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ANALYTICAL PROOF OF ONE STRONG VERSION OF THE LAW OF LARGE NUMBERS

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Abstract. An analytical proof based on the method of characteristic functions of Sakhonenko theorem on the validity of a strong option of the law of large numbers is presented in the paper: when a condition of the Lindeberg type is satisfied, the absolute value of the sum of independent random variables tends to zero on average. The mathematical expectation of the absolute value of a random variable uses the above proof through the corresponding characteristic function.

Let a "series scheme" of independent random variables (r.v.) be given in some probability space $(\Omega, \Im, P)$:

$$\{X_{n1}, ..., X_{nn}\}, \quad n = 1,\, 2, \cdots$$

and

$$S_n = X_{n1} + \ldots + X_{nn}.$$

From the point of view of summation of independent r.v. results (available in theory), we can assume, without loss of generality, that

$$EX_{nj} = 0, \;\; j = 1, 2, ..., n. \tag{1}$$

Assume that

$$D_n(\alpha) = \sum_{j=1}^{n} E \min\left(|X_{nj}|, \, |X_{nj}|^{1+\alpha}\right), \;\; \alpha > 0.$$

It is easy to check that if at certain $\alpha = \alpha_0 > 0$

$$D_n(\alpha_0) \to 0, \;\; n \to \infty,$$

then $D_n(\alpha) \to 0$, for all $\alpha > 0$.

Taking into account the latter, we assume that the following condition is met:

$$D_n = D_n(1) = \sum_{j=1}^{n} E \min\left(|X_{nj}|, \, |X_{nj}|^2\right) \to 0 \tag{D}$$

as $n \to \infty$.

In [1, p. 158] the following two versions of the law of large numbers (l.l.n.) are presented.

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The first version consists of a statement about the validity of the ordinary l.l.n. ($S_n \xrightarrow{P} 0$) when conditions (1), (D) are met.

The second version, based on stronger convergence on average, is presented as the following theorem.

Theorem A (A.I.Sakhonenko). *When conditions* (1), (D) *are met,* $E|S_n| \to 0$ *is true (or* $S_n \xrightarrow{P} 0$ *which is the same). The statement of the above theorem means, obviously, the uniform integrability of the sequence* $\{S_n,\ n \geq 1\}$*, from it follows the ordinary version of the l.l.n.* $S_n \xrightarrow{P} 0$ *so that*

$$P(|S_n| \geq \varepsilon) \leq \frac{E|S_n|}{\varepsilon} \to 0,$$

as $n \to \infty$.

The proof of Theorem A in [1, p. 159–160] was completed by the probabilistic method of "r.v.truncating". Here we give an "analytic proof" of Theorem A, based on the method of characteristic functions (ch.f.).

Assume that

$$F_{nj}(x) = P(X_{nj} < x), \quad F_n(x) = P(S_n < x)$$
$$f_{nj}(t) = Ee^{itX_{nj}}, \quad f_n(t) = Ee^{itS_n}, \quad j = 1, ..., n.$$

First we prove the following auxiliary statement.

Lemma 0.1. *For any r.v. with distribution function (d.f.)* $F(x)$

$$E|X| = \int_{-\infty}^{\infty} |x|\, dF(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1 - \operatorname{Re} f(t)}{t^2} dt, \tag{2}$$

where $f(t)$ *is the ch.f. corresponding to the d.f.* $F(x)$.

The assertion of the lemma is presented as an independent problem in [2, ch. 2, p. 409].

Proof. Changing the order of integration on the right-hand side of (2), we have

$$\begin{aligned} \int_{-\infty}^{\infty} \tfrac{1-\operatorname{Re} f(t)}{t^2} dt &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ((1 - \cos tx)\, dF(x))\, t^{-2} dt = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\tfrac{1-\cos tx}{t^2} dt\right) dF(x) = 2 \int_{-\infty}^{\infty} \left(\int_{0}^{\infty} \tfrac{1-\cos tx}{t^2} dt\right) dF(x) \end{aligned} \tag{3}$$

Furthermore

$$\int_0^{\infty} \frac{1 - \cos tx}{t^2} dt = \int_0^T + \int_T^{\infty} = I_T^{(1)} + I_T^{(2)}. \tag{4}$$

Obviously, for any $x \in R$

$$I_T^{(2)} = \int_T^{\infty} \frac{1 - \cos tx}{t^2} dt \leq 2 \int_T^{\infty} \frac{dt}{t^2} = \frac{2}{T}.$$

So

$$\sup_x I_T^{(2)}(x) \le \frac{2}{T}. \tag{5}$$

After integrating by parts, we obtain:

$$I_T^{(1)}(x) = -\int_0^T (1-\cos tx)\, d\left(\frac{1}{t}\right) = -\frac{1-\cos Tx}{T} + x\int_0^T \frac{\sin tx}{t}dt. \tag{6}$$

Taking into account the last relations (5) and (6), we obtain:

$$\int_0^\infty \left(\int_0^\infty \frac{1-\cos tx}{t^2}dt\right) dF(x) = \int_0^\infty \lim_{T\to\infty} I_T^{(1)}(x)\, dF(x) =$$

$$= \int_0^\infty x\left(\int_0^\infty \frac{\sin tx}{t}dt\right) dF(x) = \frac{\pi}{2}\int_0^\infty x dF(x). \tag{7}$$

Here the well-known equality (the Dirichlet integral) is used

$$\int_0^\infty \frac{\sin tx}{t}dt = \begin{cases} \frac{\pi}{2}, & x>0 \\ -\frac{\pi}{2}, & x<0 \end{cases}, \tag{8}$$

Applying formulas (6) and (8) in a similar way, we obtain:

$$\int_{-\infty}^0 \left(\int_0^\infty \frac{1-\cos tx}{t}dt\right) dF(x) = -\frac{\pi}{2}\int_{-\infty}^0 x dF(x) = \frac{\pi}{2}\int_{-\infty}^0 |x|\, dF(x) \tag{9}$$

Now from equalities (3), (7), (9) it follows that

$$\int_{-\infty}^\infty \frac{1-\operatorname{Re} f(t)}{t^2}dt = \int_{-\infty}^0 \left(\int_0^\infty \frac{1-\cos tx}{t}dt\right) dF(x) + \int_0^\infty \left(\int_0^\infty \frac{1-\cos tx}{t}dt\right) dF(x) =$$

$$= \frac{\pi}{2}\int_{-\infty}^0 |x| dF(x) + \frac{\pi}{2}\int_0^\infty x dF(x) = \frac{\pi}{2}\int_{-\infty}^\infty |x| dF(x).$$

Thus, formula (2) is proved. □

Now we will prove Theorem A using the lemma (or formula (2) which is the same) Assume that

$$E\,|S_n| = \frac{2}{\pi}\int_{-\infty}^\infty \frac{1-\operatorname{Re} f_n(t)}{t^2}dt = \frac{2}{\pi}\left(I_n^{(1)} + I_n^{(2)}\right), \tag{10}$$

where

$$I_n^{(1)} = \int_{|t|\le T} \frac{\operatorname{Re}(1-f_n(t))}{t^2}dt, \qquad I_n^{(2)} = \int_{|t|>T} \frac{\operatorname{Re}(1-f_n(t))}{t^2}dt, \;\; T>0.$$

It is evident that

$$I_n^{(2)} = \int\limits_{|t|>T} \left(\int\limits_{-\infty}^{\infty} (1-\cos tx)\, dF_n(x) \right) \frac{dt}{t^2} \le 2 \int\limits_{|t|>T} \frac{dt}{t^2} \le \frac{8}{T}.$$

So

$$\limsup_{n\to\infty} I_n^{(2)} \le \frac{8}{T}$$

and, by the randomness of $T > 0$ we obtain

$$I_n^{(2)} \to 0, \quad n \to \infty. \tag{11}$$

Next, we use the following equality

$$1 - f_n(t) = \prod_{j=1}^{n} 1 - \prod_{j=1}^{n} f_{nj}(t) = \sum_{j=1}^{n} (1 - f_{nj}(t)) \prod_{k=j+1}^{n} f_{nk}(t) =$$

$$= \sum_{j=1}^{n} (1 - f_{nj}(t)) \left[1 + \left(\prod_{k=j+1}^{n} f_{nk}(t) - \prod_{k=j+1}^{n} 1 \right) \right].$$

By virtue of this equality, we have

$$\frac{\operatorname{Re}(1 - f_n(t))}{t^2} = \sum_{j=1}^{n} \frac{\operatorname{Re}(1 - f_{nj}(t))}{t^2} + \psi_n(t), \tag{12}$$

where

$$\psi_n(t) = \sum_{j=1}^{n} \operatorname{Re} \left[\frac{1 - f_{nj}(t)}{t^2} \left(\prod_{k=j+1}^{n} f_{nk}(t) - \prod_{k=j+1}^{n} 1 \right) \right].$$

By Lemma 3 from [1, ch.7, §6, p. 142]

$$\left| \prod_{k=j+1}^{n} f_{nk}(t) - \prod_{k=j+1}^{n} 1 \right| \le \sum_{k=j+1}^{n} \left| E\left(e^{itX_{nk}} - 1\right) \right| =$$

$$= \sum_{k=j+1}^{n} \left| E\left(e^{itX_{nk}} - 1 - itX_{nk}\right) \right| \le 2h(t) \sum_{k=1}^{n} Eg(X_{nk}),$$

where $h(t) = \max\left(|t|, |t|^2\right), \quad g(x) = \min\left(|x|, |x|^2\right).$

So, for each $j = 1, ..., n$

$$\left| \prod_{k=j+1}^{n} f_{nk}(t) - \prod_{k=j+1}^{n} 1 \right| \le 2h(t) \left[\sum_{k=1}^{n} \left(\int\limits_{|x|\le 1} x^2 dF_{nk}(x) + \right.\right.$$
$$\left.\left. + \int\limits_{|x|>1} |x| dF_{nk}(x) \right) \right] = 2h(t)\left(M_n + L_n\right) = 2h(t) D_n \to 0. \tag{13}$$

Here

$$M_n = \sum_{k=1}^{n} \int\limits_{|x|\le 1} x^2 dF_{nk}(x), \quad L_n = \sum_{k=1}^{n} \int\limits_{|x|>1} |x|\, dF_{nk}(x).$$

Then, according to formula (6)

$$\sum_{j=1}^{n}\int_{-T}^{T}\frac{\operatorname{Re}\left(1-f_{nj}\left(t\right)\right)}{t^{2}}dt=-2\sum_{j=1}^{n}\int_{-\infty}^{\infty}\left(\int_{0}^{T}\left(1-\cos tx\right)d\left(\frac{1}{t}\right)\right)dF_{nj}\left(x\right)\leq$$

$$\leq 2\left[\sum_{j=1}^{n}\int_{-\infty}^{\infty}\left(\frac{1-\cos Tx}{T}\right)dF_{nj}\left(x\right)+\sum_{j=1}^{n}\int_{-\infty}^{\infty}x\left(\int_{0}^{T}\frac{\sin tx}{t}dt\right)dF_{nj}\left(x\right)\right] \qquad (14)$$

It is easy to see that the following estimates hold

$$\sum_{j=1}^{n}\int_{-\infty}^{\infty}\left(\frac{1-\cos Tx}{T}\right)dF_{nj}\left(x\right)=\sum_{j=1}^{n}\int_{|x|\leq 1}\left(\frac{1-\cos Tx}{T}\right)dF_{nj}\left(x\right)+$$

$$+\sum_{j=1}^{n}\int_{|x|>1}\left(\frac{1-\cos Tx}{T}\right)dF_{nj}\left(x\right)\leq T\sum_{j=1}^{n}\int_{|x|\leq 1}\frac{2\sin^{2}\frac{Tx}{2}}{4\left(\frac{Tx}{2}\right)^{2}}\cdot x^{2}dF_{nj}\left(x\right)+ \qquad (15)$$

$$+\frac{2}{T}\sum_{j=1}^{n}\int_{|x|>1}|x|\,dF_{nj}\left(x\right)\leq C\left(T\right)\left(M_{n}+L_{n}\right)=C\left(T\right)D_{n}\to 0.$$

Hereinafter, $C(T)$ denotes a positive constant (depending on T), which is different in different points.

Then

$$\sum_{j=1}^{n}\int_{-\infty}^{\infty}x\left(\int_{0}^{T}\frac{\sin tx}{t}dt\right)dF_{nj}\left(x\right)\leq\sum_{j=1}^{n}\int_{|x|\leq 1}|x|\left|\int_{0}^{T}\frac{\sin tx}{t}dt\right|dF_{nj}\left(x\right)+$$

$$+\sum_{j=1}^{n}\int_{|x|>1}|x|\left|\int_{0}^{T}\frac{\sin tx}{t}dt\right|dF_{nj}\left(x\right)\leq T\sum_{j=1}^{n}\int_{|x|\leq 1}|x|^{2}dF_{nj}\left(x\right)+ \qquad (16)$$

$$+\frac{\pi}{2}\sum_{j=1}^{n}\int_{|x|>1}|x|\,dF_{nj}\left(x\right)\leq\max\left(T,\ \frac{\pi}{2}\right)\left(M_{n}+L_{n}\right)\leq C\left(T\right)D_{n}\to 0.$$

Now from relations (14) - (16) we obtain:

$$\sum_{j=1}^{n}\int_{-T}^{T}\frac{\operatorname{Re}\left(1-f_{nj}\left(t\right)\right)}{t^{2}}dt\leq C\left(T\right)D_{n}\to 0. \qquad (17)$$

By relations (12), (13), and (17), we obtain

$$\int_{-T}^{T}\psi_{n}\left(t\right)dt\leq C\left(T\right)D_{n}^{2}\to 0. \qquad (18)$$

Finally, from relations (12), (17), and (18), we can conclude that

$$\int_{-T}^{T} \frac{\operatorname{Re}(1 - f_n(t))}{t^2} dt \leq C(T)\left(D_n + D_n^2\right).$$

Thus

$$I_n^{(1)} = \int_{-T}^{T} \frac{\operatorname{Re}(1 - f_n(t))}{t^2} dt = O\left(D_n + D_n^2\right) \to 0 \tag{19}$$

the proof of Theorem A follows from relations (11) and (19).

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