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ON HOPF LIGHTLIKE HYPER SURFACES OF INDEFINITE COSYMPLETIC MANIFOLD

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Abstract

The object of present paper is to study the properties of Hopf lightlike hypersurfaces of indefinite cosympletic manifold.

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1. Introduction

lightlike geometry have been studied by K. L. Duggal and A. Bejancu [1, 2] and D. N. Kupeli [3] with different approaches. Recently, many geometers investigated lightlike hypersurfaces by using the fundamental knowledge introduced by Duggal-Bejancu with various geometric conditions and obtained many important results. For example, D. H. Jin [4] proved a classification theorem of lightlike hypersurface M with totally umbilical screen distribution of a semi-Riemannian space form. C. Atindogbe and K. L. Duggal [5] introduced screen conformal lightlike hypersurface and proved that its induced Ricci curvature tensor is symmetric. Two monographs by Duggal-Jin [6] and Duggal-Sahin [7] contain a collection of many interesting results on lightlike hypersurfaces, and have, further, motivated other scholars to take an active role in the study of lightlike geometry. For instance see [8–11].

The object of present paper is to study the properties of Hopf lightlike hypersurfaces of indefinite cosympletic manifold.

2. Lightlike hypersurfaces

Let (M, g) be a lightlike hypersurface of $\overline{M}$. The normal bundle

T$M^{\perp}$ of M is a subbundle of the tangent bundle TM of M, of rank 1, and coincides with the radical distribution Rad(TM) = TM $\cap$ T$M^{\perp}$. Denote by F (M) the algebra of smooth functions on M and by T(E) the F (M) module of smooth sections of any vector bundle E over M.

A complementary vector bundle S(TM) of Rad(TM) in TM is non-degenerate distribution on M, which is called a screen distribution on M, such that

TM = Rad(TM) $\oplus_{orth}$ S(TM),

where $\oplus_{orth}$ denotes the orthogonal direct sum. For any null section ξ of Rad(TM), there exists a unique null section N of a unique lightlike vector bundle tr(TM) in the orthogonal complement S$(TM)^{\perp}$ of S(TM) satisfying

$\bar{g}$ (ξ, N) = 1, $\bar{g}$ (N; N) = $\bar{g}$ (N; X) = 0; $\forall$ 8 X $\in$ T(S(TM)):

We call tr(TM) and N the transversal vector bundle and the null transversal vector field of M with respect to the screen distribution S(TM), respectively.

The tangent bundle T $\bar{M}$ of $\bar{M}$ is decomposed as follow:

T $\bar{M}$ = TM $\oplus$ tr(TM) = {Rad(TM) $\oplus$ tr(TM)} $\oplus_{orth}$ S(TM):

In the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Let P be the projection morphism of TM on S(TM). Then the local Gauss and Weingartan formulas of M and S(TM) are given respectively by

(2.1) $$\bar{\nabla}_X Y = \nabla_X Y + B(X.Y)N.$$

(2.2) $$\bar{\nabla}_X N = -A_N X + \tau(X)N,$$

(2.3) $$\nabla_X PY = \nabla^*_X PY + C(X.PY)\xi.$$

$$\nabla_X \xi = -A^*_{\xi} X + \sigma(X)\xi,$$

where ∇ and ∇^* are the induced linear connections on TM and S(TM) respectively, B and C are the local second fundamental forms on TM and S(TM) respectively, A_N and A^*_{ξ} are the shape operators on TM and S(TM) respectively, and are 1-forms on TM.

The induced connection ∇ is connection of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X,Y)\eta(Z) + B(X, Z)\eta(Y),$$

where η is a 1-form such that

η (X) = $\bar{g}$ (X, N).

As B(X,Y) = $\bar{g}(\overline{\nabla}_X Y, \xi)$, so B is independent of the choice of S(TM) and satisfies

$$(2.4) \quad B(X,\xi) = 0, \forall X \in \Gamma(TM) .$$

Local second fundamental forms are related to their shape operators by

$$(2.5) \quad B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0,$$

$$(2.6) \quad C(X, PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0.$$

Denote by $\overline{R}$, R and R^* the curvature tensor of the semi-symmetric metric connection $\overline{\nabla}$ on $\overline{M}$ and the induced linear connection ∇ and ∇^* on M and S(TM) respectively. Using the Gauss-Weingarten formulas, we obtain two Gauss-Codazzi equations for M and S(TM) such that

$$\begin{aligned}(2.7) \quad \overline{R}(X,Y)Z = R(X, Y)Z &+ B(X, Z)A_N Y - B(Y, Z)A_N X \\ &+ \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &+ \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\}N,\end{aligned}$$

$$\begin{aligned}(2.8) \quad R(X,Y)PZ = R^*(X,Y)PZ &+ C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X \\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &- \tau(X)C(Y,PZ) + \sigma(Y)C(X,PZ)\}\xi.\end{aligned}$$

In case R = 0, we say that M is flat.

B.Y. Chen-K. Yano [12] introduced the notion of a semi-Riemannian manifold of quasi-constant curvature as a semi-Riemannian manifold ($\overline{M}$, $\bar{g}$) endowed with the curvature tensor $\overline{R}$ satisfying the following form:

$$\begin{aligned}(2.9) \quad \overline{R}(X,Y)Z = \lambda\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\} \\ + \mu\,\{\bar{g}(Y, Z)\theta(X)\zeta - \bar{g}(X, Z)\theta(Y)\zeta \\ + \theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y\},\end{aligned}$$

for any vector fields X, Y and Z of $\overline{M}$, where λ and μ are smooth functions, ζ is a smooth vector field and θ is a 1-form associated with ζ by θ(X) = $\bar{g}(X,\zeta)$.

Comparing the tangential and transversal components of (2.7) and (2.9), We have

$$(2.10)\quad R(X,Y)Z = \lambda\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \mu\{\bar{g}(Y,Z)\theta(X)\zeta - \bar{g}(X,Z)\theta(Y)\zeta + \theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y\} + B(Y,Z)A_N X - B(X,Z)A_N Y,$$

$$(2.11)\quad (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) = 0.$$

3. Indefinite Cosympletic Manifold

Let M be an almost contact manifold equipped with an almost contact metric structure (ϕ,ξ,η,g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a compatible Riemannian metric g satisfying

(3.1)

$$J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta,\quad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X},\bar{Y}) - \theta(\bar{X}), \theta(\bar{Y}), \theta(\zeta) = 1,$$

From this, we also have

$$J\zeta = 0,\ \theta oJ = 0, \bar{g}(J\bar{X},\bar{Y}) = -\bar{g}(\bar{X}, J\bar{Y}), \theta(\bar{X}) = \bar{g}(\bar{X},\zeta).$$

for all $X,Y \in \chi(M)$.

On an almost contact metric manifold M we may always define a 2-form ϕ by $\phi(X,Y) = g(\phi X, Y)$. (M,ϕ,ξ,η,g) is said to be an almost cosymplectic manifold [13] if the form ϕ and η are closed i. e. $d\phi = 0$ and $d\eta = 0$, where d is the operator of exterior differentiation.In particular, if the almost contact structure of an almost cosymplectic manifold is normal, then it is said to be a cosymplectic manifold [14]. As it is known, an almost contact metric structure is cosymplectic if and only if both $\nabla\eta$ and $\nabla\phi$ vanish, where ∇ is the covariant differentiation with respect to g.

For a lightlike hypersurface M of an indefinite cosympletic manifold

($\bar{M}$, $\bar{g}$), it is known [15] that J(Rad(TM)) and J(tr(TM)) are subbundles of S(TM), of rank 1 such that J(Rad(TM)) $\cap$ J(tr(TM)) = 0. Thus there exist two non-degenerate almost complex distributions D_o and D on M with respect to J, i.e., $J(D_o) = D_o$ and J(D) = D, such that

$S(TM) = J(Rad(TM)) \oplus J(tr(TM)) \oplus_{orth} D_o;$

$D = \{Rad(TM) \oplus_{orth} J(Rad(TM))\} \oplus_{orth} D_o;$

$$TM = D \oplus J(tr(TM)),$$

Consider two null vector fields U and V , and two 1-forms u and v such that

$$(3.2) \qquad U = -JN, V = J\xi.$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as X = SX + u(X)U, where u and v are 1-forms locally defined on M by

$$(3.3) \qquad u(X) = g(X, V); v(X) = g(X, U).$$

Applying J to this form, we have

$$(3.4) \qquad JX = FX + u(X)N,$$

where F is a tensor field of type (1, 1) globally defined on M by F = JoS.

Applying $\bar{\nabla}_X$ to (3.2) ⊔ (3.6) and using (2.1) ⊔ (2.4), with (3.2) ⊔ (3.6), we have

$$(3.5) \qquad B(X,U) = C(X,V),$$

$$(3.6) \qquad \nabla_X U = F(A_N X) + \tau(X)U,$$

$$(3.7) \qquad \nabla_X V = F(A^*_\xi X) - \tau(X)V,$$

$$(3.8) \qquad (\nabla_X F)Y = u(Y)A_N X - B(X,Y)U.$$

Applying $\bar{\nabla}_X$ to $g(\zeta, \xi) = 0$ and $\bar{g}(\zeta, N) = 0$, we have

$$(3.9) \qquad B(X, \zeta) = 0.$$

Thorem: Let $\bar{M}$ be an indefinite cosympletic manifold with a lightlike hypersurface M, then if F is parallel with respect to the induced connection ∇, then $\bar{M}$ and M are flat manifolds and the transversal connection of M is also flat.

Proof: If F is parallel, then by (3.8), we have

$$(3.10) \qquad u(Y)A_N X - B(X, Y)U = 0.$$

Replacing X by U and Y by V, we have $\lambda = 0$, So $\overline{M}$ is a flat manifold.
Taking Y = U in (3.10), we have

(3.11) $$A_N X = \sigma(X)U.$$

Taking scalar product with V to (3.10), we have

$$B(X, Y) = u(Y)\sigma(X),$$

i.e. $g(A^*_\xi X, Y) = g(\sigma(X)V, Y).$

As $A^*_\xi X$ and V belong to S(TM), and S(TM) is non-degenerate, so we have

(3.12) $$A^*_\xi X = \sigma(X)V.$$

Using (3.11) and (3.12) in (2.10) with $\lambda = \mu = 0$, we have

$$R(X, Y)Z = \{\sigma(Y)\sigma(X) - \sigma(X)\sigma(Y)\}u(Z)U = 0.$$

Therefore R = 0 , hence M is flat.
Using (3.11) in (3.6) and with FU = 0, we have

$$\nabla_X U = \tau(X)U$$

Using this in $\nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U = 0$, we have $d\tau = 0$.
Hence transversal connection of M is flat.

4. Hopf lightlike hypersurfaces

Definition: The canonical structure vector field U is called principal [16], with respect to the shape operator A^*_ξ , if there exists a smooth function f such that

(4.1) $$A^*_\xi U = fU.$$

A lightlike hypersurface M of an indefinite almost complex manifold $\overline{M}$ is said to be a Hopf lightlike hypersurface [16] if it admits a principal canonical structure vector field U, with respect to the shape operator A^*_ξ.
Taking scalar product with X to (4.1) and with (3.5), we have

(4.2)
$$B(X, U) = f\,\nu(X), \quad C(X, V) = f\,\nu(X), \quad \sigma(X) = f\,\nu(X).$$

Theorem 4.1 : Let $\overline{M}$ be an indefinite cosympletic manifold with a Hopf lightlike hypersurface M. Then $\overline{M}$ is a flat manifold.
Proof: Replacing X by ζ in (4.2) and with (3.9) , we have

$$B(U,\zeta) = \mathrm{f}\ \nu(\zeta) = -\mathrm{f}\ \theta(\mathrm{JN}) = 0.$$

Therefore $\lambda = 0,$ so $\overline{M}$ is a flat manifold.

Theorem 4.2 : Let $\overline{M}$ be an indefinite cosympletic manifold with a Hopf lightlike hypersurface M. If F is parallel with respect to induced connection ∇ of M, then $\mathrm{f} = 0$ and S(TM) is totally geodesic in M.

Proof: As M is Hopf lightlike hypersurface , by (3.11) and (4.2), we have

(4.3) $$A_N X = \mathrm{f}\ \nu(\mathrm{X})\,\mathrm{U}.$$

Taking scalar product with Y to (3.12) and with (4.2) , we have

$$B(X,Y) = \mathrm{f}\ \nu(X)\,u(Y).$$

Replacing X by V and Y by U and X by U and Y by V one by one, we get

$$B(V,U) = \mathrm{f}, \qquad \mathrm{B(U,V)} = 0.$$

Therefore $f = 0.$

Hence by (4.2) , we have $A_N = 0$ and S(TM) is totally geodesic in M.

Theorem 4.3 : Let $\overline{M}$ be an indefinite cosympletic manifold with a Hopf lightlike hypersurface M. If U is parallel with respect to induced connection ∇ of M, then S(TM) is an integrable distribution .

Proof: As M is Hopf lightlike hypersurface , by (3.11) and (4.2), we have

$$A_N X = \mathrm{f}\ \nu(\mathrm{X})\,\mathrm{U}.$$

Taking scalar product with Y to this equation , we have

$$g(A_N X, Y) = \mathrm{f}\ \nu(X)\,\nu(Y).$$

So A_N is self adjoint linear operator with respect to g. So by (2.6) C is symmetric on S(TM) . By (2.3), we have

$$\eta([X,Y]) = C(X,Y) - C(Y,X) = 0,\ \forall X, Y \in \Gamma(S(TM)).$$

Therefore $[X,Y] \in \Gamma(S(TM))$ for any $X, Y \in \Gamma(S(TM))$.

Hence S(TM) is an integrable distribution .

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