [15] and Jones [16] studied the propagation of Love and Rayleigh waves in poroelastic half-space while Gardener [17] discussed the extensional vibration of a fluid-saturated porous cylinder. Paria [18] and Chakraborty [19, 20] considered some axi-symmetric dynamic response of half-space and spherical cavity. Paul [21, 22] studied the disturbance produced in a poroelastic half-space for different types of stationary or moving surface loads. Tajuddin and his co-workers [23-

37] discussed various specific problems in this field. Sanyal and Basu Mallik [38 – 40] also gave some contributions in porous media.

Now bony materials which are the subject of the present investigation were discussed by Nowinski and Davis [41-43]. In these works, it was suggested that the theory of poroelasticity should be applied to the stress and strain analysis of bones. To give an illustration, the compressive strength of bones in a dry state is about 50% higher than that of the wet bones, with the ratio 1:2 for the percentage of elongation of dry and wet samples respectively [44]. Likewise, the removal of moisture completely suppresses the plastic properties of bones.

In our investigation, we assume that the volume concentration of pores (porosity) in the bulk material is uniform so that the material may be regarded as quasi-homogeneous and quasi-isotropic. Also we consider that in the two-phase solid-fluid system, the solid skeleton is linearly and perfectly elastic and undergoing small deformations. The liquid phase is a perfect fluid and the pores are interconnected. With these assumptions, we apply Biot's theory to the propagation of longitudinal waves in bony materials having the form of long cylindrical bars of circular cross-section. To solve the problem, the four coupled differential equations are reduced to the solution of a single ordinary differential equation with variable coefficients and a regular singular point and this equation is solved by Frobenius method. External loading on the curved surface is supposed to be absent and the surface is assumed to be permeable to the flow of pore fluid.

2. General Equations

Let us consider an infinite circularly cylindrical bar of radius a of poroelastic material with its longitudinal axis coinciding with the *z*-axis of the cylindrical coordinates (r, θ, z). Then the equations governing the wave propagation in poroelastic bodies are [1]

$$N\nabla^2\mathbf{u} + \boldsymbol{\nabla}[(A + N)e + Q\epsilon] = \frac{\partial^2}{\partial t^2}(\rho_{11}\,\boldsymbol{u} + \rho_{12}\,\mathbf{U})$$

$$\boldsymbol{\nabla}(Qe + R\epsilon) = \frac{\partial^2}{\partial t^2}(\rho_{12}\,\boldsymbol{u} + \rho_{22}\,\mathbf{U})$$

(1)

Where e and ϵ are the dilatations of the solid and liquid phases respectively,

$$e = .\boldsymbol{u}\,,\ \epsilon = \boldsymbol{\nabla}.\mathbf{U}$$

(2)

with $\mathbf{u}$ and $\mathbf{U}$ as the displacement vectors. Theρ's are the mass coefficients s.t. the sums $\rho_{11} + \rho_{12}$ and $\rho_{12} + \rho_{22}$ represent the mass of solid and mass of fluid per unit volume of the bulk material respectively. A, N, Q, R are the material coefficients.

In the axisymmetric case, equations (1) become

$$(A + 2N)\frac{\partial e}{\partial r} + 2N\frac{\partial w_\theta}{\partial z} + Q\frac{\partial \epsilon}{\partial r} = \rho_{11}\frac{\partial^2 u_r}{\partial t^2} + \rho_{12}\frac{\partial^2 U_r}{\partial t^2},$$

$$(A + 2N)\frac{\partial e}{\partial z} - \frac{2N}{r}\frac{\partial}{\partial r}(rw_\theta) + Q\frac{\partial \epsilon}{\partial z} = \rho_{11}\frac{\partial^2 u_z}{\partial t^2} + \rho_{12}\frac{\partial^2 U_z}{\partial t^2},$$

(3a)

$$Q\frac{\partial e}{\partial r} + R\frac{\partial \epsilon}{\partial r} = \rho_{12}\frac{\partial^2 u_r}{\partial t^2} + \rho_{22}\frac{\partial^2 U_r}{\partial t^2},$$

$$Q\frac{\partial e}{\partial z} + R\frac{\partial \epsilon}{\partial z} = \rho_{12}\frac{\partial^2 u_z}{\partial t^2} + \rho_{22}\frac{\partial^2 U_z}{\partial t^2}$$

(3b)

with

$$e = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\,,\quad \epsilon = \frac{1}{r}\frac{\partial}{\partial r}(rU_r) + \frac{\partial U_z}{\partial z},\quad w_\theta = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right).$$

(3c)

The stress-strain relations are

$$\sigma_{rr} = 2Ne_{rr} + Ae + Q\epsilon\,,$$

$$\sigma_{\theta\theta} = 2Ne_{\theta\theta} + Ae + Q\epsilon\,,$$

(4)

$$\sigma_{rz} = 2\mathrm{N}e_{rz} ,$$
$$s = Qe + R\epsilon$$

where s is a quantity proportional to the fluid pressure.

Let us now assume that a train of sinusolidal wave propagates along the *z*-axis of the cylinder so that

$$[u_r , u_z , U_r , U_z ,] = [R_1(r),\ R_2(r), R_3(r), R_4(r)] \exp \{i(\gamma z + pt)\}$$
(5)

Where p is the wave frequency, $\frac{p}{\gamma}$ is the phase velocity and R_i(r), (I = 1, 2, 3, 4) are functions of r to be determined.

Keeping (5) in mind equations (3) can be written as

$$(A + 2N)\frac{\partial e}{\partial r} + 2N\, i\gamma w_\theta + Q\frac{\partial \epsilon}{\partial r} = -\rho_{11}\, p^2 u_r - \rho_{12}\, p^2 U_r,$$
$$(A + 2N)\, i\gamma e - \frac{2N}{r}\frac{\partial}{\partial r}(r w_\theta) + Q\, i\gamma\epsilon = -\rho_{11}\, p^2 u_z - \rho_{12}\, p^2 U_z, \quad (6a)$$
$$Q\frac{\partial e}{\partial r} + R\frac{\partial \epsilon}{\partial r} = -\rho_{12}\, p^2 u_r - \rho_{22}\, p^2 U_r,$$
$$Q\frac{\partial e}{\partial z} + R\frac{\partial \epsilon}{\partial z} = -\rho_{12}\, p^2 u_z - \rho_{22}\, p^2 U_z$$
(6b)

A longer manipulation of the above equations leads to

$$\frac{\partial^2 e}{\partial r^2} + \frac{1}{r}\frac{\partial e}{\partial r} + (\beta_1 p^2 - \gamma^2)e + \beta_2 p^2 \epsilon = 0$$
$$\frac{\partial^2 \epsilon}{\partial r^2} + \frac{1}{r}\frac{\partial \epsilon}{\partial r} + (\beta_1 p^2 - \gamma^2)\epsilon + \beta_4 p^2 e = 0$$
(7)

where

$$\beta_1 = \frac{Q\rho_{12} - R\rho_{11}}{\alpha},\ \beta_2 = \frac{Q\rho_{22} - R\rho_{12}}{\alpha},$$
$$\beta_3 = \frac{(A+2N)\rho_{22} - Q\rho_{12}}{-\alpha},\ \ \beta_4 = \frac{(A+2N)\rho_{12} - Q\rho_{11}}{-\alpha}$$
(8)

$$\alpha = Q^2 - R(A + 2N).$$

Employing the expressions (5), we obtain from (7)

$$S_1'' + \frac{1}{r} S_1' + \gamma_1 S_1 + \beta_2 p^2 S_2 = 0,$$
(9a)

$$S_2'' + \frac{1}{r} S_2' + \gamma_2 S_2 + \beta_4 p^2 S_1 = 0,$$
(9b)

where

$$S_1 = R_1' + \frac{1}{r} R_1 + i\gamma R_2 ,$$
$$S_2 = R_3' + \frac{1}{r} R_3 + i\gamma R_4$$
(10)

$$\gamma_1 = \beta_1 p^2 - \gamma^2,\ \ \gamma_2 = \beta_3\, p^2 - \gamma^2.$$

Noting from (9a) that

$$S_2 = -\frac{1}{\beta_2 p^2}\left[S_1'' + \frac{1}{r} S_1' + \gamma_1 S_1 \right]$$
(11)

we have from (9b)

$$r^3 S_1^{iv} + 2r^2 S_1''' + [(\gamma_1 + \gamma_2)r^3 - r]S_1'' + [(\gamma_1 + \gamma_2)r^2 + 1]S_1' + (\gamma_1\gamma_2 - \beta_2\beta_4 p^2)r^3 S_1 = 0$$
(12)

Since all the coefficients of the basic equation are finite single-valued and continuous throughout the domain of definition $0 \le r \le a$, the only singular points which may occur within that interval are the zeros of the leading coefficients. For the present discussion, we write equation (12) in a symbolic and symmetric form

$$r^4 \varphi^{iv} + r^3 P_1(r)\, \varphi''' + r^2 P_2(r)\, \varphi'' + r\, P_3(r)\, \varphi' + P_4(r)\, \varphi = 0$$
(13)

Where $\varphi = \varphi(r)$ represents the function S_1. Clearly, $P_1(r)$, $P_2(r)$, ... are analytic in the neighbourhood of the origin r = 0 since these are algebraic expressions.

Now $r^{-s} P_s(r)$, (s = 1, 2, 3, 4), are of order r^{-s} as r $\to$ 0, since $P_s(r)$ remain bounded throughout this process. But this is the necessary and sufficient condition in order that r = 0 be a regular singular point. Hence equation (12) possesses a fundamental set of solutions regular at the origin and it is possible to obtain at least some of the solutions in the form of power series by using Frobenius method.

The point r = 0 being regular the associated indical equation must be a polynomial of degree n = 4 in the exponent of the power solution. Some of these exponents (roots) may be equal or differ by an integer so that the number of particular solutions of the type considered may fall short of n = 4. However, by differentiating equation (9), we have raised the order of the equation and this has introduced additional solutions which are not necessary solutions of the original problem. In fact, we will see later that to satisfy the boundary conditions of the problem, it is sufficient to know one particular solution of equation (12).

3. Solutions of Problem

For the solution of (12) we take

$$S_1(r) = \sum_{k=0}^{\infty} a_k r^{k+\delta}, \qquad (a_0 \neq 0),$$
(14)

Where the parameter δ is determined from the indicial equation

$$\delta^2(\delta-2)^2 = 0.$$
(15)

With double roots δ_1 = 0, δ_2 = 2. Since the roots differ by an integers, they provide a single independent solution, say

$$S_1(r) = \sum_{k=0}^{\infty} a_k r^k.$$
(16)

In order that the function S_2(r) to be finite at r = 0, we should have $a_1 \equiv 0$ and the general recurrence formula becomes

$$a_n = \frac{-a_{n-2}[2(\gamma_1+\gamma_2)(n-2)] - a_{n-4}(\gamma_1\gamma_2 - \beta_2\beta_4 p^4)}{[3n(n-1)(n-2) - n(n-1) + n]},$$
(17)

with n = 2, 4, 6, ..., all the coefficients with an odd subscript vanishing. It may be noted from (17) that a_0, a_2 remain undetermined. These have to be determined from the boundary conditions on the curved surface of the bar.

Let us first express the solid and liquid phase dilatation in terms of the functions S_1 and S_2 as

$$(e, \epsilon) = (S_1, S_2) exp\{i(\gamma z + pt)\}$$
(18)

Inserting these into the equations of (6a) we obtain after some rearrangement

$$\frac{\partial E}{\partial r} = \rho_1^* u_r - Ni\gamma \frac{\partial u_z}{\partial r},$$

$$i\gamma E = \rho_2^* u_z - \frac{Ni\gamma}{r} \frac{\partial}{\partial r}(r u_r) + \frac{N}{r}\frac{\partial u_r}{\partial r} + N\frac{\partial^2 u_z}{\partial r^2}$$
(19a)

where

$$E = -\left[\left(P - \frac{\rho_{11}}{\rho_{12}} Q\right) e + \left(Q - \frac{\rho_{12}}{\rho_{22}} R\right)\epsilon\right],$$

$$\rho_1^* = \rho_{11}p^2 - \frac{\rho_{12}^2}{\rho_{22}} p^2 - N\gamma^2,$$
(19b)

$$\rho_2^* = \rho_{11}p^2 - \frac{\rho_{12}^2}{\rho_{22}} p^2, \ p = A + 2N.$$

Eliminating u_r from equation (19a) and ordering we have

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{{\rho_2}^*}{\frac{N^2\gamma^2}{{\rho_1}^*}+N} u_z = \frac{i\gamma E}{\frac{N^2\gamma^2}{{\rho_1}^*}+N} + \frac{i\gamma}{N\gamma^2+{\rho_1}^*}\frac{1}{r}\frac{\partial E}{\partial r} + \frac{i\gamma}{N\gamma^2+{\rho_1}^*}\frac{\partial^2 E}{\partial r^2}$$

(20)

So that by means of the transformation

$$E^* = -\left[\left(P - \frac{\rho_{12}}{\rho_{22}} Q\right)S_1 + \left(Q - \frac{\rho_{12}}{\rho_{22}} R\right)S_2\right]$$

(21)

Equation (20) is reduced to

$$r^2\frac{d^2R_2}{dr^2} + r\frac{dR_2}{dr} + \lambda^2 r^2 R_2 = G(r)$$

(22a)

with

$$G(r) = b_1 r^2 E^* + b_2 r\frac{dE^*}{dr} + b_2 r^2\frac{d^2E^*}{dr^2}$$

(22b)

and

$$\lambda^2 = \frac{{\rho_2}^*}{\frac{N^2\gamma^2}{{\rho_1}^*}+N},\ b_1 = \frac{i\gamma}{\frac{N^2\gamma^2}{{\rho_1}^*}+N},\ b_1 = \frac{i\gamma}{N\gamma^2+{\rho_1}^*}$$

(22c)

Introducing the new variable x = λr, equation (22a) becomes

$$x^2{R_2}'' + x{R_2}' + x^2R_2 = G(x)$$

(23)

where prime denotes differentiation w. r. t. x

The C. F. of the solution of (23) is

$$R_2 = C_1 J_o(x) + C_2 Y_o(x)$$

(24)

P. I. of the nonhomogeneous equation (23) can be easily obtained by the method of variation of parameter as

$$R_{2\text{part}} = -\frac{\pi}{2}J_o(x)\int xY_o(x)\,G(x)dx + \frac{\pi}{2}Y_o(x)\int xJ_o(x)\,G(x)dx\,.$$

Thus by retransformation to the variable, the complete solution of (23) in its final form is

$$R_2 = C_1J_o(\lambda r) + C_2Y_o(\lambda r) - \frac{\pi}{2}J_o(\lambda r)\,\lambda^2\int rY_o(\lambda r)G(r)dr + \frac{\pi}{2}Y_o(\lambda r)\,\lambda^2\int rJ_o(\lambda r)G(r)dr.$$

(25)

In order to guarantee the boundedness of R_2(r) at the point r = 0, we investigate the behavior of this function at r = 0. Confining ourselves to the terms involving the coefficients a_0 and a_2, we obtain from (22b)

$$G(r) = r^2\left[b_1\left\{q_1(a_0 + a_2 r^2) - \frac{q_2}{\beta_2 p^2}(\gamma_1 a_0 + 4a_2 + \gamma_1 a_2 r^2)\right\} + 4b_2\left(q_1 - \frac{q_2\gamma_1}{\beta_2 p^2}\right)a_2\right]$$ (26a)

where

$$q_1 = -P + \frac{\rho_{12}}{\rho_{22}}Q,\ q_2 = -Q + \frac{\rho_{12}}{\rho_{22}}R.$$

(26b)

It is seen that in the interval $0 \le r \le a$. G(r) remains bounded with its upper bound, say M. On the other hand

$$\int r\,Y_o(\lambda r)dr = \frac{r}{\lambda}Y_1(\lambda r),\ \int r\,J_o(\lambda r)dr = \frac{r}{\lambda}J_1(\lambda r)$$

So that in the neighbourhood of the point r = 0 the integrals in (25) behave like $r^2\left(\gamma + ln\frac{\lambda r}{2}\right)$, (dropping the bounded factors), and r^2 respectively ($\gamma =$

Euler's constant). It follows that the third and fourth terms in (25) behave like $r^2\left(\gamma + ln\frac{\lambda r}{2}\right)$ and consequently tend to zero as r → 0.

Since $Y_o(\lambda r)$ is unbounded at r = 0, so the boundedness of $R_2(0)$ requires that $C_2 \equiv 0$. Thus we are left with three unknown integration constants a_0, a_2 and C_1 to be determined from the boundary conditions.

4. Boundary Conditions — Frequency Equation

By hypothesis, the curved surface of the bar should be free from external load and permeable for the fluid, so that

$$\sigma_{rr} = \sigma_{rz} = s = 0 \quad \text{at} \quad r = a. \tag{27}$$

Substitution of (4) into the foregoing conditions gives the following system of three equations for the three unknown coefficients a_0, a_2 and C_1

$$2NR_1{}' + AS_1 + QS_2 = 0,$$

$$R_1 i\gamma + R_2{}' = 0,$$

$$QS_1 + RS_2 = 0. \tag{28}$$

Using (19a) and then (5) we obtain from (28)

$$R_1 = \frac{1}{\rho_1{}^*} Ni\gamma \qquad R_2{}' + \frac{1}{\rho_1{}^*} E^{*\prime} \tag{29}$$

where

$$E^{*\prime} = q_1(2a_2 r + 4a_4 r^3 + \ldots\ldots) - \frac{q_2}{\beta_2 p^2}[(2\gamma_1 a_2 + 16\, a_4)r + 4(36a_6 + \gamma_1 a_4)r^3 + \ldots\ldots] \tag{30}$$

We may now cast the stress boundary conditions (28) into the form

$$2N^2 i\gamma\lambda\,[-C_1 J_1{}^1(\lambda r) + \frac{\pi}{2}\lambda^2 J_1{}'(\lambda r)\int G(r)\, Y_o(\lambda r)\, dr + \frac{\pi}{2}\lambda^2 J_1(\lambda r)\, G(r)\, Y_o(\lambda r) - \frac{\pi}{2}\lambda^2 Y_1{}'(\lambda r)\int G(r)\, J_o(\lambda r) dr - \frac{\pi}{2}\lambda^2 Y_1(\lambda r) G(r) J_o(\lambda r)] + 2NE^{*\prime\prime} + P_1{}^* QS_2(r) = 0 \text{ at } r = a,$$

$$\left(1 - \frac{N\gamma^2}{\rho_1{}^*}\right)\lambda\,[-C_1 J_1(\lambda r) + \frac{\pi}{2}\lambda^2 J_1(\lambda r)\int G(r)\, Y_o(\lambda r)\, dr - \frac{\pi}{2}\lambda^2 Y_1(\lambda r)\int G(r)\, J_o(\lambda r) dr] + \frac{i\gamma E^{*\prime}}{\rho_1{}^*} = 0 \text{ at } r = a \tag{31}$$

$$Q\, S_1(r) + Q\, S_2(r) = 0 \text{ at } r = a.$$

A nontrivial solution of this system requires the vanishing of the determinant of the system. The computations may be simplified by first solving equation $(37)_2$ w. r. t. $C_1 J_1(\lambda r)$ and then differentiating w.r.t. r and inserting into equation $(37)_1$. Then we have

$$\left[\frac{2N^2\gamma^2}{(\rho_1{}^* - N\gamma^2)} + 2N\right]E^{*\prime\prime} + Q\rho_1{}^* S_2(r) = 0 \text{ at } r = a,$$

$$Q\, S_1(r) + R\, S_2(r) = 0 \text{ at } r = a.$$

For a first approximation we retain only two terms in the series expansions involving a_0 and a_2 and rewrite above equations in a more symmetric form

$$-\frac{Q\rho_1{}^*\gamma_1}{\beta_2 p^2}a_0 + \left[\left(\frac{4N\gamma^2}{\rho_1{}^* - N\gamma^2} + 4N\right)\left(q_1 - \frac{q_2\gamma_1}{\beta_2 p^2}\right) - \frac{Q\rho_1{}^*}{\beta_2 p^2}(4 + \gamma_1 a^2)\right]a_2 = 0,$$

$$\left(Q - \frac{R\gamma_1}{\beta_2 p^2}\right)a_0 + \left(Qa^2 - \frac{4R}{\beta_2 p^2} - \frac{\gamma_1 a^2 R}{\beta_2 p^2}\right)a_2 = 0.$$

Vanishing of the determinant of this system now yields, after some transformations

$$p^4\varphi_1 - p^2\gamma^2\varphi_2 + \gamma^2\varphi_3 \tag{32}$$

Where

$$\varphi_1 = \left(\rho_{11} - \frac{\rho_{12}{}^2}{\rho_{22}}\right) Q^2\beta_2 + (\beta_1 q_2 - \beta_2 q_1)(Q\beta_2 - R\beta_1)N,$$

$$\varphi_2 = N[(Q\beta_2 - 2R\beta_1)q_2 + R\beta_2 q_1 + 2Q^2\beta_2],$$

(33)

$$\varphi_3 = -NRq_2.$$

Using the notations

$$c = \frac{p}{\gamma}, \psi_1 = \frac{\varphi_2}{\varphi_1}, \psi_3 = \frac{\varphi_3}{\varphi_1}$$

(34)

The equation (33) is converted into a simpler form

$$c^4 - \psi_1 c^2 + \psi_2 = 0.$$

(35)

From which we obtain immediately two velocities of propagation of the longitudinal waves, c_1 and c_2, where

$$c_1{}^2 = \frac{\psi_1 + \sqrt{\psi_1{}^2 - 4\psi_2}}{2}, \quad c_2{}^2 = \frac{\psi_1 - \sqrt{\psi_1{}^2 - 4\psi_2}}{2}$$

(36)

The existence of two wave velocities in the bar agrees with a similar finding of Biot for an infinite poroelastic space [1]

An inspection of the intricate representation of the symbols ψ_1 and ψ_2 in terms of the material constants A, N, Q, R by means of the relations (34), (33), (26b) and (8), indicates that even the first approximation to the wave velocities in poroelastic bodies is highly complicated. This difficulty is aggravated by the fact that the material coefficients of the bony materials are in a great measure unknown. However, we attempt a rough analysis using the meager data available and following Biot's comments on the elastic coefficients of the consolidation theory [1]. As contended in [2], the constants A, N, Q R may be represented as follows:

$$A - \frac{Q^2}{R} = \lambda, N = \mu,$$

$$Q = \frac{f\left(1-f-\frac{\zeta}{\kappa}\right)}{\xi + \zeta - \frac{\zeta^2}{\kappa}}, R = \frac{f^2}{\xi + \zeta - \frac{\zeta^2}{\kappa}} \tag{37}$$

where λ, μ are Lame constants under condition of pore pressure, f the porosity, κ and ζ the coefficients of jacketed and unjacketed compressibility respectively and ξ the coefficient of fluid content.

Noting that κ is the inverse of the bulk modulus under conditions of constant pore pressure one obtains

$$\kappa = \frac{3(1-2\sigma)}{E} \tag{38}$$

where E is Young's modulus and σ is Poisson's ratio. Taking values of these to be 3×10^6 and 0.28 [43], an approximate value of κ is

$$\kappa = 0.44 \times 10^{-6}\ in^2/lb. \tag{39}$$

Typical values of porosity of compact bone are given to range from 8 – 22 percent [43]. We take the value of 14 percent.

It can be shown that [43] $f \leq 1 - \frac{\zeta}{\kappa} \leq 1$

so that in the present case $0 \leq \zeta \leq 0.80\kappa$ and using (39), we have

$$\zeta \approx 0.22 \times 10^{-6}\ in^2 / \text{lb}.$$

(40)

We also suppose the fluid to be incompressible for which $\xi = -f\zeta$ so that using the present values

$$\xi \approx -0.0308 \times 10^{-6}\ in^2 / \text{lb}.$$

(41)

with these values, the coefficients A, N, Q, R can be calculated.

Now the only additional numerical values required are for the mass parameters. As a reasonable approximation, we take the mass coupling parameter ρ_{12} to be equal to zero. The total mass of both solid and fluid constituent per unit volume of bulk material is then $\rho = \rho_{12} + \rho_{22}$. Assuming the fluid mass density to be that of water and a value of ρ to be [43] $\rho = 3.7\ slug/ft^3$, we obtain finally

$$\rho_{11} = 1.65 \times 10^{-4}\ \text{lb}\ sec^2 / in^4,\ \rho_{22} = 0.14 \times 10^{-4}\ \text{lb}\ sec^2 / in^4. \qquad (42)$$

Substitution of all the foregoing approximate values into expressions (36) yields two wave velocities in the bony material equal to

$$c_1 = 1.40 \times 10^{-5}\text{ips},\ c_2 = 0.97 \times 10^5\text{ips}.$$

It is interesting to note the two velocities so calculated bracket the velocities found by McElhaney [45] on the basis of his experimental work. In fact, McElhaney's values are 1.21 × 10^5 ips for fresh bovine and the embalmed human femur bone, respectively.

5. Concluding Remarks

As mentioned earlier, the entire set of the material coefficients of bony material is in a great measure unknown and the foregoing numerical evalution may only be considered as a rough estimate. It seems that the next step in examination of the applicability of the theory requires extensive experimental work which could provide more reliable data.

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