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ASYMPTOTIC ANALYSIS OF MARKOVIAN RETRIAL QUEUE WITH UNRELIABLE SERVER AND MULTIPLE TYPES OF OUTGOING CALLS

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Abstract. In this paper, we consider markovian retrial queue with two-way communication and unreliable server. Input process is Poisson with constant rate. Incoming calls that find the server busy join the orbit and reattempt to get the service after an exponentially distributed delay. In its idle time the server makes outgoing calls. There are multiple types of outgoing calls in the system. Service durations and rates of making outgoing calls are different and depend on type of outgoing call. The unreliability of the server is characterised by breakdown and restoration periods and its durations are exponentially distributed with parameters depending on the server state.

1. Introduction

Retrial queues are the models of various telecommunication systems without losses. Instead of leaving the system incoming customer that finds the server busy repeat the request for service after some random delay. Such models are represented in monographs [1], [2].

Retrial behaviour is common for call centers and retrial queues are videly used in this area [3], [4]. Two-way communication in retrial queues is a phenomenon arised from the constraints of the call centers functioning. Call centers are provide both incoming and outgoing calls to increase the productivity of the system. Two-way communication models have become widespread recently [5], [6]. Such models have various modifications depending on real systems functioning conditions such as finite capacity of input, server-orbit interaction [7], [8], etc.

In this paper we concider two modifications of two-way communication retrial queues: multiple types of outgoing calls and unreliable server. To research the markovian retrial queue with aforementioned modifications we use asymptotic analysis method under low rates of outgoing calls limit condition.

2. Mathematical Model

We consider single server retrial queue with multiple types of outgoing calls. Input process is a stationary Poisson process with rate λ. Incoming calls occupy the server for an exponentially distributed time with rate μ_1. Calls that find the server busy join the orbit and repeat their attempt to take the server after an

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exponentially distributed delay with rate σ. In its idle time the server makes outgoing calls of type n with rate α_n and provides the service for an exponentially distributed time with parameter μ_n. For convenience, we number the types of outgoing calls from 2 to N.

Let $k(t)$ denotes the state of the server at the time $t \geq 0$ as follows: 0 if the server is idle, 1 if an incoming call is in service, n if an outgoing call of type $n = \overline{2, N}$ is in service, $N+1$ if the server is in restoration mode. We denote γ_0 is the rate of breakdowns in state 0, γ_1 is the rate of breakdowns in state 1 and γ_2 is the rate of restorations. We assume that when the outgoing call is in service there is no breakdowns as the server calls the customer itself. If the server is busy at the moment of breakdown serving customer joins the orbit.

Let $i(t)$ denotes the number of calls in the orbit at the moment $t \geq 0$. It is easy to see that two-dimensional process $\{i(t), k(t)\}$ forms a continuous time Markov chain.

Let $P\{i(t) = i, k(t) = k\} = P_k(i,t)$ denotes the probability distribution of the process $\{i(t), k(t)\}$, then it is the solution of Kolmogorov's system of equations. We present the system in stationary regime

$$
\begin{gathered}
-\left(\lambda + i\sigma + \sum_{n=2}^{N} \alpha_n + \gamma_0\right) P_0(i) + \sum_{k=1}^{N} \mu_k P_k(i) + \gamma_2 P_{N+1}(i) = 0, \\
-(\lambda + \mu_1 + \gamma_1)P_1(i) + \lambda P_1(i-1) + \lambda P_0(i) + (i+1)\sigma P_0(i+1) = 0, \\
-(\lambda + \mu_n)P_n(i) + \lambda P_n(i-1) + \alpha_n P_0(i) = 0, \ n = \overline{2, N}, \\
-(\lambda + \gamma_2)P_{N+1}(i) + \lambda P_{N+1}(i-1) + \gamma_0 P_0(i) + \gamma_1 P_1(i-1) = 0. \qquad (2.1)
\end{gathered}
$$

Let $H_k(u)$ denotes the partial characteristic functions $H_k(u) = \sum\limits_{i=0}^{\infty} e^{jui} P_k(i)$, $k = \overline{0, N+1}$, where $j = \sqrt{-1}$. Multiplying equations of system by e^{jui} and taking the sum over i yelds

$$
\begin{gathered}
-\left(\lambda + \gamma_0 + \sum_{n=2}^{N} \alpha_n\right) H_0(u) + j\sigma H_0'(u) + \sum_{k=1}^{N} \mu_k H_k(u) + \gamma_2 H_{N+1}(u) = 0, \\
(\lambda(e^{ju} - 1) - \mu_1 - \gamma_1)H_1(u) + \lambda H_0(u) - j\sigma e^{-ju} H_0(u) = 0, \\
(\lambda(e^{ju} - 1) - \mu_n)H_n(u) + \alpha_n H_0(u) = 0, \ n = \overline{2, N}, \\
(\lambda(e^{ju} - 1) - \gamma_2)H_{N+1}(u) + \gamma_0 H_0(u) + \gamma_1 e^{ju} H_1(u) = 0. \qquad (2.2)
\end{gathered}
$$

We also present an additional equation obtained by summing up equations of the system (2.2)

$$
j\sigma e^{-ju} H_0'(u) + (\lambda + \gamma_1)H_1(u) + \lambda \sum_{n=2}^{N+1} H_n(u) = 0. \qquad (2.3)
$$

The system (2.2) and equation (2.3) we will use for the further analysis. The aim of the research is to obtain the steady state characteristic function of the process $i(t)$, which is

$$H(u) = \sum_{k=0}^{N+1} H_k(u).$$

The main contribution of this research is the solution of the system (2.2) by using an asymptotic analysis method under low rates of outgoing calls limit condition.

3. Asymptotic Analysis

To match the asymptotic condition we denote $\mu_n = \mu\nu_n$ in the system (2.2)

$$-\left(\lambda + \gamma_0 + \sum_{n=2}^{N} \alpha_n\right) H_0(u) + j\sigma H_0'(u) + \mu_1 H_1(u) + \sum_{k=2}^{N} \mu\nu_k H_k(u) + \gamma_2 H_{N+1}(u) = 0,$$
$$(\lambda(e^{ju}-1) - \mu_1 - \gamma_1)H_1(u) + \lambda H_0(u) - j\sigma e^{-ju} H_0(u) = 0,$$
$$(\lambda(e^{ju}-1) - \mu\nu_n)H_n(u) + \alpha_n H_0(u) = 0,\ n = \overline{2,N},$$
$$(\lambda(e^{ju}-1) - \gamma_2)H_{N+1}(u) + \gamma_0 H_0(u) + \gamma_1 e^{ju} H_1(u) = 0, \tag{3.1}$$

then the limit condition takes the form $\mu \to 0$.

Theorem 3.1. *If $i(t)$ is the number of customers in the orbit described system then the following equality is true*

$$\lim_{\mu\to 0} \mathbb{E} e^{ju\mu i(t)} =$$
$$= \left(\sum_{n=2}^{N} \frac{\alpha_n}{\mu_n}\right)^{-1} \prod_{k=2}^{N} \left(1 - ju\frac{\lambda}{\mu_k}\right)^{-\frac{\alpha_k \gamma_2(\mu_1+\gamma_1)}{\sigma(\mu_1\gamma_2 - \lambda\gamma_2 - \lambda\gamma_1)}} \sum_{n=2}^{N} \frac{\alpha_n}{\mu_n}\left(1 - ju\frac{\lambda}{\mu_n}\right)^{-1}.$$

Proof. In the system (3.1) and the equation (2.3) we denote $\mu = \varepsilon$ and introduce the following notations

$$u = \varepsilon w,\ H_0(u) = \varepsilon F_0(w,\varepsilon),\ H_k(u) = F_k(w,\varepsilon),$$

in order to obtain the system of equations

$$-\left(\lambda + \gamma_0 + \sum_{n=2}^{N} \alpha_n\right) \varepsilon F_0(w,\varepsilon) + j\sigma\frac{\partial F_0(w,\varepsilon)}{\partial w} + \mu_1 F_1(w,\varepsilon) +$$
$$+\varepsilon \sum_{n=2}^{N} \nu_n F_n(w,\varepsilon) + \gamma_2 F_{N+1}(w,\varepsilon) = 0,$$
$$(\lambda(e^{jw\varepsilon}-1) - \mu_1 - \gamma_1)F_1(w,\varepsilon) + \varepsilon\lambda F_0(w,\varepsilon) - j\sigma e^{-jw\varepsilon}\frac{\partial F_0(w,\varepsilon)}{\partial w} = 0,$$
$$(\lambda(e^{jw\varepsilon}-1) - \varepsilon\nu_n)F_n(w,\varepsilon) + \varepsilon\alpha_n F_0(w,\varepsilon) = 0,\ n = \overline{2,N},$$
$$(\lambda(e^{jw\varepsilon}-1) - \gamma_2)F_{N+1}(w,\varepsilon) + \varepsilon\gamma_0 F_0(w,\varepsilon) + \gamma_1 e^{jw\varepsilon} F_1(w,\varepsilon) = 0,$$
$$j\sigma e^{-jw\varepsilon}\frac{\partial F_0(w,\varepsilon)}{\partial w} + (\lambda+\gamma_1)F_1(w,\varepsilon) + \lambda\sum_{n=2}^{N+1} F_n(w,\varepsilon) = 0. \tag{3.2}$$

Then we use Taylor's decompositions

$$e^{jw\varepsilon} = 1 + jw\varepsilon + o(\varepsilon),$$

$$e^{-jw\varepsilon} = 1 - jw\varepsilon + o(\varepsilon),$$

and take the limit by $\varepsilon \to 0$ in the system (3.2)

$$\begin{gathered} j\sigma F_0'(w) + \mu_1 F_1(w) + \gamma_2 F_{N+1}(w) = 0, \\ -(\mu_1 + \gamma_1)F_1(w) - j\sigma F_0'(w) = 0, \\ (jw\lambda - \nu_n)F_n(w) + \alpha_n F_0(w) = 0, \\ -\gamma_2 F_{N+1}(w) + \gamma_1 F_1(w) = 0, \\ j\sigma F_0'(w) + (\lambda + \gamma_1)F_1(w) + \lambda \sum_{n=2}^{N+1} F_n(w) = 0. \end{gathered} \tag{3.3}$$

From each of the equations of the system (3.3) we obtain the expressions

$$j\sigma F_0'(w) = -\mu_1 F_1(w) - \gamma_2 F_{N+1}(w), \tag{3.4}$$

$$j\sigma F_0'(w) = -(\mu_1 + \gamma_1)F_1(w), \tag{3.5}$$

$$F_n(w) = \frac{\alpha_n}{\nu_n - jw\lambda} F_0(w), \; n = \overline{2, N}, \tag{3.6}$$

$$F_{N+1}(w) = \frac{\gamma_1}{\gamma_2} F_1(w), \tag{3.7}$$

$$j\sigma F_0'(w) = -(\lambda + \gamma_1)F_1(w) - \lambda \sum_{n=2}^{N+1} F_n(w). \tag{3.8}$$

We equal the right parts of the expressions (3.5) and (3.8)

$$(\mu_1 + \gamma_1)F_1(w) = (\lambda + \gamma_1)F_1(w) + \lambda \sum_{n=2}^{N+1} F_n(w),$$

then we obtain the expression for $F_1(w)$

$$F_1(w) = \frac{\lambda}{\mu_1 - \lambda} \sum_{n=2}^{N+1} F_n(w). \tag{3.9}$$

The obtained expression we rewrite as

$$F_1(w) = \frac{\lambda}{\mu_1 - \lambda} \sum_{n=2}^{N} F_n(w) + F_{N+1}(w).$$

Using the expressions (3.6) and (3.7) we transform the obtained equation

$$F_1(w) = \frac{\lambda}{\mu_1 - \lambda} \sum_{n=2}^{N} \frac{\alpha_n}{\nu_n - jw\lambda} F_0(w) + \frac{\lambda\gamma_1}{\gamma_2(\mu_1 - \lambda)} F_1(w),$$

then the expression for $F_1(w)$ is given as follows

$$F_1(w) = \frac{\lambda\gamma_2}{\mu_1\gamma_2 - \lambda\gamma_2 - \lambda\gamma_1} \sum_{n=2}^{N} \frac{\alpha_n}{\nu_n - jw\lambda} F_0(w). \tag{3.10}$$

We derive the differential equation for the function $F_0(w)$ using the expressions (3.5) and (3.10)

$$j\sigma F_0'(w) = -\frac{\lambda\gamma_2(\mu_1+\gamma_1)}{\mu_1\gamma_2-\lambda\gamma_2-\lambda\gamma_1}F_0(w)\sum_{n=2}^{N}\frac{\alpha_n}{\nu_n - jw\lambda}. \tag{3.11}$$

The solution of the equation (3.11) is given by

$$F_0(w) = C\prod_{n=2}^{N}\left(1 - jw\frac{\lambda}{\nu_n}\right)^{-\frac{\alpha_n\gamma_2(\mu_1+\gamma_1)}{\sigma(\mu_1\gamma_2-\lambda\gamma_2-\lambda\gamma_1)}}, \tag{3.12}$$

where C is the integration constant.

Thus, using (3.10) and (3.12) we can write the explicit expression for $F_1(w)$ up to an integration constant C

$$F_1(w) = C\frac{\lambda\gamma_2}{\mu_1\gamma_2-\lambda\gamma_2-\lambda\gamma_1}\prod_{k=2}^{N}\left(1 - jw\frac{\lambda}{\nu_k}\right)^{-\frac{\alpha_k\gamma_2(\mu_1+\gamma_1)}{\sigma(\mu_1\gamma_2-\lambda\gamma_2-\lambda\gamma_1)}}\times$$
$$\times\sum_{n=2}^{N}\frac{\alpha_n}{\nu_n}\left(1 - jw\frac{\lambda}{\nu_n}\right)^{-1}. \tag{3.13}$$

For the functions $F_n(w)$, $n=\overline{2,N}$ we write the expression using (3.6) and (3.12)

$$F_n(w) = C\frac{\alpha_n}{\nu_n}\left(1 - jw\frac{\lambda}{\nu_n}\right)^{-1}\prod_{k=2}^{N}\left(1 - jw\frac{\lambda}{\nu_k}\right)^{-\frac{\alpha_k\gamma_2(\mu_1+\gamma_1)}{\sigma(\mu_1\gamma_2-\lambda\gamma_2-\lambda\gamma_1)}}. \tag{3.14}$$

Finally, the expression for the function $F_{N+1}(w)$ we obtain using (3.7) and (3.13)

$$F_{N+1}(w) = C\frac{\lambda\gamma_1}{\mu_1\gamma_2-\lambda\gamma_2-\lambda\gamma_1}\prod_{k=2}^{N}\left(1 - jw\frac{\lambda}{\nu_k}\right)^{-\frac{\alpha_n\gamma_2(\mu_1+\gamma_1)}{\sigma(\mu_1\gamma_2-\lambda\gamma_2-\lambda\gamma_1)}}\times$$
$$\times\sum_{n=2}^{N}\frac{\alpha_n}{\nu_n}\left(1 - jw\frac{\lambda}{\nu_n}\right)^{-1}. \tag{3.15}$$

For the sought characteristic function $H(u)$ we can write an approximate equality

$$H(u) = \sum_{k=0}^{N+1}H_k(u) \approx \sum_{k=1}^{N+1}F_k(w).$$

We denote

$$\Phi(w) = \sum_{k=1}^{N+1}F_k(w)$$

and obtain the expression

$$\Phi(w) = C\frac{\mu_1\gamma_2}{\mu_1\gamma_2-\lambda\gamma_2-\lambda\gamma_1}\prod_{k=2}^{N}\left(1 - jw\frac{\lambda}{\nu_k}\right)^{-\frac{\alpha_k\gamma_2(\mu_1+\gamma_1)}{\sigma(\mu_1\gamma_2-\lambda\gamma_2-\lambda\gamma_1)}}\times$$

$$\times \sum_{n=2}^{N} \frac{\alpha_n}{\nu_n} \left(1 - jw\frac{\lambda}{\nu_n}\right)^{-1}.$$

To refine the constant C we use the condition

$$\Phi(0) = 1,$$

then the value of C is given as follows

$$C = \left[\frac{\mu_1\gamma_2}{\mu_1\gamma_2 - \lambda\gamma_2 - \lambda\gamma_1} \sum_{n=2}^{N} \frac{\alpha_n}{\nu_n}\right]^{-1}.$$

Thereby, we obtain the sought function $\Phi(w)$

$$\Phi(w) =$$

$$= \left(\sum_{n=2}^{N} \frac{\alpha_n}{\nu_n}\right)^{-1} \prod_{k=2}^{N} \left(1 - jw\frac{\lambda}{\nu_k}\right)^{-\frac{\alpha_k\gamma_2(\mu_1+\gamma_1)}{\sigma(\mu_1\gamma_2-\lambda\gamma_2-\lambda\gamma_1)}} \sum_{n=2}^{N} \frac{\alpha_n}{\nu_n} \left(1 - jw\frac{\lambda}{\nu_n}\right)^{-1}. \quad (3.16)$$

Making the reverse substitutions $w = \frac{u}{\varepsilon}$, $\varepsilon = \mu$ and taking into account $\mu_k = \mu\nu_k$ we obtain

$$H(u) \approx \left(\sum_{n=2}^{N} \frac{\alpha_n}{\mu_n}\right)^{-1} \prod_{k=2}^{N} \left(1 - ju\frac{\lambda}{\mu_k}\right)^{-\frac{\alpha_k\gamma_2(\mu_1+\gamma_1)}{\sigma(\mu_1\gamma_2-\lambda\gamma_2-\lambda\gamma_1)}} \sum_{n=2}^{N} \frac{\alpha_n}{\mu_n} \left(1 - ju\frac{\lambda}{\mu_n}\right)^{-1} \quad (3.17)$$

Then the limit equality defined in theorem holds. □

Theorem 1 defines asymptotic characteristic function of the number of customers in the orbit. The probability distribution of the process $i(t)$ can be obtained using inverse Fourier transform by the formula

$$P(i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jui} H(u) du. \quad (3.18)$$

4. Conclusion

We have considered markovian retrial queue with unreliable server and multiple types of outgoing calls. Using asymptotic analysis method under low rates of outgoing calls limit condition we have built an approximation for the probability distribution (3.18) of the number of calls in the orbit.

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