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ODD PERIODIC OSCILLATIONS FOR COMB-DRIVE FINGERS MEMS WITH CUBIC STIFFNESS

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Abstract. In this work we study the existence of transverse odd periodic oscillations of the movable electrode in a Comb-drive finger device, when a cubic nonlinear mechanical stiffness and a periodic input voltage are considered in the modeling. We prove the existence of a family of odd periodic responses with a prescribed number of zeros. The results are obtained by means of tools as the shooting method, the Sturm Comparison theory and the truncation technique. Some numerical examples are provided in order to illustrate the existence results and give an insight about the stability properties.

1. Introduction

The comb-drive devices are commonly used in MEMS (microelectromechanical systems) as sensing and actuation mechanisms. These micro-scale devices are based on two comb structures, one fixed and the other movable. Since each comb structure has electrodes called fingers, the device can have interdigitated or non-interdigitated fingers. Additionally, the movable comb is attached to flexures or flexible structures (tethers) that act like springs in order to limit the movement. In particular, they can be stiff to prevent some undesired motions [18]. Precisely, this is a sort of the nonlinear mechanical stiffness.

We notice that the presence of linear and nonlinear (constant or time-varying) stiffness coefficients is common in some models for oscillators. There is active research about the nonlineal stifness effects in MEMS based on comb-drives. It is worthy to mention [17, 5, 1] where authors study chaos in MEMS through models with cubic stiffness coefficients (via a nonlinear version of the Mathieu Equation) and consider applications to secure communication schemes based on synchronized chaos, they present a method to independently tune both stiffness coefficients in a MEMS oscillator with linear and cubic restoring forces, without affecting some important factors as the resonant frequency [1].

Additionally, authors in [15, 4, 6, 9] consider nonlinearities of structures and applications as MEMS-based filters, gyroscopes and folded-MEMS Comb-drive resonators. We notice that most of these works consider micro devices that utilize

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non-interdigitated comb-drives along with the nonlinear stiffness effects. Nevertheless, we will focus on the in-plane interdigitated ones which provide the basis of some MEMS with applications as accelerometers [12, 18], particularly, in the transverse movement of their fingers when a cubic nonlinear mechanical stiffness term is considered in the model. Figure 1 shows the classical configuration of the comb-drive devices in this paper.

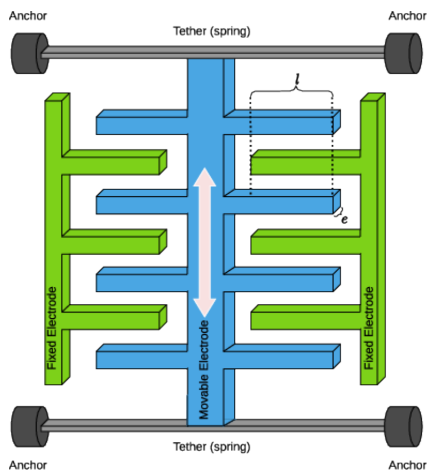


FIGURE 1. Scheme of a transverse in-plane Comb-drive actuator.

Next we introduce a dimensional second order ODE that describes the dynamics of the movable finger sandwiched between two fixed fingers in this Comb-drive device. We consider a spring-mass model where x denotes the displacement of the movable finger in the transverse direction to the longitudinal axis of the stationary electrode, thus it is ruled by

$$\ddot{x} + \hat{\omega}^2 x + \frac{c}{\mathfrak{m}} x^3 = \frac{4dhxV^2(t)}{\left(d^2 - x^2\right)^2}, \tag{1.1}$$

where $V(t)$ denotes the input voltage, $\mathfrak{m}$ denotes the mass of the movable electrode, d denotes the gap between the movable finger and the fixed electrode, k and c are the linear and nonlinear mechanical stiffness coefficients respectively, $\hat{\omega}^2 = k/\mathfrak{m}$ and $h = \varepsilon el/2\mathfrak{m} > 0$ for ε the dielectric constant of vacuum, e the width of the electrodes and l the length of the electrodes in the interacting zone (overlapping zone) [18].

We notice that in appropriate units of distance and time, the spring-mass model (1.1) can be written as (see [13, 18])

$$\ddot{x} + x\left(1 + \alpha x^2 - \frac{4\beta\mathcal{V}^2(t)}{(1-x^2)^2}\right) = 0, \qquad |x| < 1, \tag{1.2}$$

where α and β are positive physical constants given by

$$\alpha = \frac{cd^2}{k}, \quad \beta = \frac{el\epsilon}{2kd^3}. \tag{1.3}$$

We also consider a positive T-periodic AC-DC voltage of the form

$$\mathcal{V}(t) = \mathcal{V}_0 + \delta q(t), \tag{1.4}$$

where $\mathcal{V}_0 > 0$, $q \in C(\mathbb{R}/T\mathbb{Z})$ is an even function such that $\int_0^T q(t)dt = 0$, and $\delta \in \left]0, -\frac{\mathcal{V}_0}{q_m}\right]$ for $q_m = \min_{t\in\mathbb{R}} q(t) < 0$. For example, we can set $q(t) = \cos(\omega t)$ with $\omega = \frac{2\pi}{T}$. Moreover, we define

$$\mathcal{V}_m := \min_{t\in[0,T]} \mathcal{V}(t), \quad \mathcal{V}_M := \max_{t\in[0,T]} \mathcal{V}(t).$$

Regarding to the more basic dynamical aspects like existence of periodic responses for the model (1.2), it is worthy to mention [13] where authors provide sufficient conditions (in terms of the system parameters) to determine the existence and the stability properties of constant sign periodic responses, when a periodic and variable voltage is supplied. More precisely, the dynamics depends on the cubic stiffness coefficient α, thus if it is greater than certain constant it is possible to get a stable positive sign (lateral) periodic response. In [7] the existence of even periodic solutions for the corresponding linear model ($\alpha = 0$) and with a prescribed number of zeros is studied by means of topological techniques.

Motivated for these considerations, in this work we concern about non-constant sign periodic responses for (1.1), when an even periodic input voltage is supplied and the damping effects are neglected. Specifically, we are interested in its odd periodic responses. Thus the main purpose of this paper is to determine sufficient conditions over the system parameters that lead to odd periodic responses with a prescribed number of oscillations. Here the approach is more elementary than the one of the papers mentioned above, and it is based on the ideas of [14]. More precisely, the theorems in [14] cannot be directly applied in this case, thus the model is modified through a truncation process by using *a priori bounds*, in order to obtain an equivalent equation with the same periodic solutions.

This paper is organized as follows. In section 2 we present our main results regarding to the existence of odd periodic responses with a prescribed number of zeros for the Comb-drive finger model with cubic stiffness (Theorem 1 and Corollary 1 in section 2). This is complemented in section 4 with some numerical simulations that support theoretical results and let us give an insight about the stability properties from a numerical point of view. In section 3 we prove our main results and in section 5 we present some concluding remarks about this work.

2. Main Results

In order to present the main results of this paper we first consider a key result due to R. Ortega for symmetric driven oscillators [14].

Let $G \in C^{0,1}([0, L]\times\mathbb{R})$ for some $L > 0$, where $C^{0,1}([0, L]\times\mathbb{R})$ denotes the space of all functions that are continuous in its first variable and have partial derivative with respect to its second variable continuous on $\mathbb{R}$. We will consider the following Dirichlet problem

$$\begin{cases} \ddot{x} + xG(t, x) = 0, \\ x(0) = x(L) = 0. \end{cases} \tag{2.1}$$

Thus, the variational equation associated to (2.1) at $x = 0$ is given by

$$\ddot{u} + G(t,0)u = 0. \tag{2.2}$$

Moreover, if $\phi(t)$ denotes the solution of (2.2) such that $\phi(0) = 0$ and $\dot{\phi}(0) = 1$ then we define ν_ϕ as the cardinality of the set

$$\{t \in]0, L[\ \mid \phi(t) = 0\}. \tag{2.3}$$

Notice that by uniqueness $\phi(t)$ has isolated zeros, therefore ν_ϕ is well defined. Next we present an equivalent form of Proposition 4 in [14]. The hypothesis (7) in this paper has been substituted by the equivalent condition $xG(t,x)$ bounded.

Proposition 2.1 (Ortega's principle)**.** *Consider the Dirichlet problem* (2.1) *such that the non-linearity $xG(t,x)$ is bounded, and $G(t,x) < G(t,0)$ for all $t \in \mathbb{R}$ and $x \neq 0$. Then the problem* (2.1) *has a solution with $N \geq 0$ zeros in $]0, L[$ if and only if $\nu_\phi > N$.*

On the other hand, for a positive integer m it is a well known fact that the problem of finding odd mT-periodic solutions for the second order differential equation

$$\ddot{x} + xG(t,x) = 0, \tag{2.4}$$

where $G \in C^{0,1}(\mathbb{R}/T\mathbb{Z} \times \mathbb{R})$ verifies the following symmetries

$$G(-t,x) = G(t,x), \qquad G(t,-x) = G(t,x), \tag{2.5}$$

can be reduced to solve the Dirichlet problem in (2.1) for $L = mT/2$, [8, 2]. Notice that (1.2) has the form of equation (2.4). Therefore the associated variational equation at $x = 0$ is (see (2.2))

$$\ddot{u} + (1 - 4\beta\mathcal{V}^2(t))u = 0. \tag{2.6}$$

Besides, if $u_0(t)$ denotes the solution of (2.6) such that $u_0(0) = 0$ and $\dot{u}_0(0) = 1$, then we define ν_0 as the cardinality of the set (2.3) for the solution $u_0(t)$ and $L = mT/2$.

Proposition 2.1 along with a truncation of (1.2) lead us to our main result about the existence of odd periodic solutions for the equation (1.2). We focus on the odd periodic solutions having positive initial velocities because from the symmetries (2.5) it deduces that if $x(t)$ is a periodic solution of (2.4) then $-x(t)$ also is.

Theorem 2.2. *Consider $m > 0$ and $N \geq 0$ two integers numbers, and assume that the following conditions hold*

$$a)\ \mathcal{V}_M < \sqrt{\frac{1}{4\beta}}, \qquad b)\ \alpha \leq 8\beta\mathcal{V}_m^2.$$

Then the equation (1.2) *has an odd mT-periodic solution φ_N with N zeros in $\left]0, \frac{mT}{2}\right[$ such that $\dot{\varphi}_N(0) > 0$ if and only if $\nu_0 > N$. Moreover, if $\nu_0 > 0$ then φ_0 is the unique odd mT-periodic solution of* (1.2) *positive in $\left]0, \frac{mT}{2}\right[$ and therefore has minimal period mT.*

From the Sturm Comparison theory it is straightforward to deduce the following result

Corollary 2.3. *Consider m and N_0 two positive integers numbers, and assume that conditions of Theorem 2.2 hold. Assume that $T > \frac{2\pi(N_0+1)}{m}$ and*

$$\mathcal{V}_m,\ \mathcal{V}_M \in \left[\sqrt{\frac{1-\left[\dfrac{2\pi(N_0+1)}{mT}\right]^2}{4\beta}},\ \sqrt{\frac{1-\left[\dfrac{2\pi N_0}{mT}\right]^2}{4\beta}} \right].$$

Then $\nu_0 = N_0$ or $\nu_0 = N_0+1$, and (1.2) has an odd mT-periodic solution φ_N with N zeros in $\left]0, \frac{mT}{2}\right[$ for each $N = 0, 1, 2, \ldots, N_0 - 1$.

3. Proofs

Proof of Theorem 2.2. The proof is presented in several steps as follows.

Bounding all periodic solutions: Assume that hypotheses a) and b) of Theorem 2.2 hold, and consider $m > 0$ an integer number. Then every mT periodic solution $\varphi(t)$ of (1.2) satisfies

$$|\varphi(t)| \le R_0, \ \text{ for all } t \in \mathbb{R}, \tag{3.1}$$

where $R_0 \coloneqq r_0 + \epsilon_0$, for r_0 the unique positive real solution of $\phi(x) = 4\beta\mathcal{V}_m^2$ in $]0,1[$, with

$$\phi(x) := (\alpha x^2 + 1)(1 - x^2)^2$$

and ϵ_0 a fixed positive constant such that $R_0 < 1$.

In order to prove this, we consider the following lemma which is a standard way to get *a priori bounds* for periodic solutions and has similar conditions to those known in the literature as Hartman conditions (see [3]). An analogous result for *a priori bounds* of periodic solutions in a transverse comb-drive device whenever $\alpha = 0$ was first established in Proposition 2 of [7].

Lemma 3.1. *Consider $h : \mathbb{R} \times]a,b[\to \mathbb{R}$ a continuous and locally Lipschitz function, and the following differential equation*

$$\ddot{x} = h(t, x). \tag{3.2}$$

Assume that h is T-periodic in t for each fixed x, and that there exist $R_1, R_2 \in]a,b[$ with $R_1 < R_2$ such that for all $t \in \mathbb{R}$: $h(t,x) < 0$ for all $x \in]a, R_1]$ and $h(t,x) > 0$ for all $x \in [R_2, b[$. Then any T-periodic solution $\psi(t)$ of (3.2) satisfies for all $t \in \mathbb{R}$ that

$$R_1 \le \psi(t) \le R_2.$$

Proof of Lemma 3.1. Assume that hypotheses of the Lemma hold and that $\psi(t)$ denotes a T-periodic solution of (3.2) such that $a < \psi(t) < b$, $\forall t$.

If we assume that $a < \psi(\underline{t}) < R_1$ for some $\underline{t} \in \mathbb{R}$ we obtain a contradiction since $\psi(t)$ is continuous and hence there exists t^* such that

$$\min_{t\in[0,T]} \psi(t) = \psi(t^*) \le \psi(\underline{t}) < R_1, \text{ and therefore } 0 \le \ddot{\psi}(t^*) = h(t^*, \psi(t^*)) < 0.$$

Then for all $t \in \mathbb{R}$ we have that $\psi(t) \ge R_1$. An analogous reasoning can be used to prove that $\psi(t) \le R_2$. □

Now, we shall present some important facts about $\phi(x)$ in the first place. Notice that the function $\phi(x)$ defined above is an even continuously differentiable function that vanishes only at $x = \pm 1$. Moreover, since the hypothesis *a*) is equivalent to $4\beta\mathcal{V}_M^2 < 1$ and by definition $0 < \mathcal{V}_m < \mathcal{V}_M$, we have that $0 < \alpha \leq 8\beta\mathcal{V}_m^2$ implies that $0 < \alpha < 2$. Therefore, the function $\phi'(x) = 2x(1-x^2)(\alpha - 2 - 3\alpha x^2)$ vanishes only at $x = \pm 1$ and $x = 0$, and thus it is not difficult to verify that hypothesis *b*) implies that the function $\phi(x)$ has a maximum value at $x = 0$ in the interval $]-1, 1[$ with $\phi(0) = 1$. Figure 2 illustrates the general form of function $\phi(x)$ under hypothesis *a*) and *b*). Since the function $\phi(x)$ is monotone increasing in the interval $]-1, 0[$ and monotone decreasing in the interval $]0, 1[$, and $0 < 4\beta\mathcal{V}_m^2 < 1$, there exists a unique positive real solution in $]0, 1[$ of equation $\phi(x) = 4\beta\mathcal{V}_m^2$ which will be denoted by r_0. Notice that $-r_0$ is also a solution because $\phi(x)$ is an even function. Figure 2 shows the location of r_0 and $-r_0$, which can be understood as the x-coordinates of the intersection between the graph of function $\phi(x)$ and the graph of the horizontal line $y = 4\beta\mathcal{V}_m^2$ for $x \in]-1, 1[$.

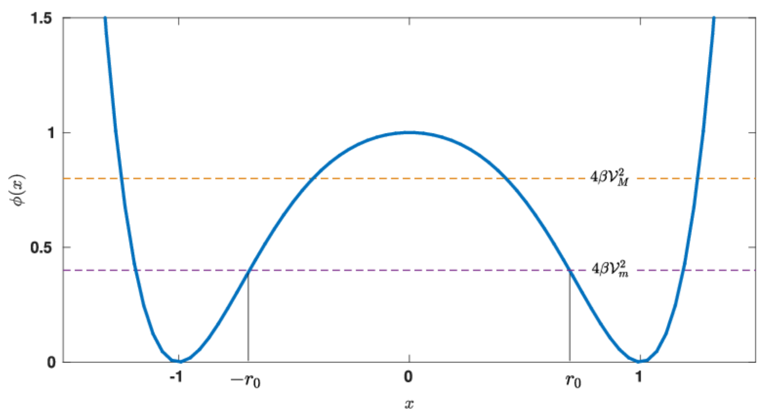


FIGURE 2. Graph of function $\phi(x)$ for $0 < \alpha \leq 2$ and location of the root r_0.

Now let us fix $\epsilon_0 > 0$ such that $R_0 \coloneqq r_0 + \epsilon_0 < 1$. Then we can define $R_2 \coloneqq R_0 > r_0 > 0$ and $R_1 \coloneqq -R_0 < -r_0 < 0$. It is not difficult to check that the hypotheses of Lemma 3.1 hold for $x \in]-1, 1[$ and

$$g(t, x) = x\left(\frac{4\beta\mathcal{V}^2(t)}{(1-x^2)^2} - (\alpha x^2 + 1)\right),$$

where $g(\cdot, x)$ is mT-periodic for each x. Notice that $g(t, -x) = -g(t, x)$, and since

$$g(t, x) = x\left(\frac{4\beta\mathcal{V}^2(t) - \phi(x)}{(1-x^2)^2}\right),$$

we have that for all $t \in \mathbb{R}$

$$x\left(\frac{4\beta\mathcal{V}_m^2 - \phi(x)}{(1-x^2)^2}\right) \leq g(t, x), \quad \text{if} \quad x \in]0, 1[,$$

and

$$g(t,x) \leq x\left(\frac{4\beta\mathcal{V}_m^2 - \phi(x)}{(1-x^2)^2}\right), \quad \text{if } \ x \in\,]-1,0[\,.$$

Moreover, definitions of R_1 and R_2 imply that $4\beta\mathcal{V}_m^2 > \phi(x)$ whenever $x \in\,]-1,R_1] \cup [R_2,1[$ (see figure 2). Therefore $g(t,x) > 0$ for all $t \in \mathbb{R}$ and $x \in [R_2,1[$, and $g(t,x) < 0$ for all $t \in \mathbb{R}$ and $x \in\,]-1,R_1]$. So that the mT-periodic solutions of (1.2) have their range contained in $[-R_0,R_0]$.

An equivalent modified differential equation: In order to tackle the difficulties derived from the singularities in equation (1.2), we introduce a truncated equation which will have the same periodic solutions of (1.2) (see [3] for more about the truncation technique in a boundary value problem). For that purpose we need to consider an odd increasing function $\mathcal{T}(x)$ such that $\mathcal{T} \in C^1(\mathbb{R})$, $|\mathcal{T}(x)| < 1$ for all $x \in \mathbb{R}$ and $\mathcal{T}(x) = x$ whenever $|x| \leq R_0$. Thus a particular definition, with R_0 defined as above, is the following. Consider

$$\mathcal{T}_\mu(x) = \begin{cases} x & \text{if } \ 0 \leq x \leq R_0, \\ \dfrac{(1+\mu R_0)(x-R_0)+R_0}{\mu(x-R_0)+1} & \text{if } \ x > R_0. \end{cases}$$

where $\mu > \frac{1}{1-R_0}$. Notice that $\mathcal{T}_\mu(x)$ is a C^1-function on its domain with $\mathcal{T}_\mu'(R_0) = 1$ since

$$\mathcal{T}_\mu'(x) = \begin{cases} 1 & \text{if } \ 0 \leq x \leq R_0, \\ \dfrac{1}{(\mu(x-R_0)+1)^2} & \text{if } \ x > R_0. \end{cases}$$

Moreover, $\mathcal{T}_\mu(x)$ is a monotone increasing function with $\sup \mathcal{T}_\mu = R_0 + \frac{1}{\mu} < 1$. Therefore we can set

$$\mathcal{T}(x) = \begin{cases} \mathcal{T}_\mu(x) & \text{if } \ x \geq 0, \\ -\mathcal{T}_\mu(-x) & \text{if } \ x < 0. \end{cases} \tag{3.3}$$

Thus, the truncated equation for (1.2) is given by

$$\ddot{z} = H(t,z),\; H(t,z) = -z\hat{G}(t,z) \tag{3.4}$$

where

$$\hat{G}(t,z) = \begin{cases} \dfrac{\mathcal{T}(z)}{z} G(t,\mathcal{T}(z)) & \text{if } z \neq 0, \\ G(t,0) & \text{if } z = 0. \end{cases}$$

and

$$G(t,x) = 1 + \alpha x^2 - \frac{4\beta\mathcal{V}^2(t)}{(1-x^2)^2}.$$

It is not difficult to prove that $H \in C^{0,1}(\mathbb{R}\times\mathbb{R})$. On the other hand, notice that (3.4) and (1.2) coincide for $|z| \leq R_0$. Moreover, equation (3.4) satisfies the hypotheses of Lemma 3.1 since for $z \geq R_0$ we have that $R_0 \leq \mathcal{T}(z) < 1$. Thus the definition of r_0 and R_0 imply that for $z \geq R_0$

$$G(t,\mathcal{T}(z)) = \frac{\phi(\mathcal{T}(z)) - 4\beta\mathcal{V}^2(t)}{(1-\mathcal{T}^2(z))^2} < 0. \tag{3.5}$$

Therefore $H(t,z) > 0$ for $z \geq R_0$. On the other hand, $-1 < \mathcal{T}(z) \leq -R_0$ whenever $z \leq -R_0$, thus $G(t, \mathcal{T}(z)) < 0$ again, and therefore $H(t,z) < 0$ for $z \leq -R_0$. Lemma 3.1 implies that all mT-periodic solution of (3.4) has its range contained in $[-R_0, R_0]$, then both equations have the same periodic solutions. So, we shall apply Proposition 2.1 to the Dirichlet problem

$$\begin{cases} \ddot{z} + z\hat{G}(t,z) = 0, \\ z(0) = z(L) = 0. \end{cases}$$

in order to reach the conclusion.

Towards to the hypotheses of Ortega's Principle: The function $H(t,z)$ is bounded since for all $(t,z) \in \mathbb{R}^2$ we have by construction that

$$|H(t,z)| \leq H_\infty := \gamma \left(1 + \alpha\gamma^2 + \frac{4\beta\mathcal{V}_M^2}{(1-\gamma^2)^2}\right), \tag{3.6}$$

for $\gamma = R_0 + \frac{1}{\mu}$.

On the other hand, consider the following function for $w \in [0,1[$

$$S(w) = \frac{2-w}{(1-w)^2}.$$

Hence we have that

$$S'(w) = \frac{3-w}{(1-w)^3} > 0,$$

and thus $S(w)$ reaches to its minimum value at $w = 0$ with $S(0) = 2$. Then $0 < \alpha \leq 8\beta\mathcal{V}_m^2$ leads to

$$0 < \alpha \leq 4\beta\mathcal{V}_m^2 \min_{u \geq 0} S(u).$$

Thus for $z \neq 0$ we obtain that

$$0 < \alpha < 4\beta\mathcal{V}^2(t)\frac{(2-z^2)}{(1-z^2)^2},$$

$$\frac{\alpha z^2(1-z^2)^2 + 4\beta\mathcal{V}^2(t)(z^4 - 2z^2)}{(1-z^2)^2} < 0,$$

$$\alpha z^2 + 4\beta\mathcal{V}^2(t)\left(1 - \frac{1}{(1-z^2)^2}\right) < 0.$$

In consequence, for all $t \in \mathbb{R}$ and $z \neq 0$ such that $|z| < R_0$ we have that

$$\hat{G}(t,z) = 1 + \alpha z^2 - \frac{4\beta\mathcal{V}^2(t)}{(1-z^2)^2} < 1 - 4\beta\mathcal{V}^2(t) = \hat{G}(t,0),$$

and for $|z| \geq R_0$ the above inequality is trivial because $sign\, \hat{G}(t,z) = sign\, G(t, \mathcal{T}(z)) = -1$ (see (3.5) and the subsequent claim), and $\hat{G}(t,0)$ is positive by hypothesis a) of Theorem 2.2.

Towards the uniqueness: For $x \in]0,1[$ we have that the condition *b)* implies that

$$2\alpha - \frac{16\beta\mathcal{V}_m^2}{(1-x^2)^3} < 2\alpha - 16\beta\mathcal{V}_m^2 \leq 0,$$

because $s(x) = (1 - x^2)^3$ reaches to its maximum value at $x = 0$ and $s(0) = 1$. Notice that Proposition 5 in [14] is still true for an open z-interval of positive numbers. Hence, a direct application of this result leads to the conclusion since for all $t \in \mathbb{R}$ and $x \in]0, 1[$ we have that

$$G_x(t, x) = x\left(2\alpha - \frac{16\beta\mathcal{V}^2(t)}{(1 - x^2)^3}\right) \leq x\left(2\alpha - \frac{16\beta\mathcal{V}_m^2}{(1 - x^2)^3}\right) < 0.$$

□

Remark 3.2. We notice that the last proof requires to find *a priori bounds* of periodic solutions for both the original equation and its equivalent modified equation, nevertheless, singularities of the original equation prevent the direct application of known results for *a priori bounds* using the so called Hartman conditions [3]. For this reason we have introduced the Lemma 3.1.

Remark 3.3. The main results in this section could be dependent on the particular truncating that is employed with the required properties, and therefore they provide certain flexibility for searching odd periodic oscillations.

4. Numerical Examples

In this section we provide numerical simulations in order to illustrate the main results of this paper. We consider the following data for the truncated differential equation (3.4) with $\mu = 1.0 \times 10^3$, $\epsilon \approx 8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$, $\mathfrak{m} = 9.33 \times 10^{-11}\,\mathrm{kg}$, $l = 100\,\mathrm{\mu m}$, $e = 10.0\,\mathrm{\mu m}$, $d = 1.0\,\mathrm{\mu m}$, $k = 1.0\,\mathrm{N\,m^{-1}}$ and $c = 3.10 \times 10^9\,\mathrm{N\,m^{-3}}$.

Thus $\alpha = 3.10 \times 10^{-3}$ and $\beta \approx 4.427 \times 10^{-3}\,\mathrm{V^{-2}}$. Furthermore, we fix the inputs m, N_0 for Corollary 2.3 and consider the period

$$T = 2\pi\left(\frac{m}{N_0 + 1} - \theta_\varepsilon\right)^{-1} \text{ with } \theta_\varepsilon = 0.1,$$

and $\mathcal{V}_m$, $\mathcal{V}_M$ as the extremes values of the interval given by Corollary 2.3.

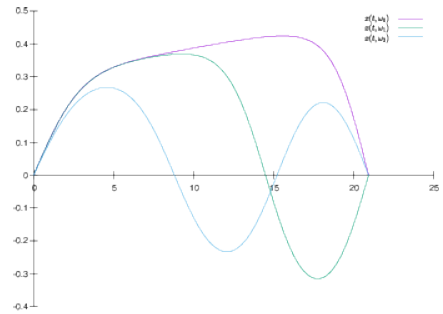
The panel figure 3 sums up the results regarding to the case $m = 1$ and $N_0 = 3$ as follows. Table in 3a shows the initial velocities $\omega_N = \dot{\varphi}_N(0)$. Figure 3b shows the graph of the solutions $\varphi_N(t)$ in the interval $\left[0, \frac{mT}{2}\right]$ for $N = 0, 1, 2$. Finally, figure 3c shows the associated Stroboscopic map on the left, and a close-up to some regions of interest that contain the fixed points corresponding to the odd T-periodic solutions on the right. The Stroboscopic map, was performed using the Taylor method with automatic differentiation, with relative and absolute error of order 10.0×10^{-12}, this method offers high precision and is very fast [10].

We recall that those ω_N in table 3a agree with the critical velocities for which the number of zeros of the solution of the associated Dirichlet problem in $\left]0, \frac{mT}{2}\right[$ for the truncated differential equation changes. In order to compute them we have performed a strategy of bisection.

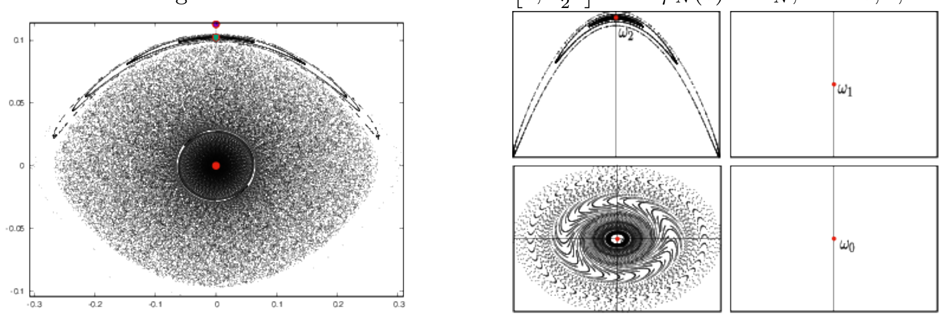
On the other hand, table 1 shows the values of N_0 and N for which the odd mT-periodic solutions of equation (1.2) $\varphi_N(t)$ obtained in Corollary 2.3 are linearly Lyapunov stable. This conclusion follows from the numerical evidence provided by the Stroboscopic map, since we can observe the appearance of invariant closed curves around of fixed points of the form $(0, \omega_N)$ for some N (see figure 3c).

N	ω_N
2	$1.025\,594\,466\,101\,665\,5 \times 10^{-1}$
1	$1.129\,623\,793\,301\,107\,2 \times 10^{-1}$
0	$1.130\,395\,831\,096\,393\,5 \times 10^{-1}$

(A) Critical velocities ω_N for which the number of zeros changes.



(B) Solutions $\varphi_N(t)$ in the interval $\left[0, \frac{mT}{2}\right]$ with $\dot{\varphi}_N(0) = \omega_N$, $N = 0, 1, 2$.



(C) Stroboscopic map for $m = 1,\ N_0 = 3$

FIGURE 3. Table and graphics about of critical velocities for $m = 1, N_0 = 3$.

Notice that because of the Hamiltonian character of (1.2) its associated Stroboscopic map is area preserving, and thus the stability of its fixed points is marginal or neutral type and can be studied by observing simple closed invariant curves around of them (see [16]). Additionally, we complement this study by computing the Floquet multipliers for the linearization at each periodic solution through a numerical software, for example *Auto* (see [11] for more details). Hence, for the cases in table 1 we obtain that the Floquet multipliers are complex numbers with modulus equals to 1, and thus the corresponding solutions are elliptic or linearly stable. The other solutions have multipliers outside of unit circle and therefore are unstable.

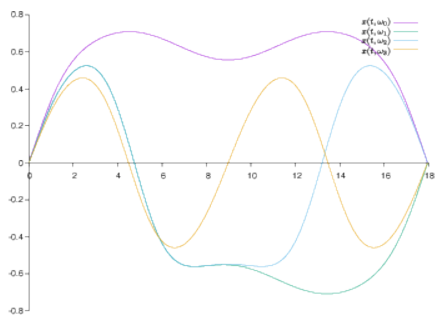
We have also included the case $m = 4$, $N_0 = 4$ in panel figure 4. Table in 4a shows the initial velocities $\omega_N = \dot{\varphi}_N(0)$. Figure 4b shows the graph of the solutions $\varphi_N(t)$ in the interval $\left[0, \frac{mT}{2}\right]$, for $N = 0, 1, 2, 3$. Finally, figure 4c shows the associated Stroboscopic map on the left, and a close-up to the fixed points corresponding to the odd mT-periodic solutions on the right.

m	N_0	N
1	3	2
	4	3
	5	3
	6	4
	7	6

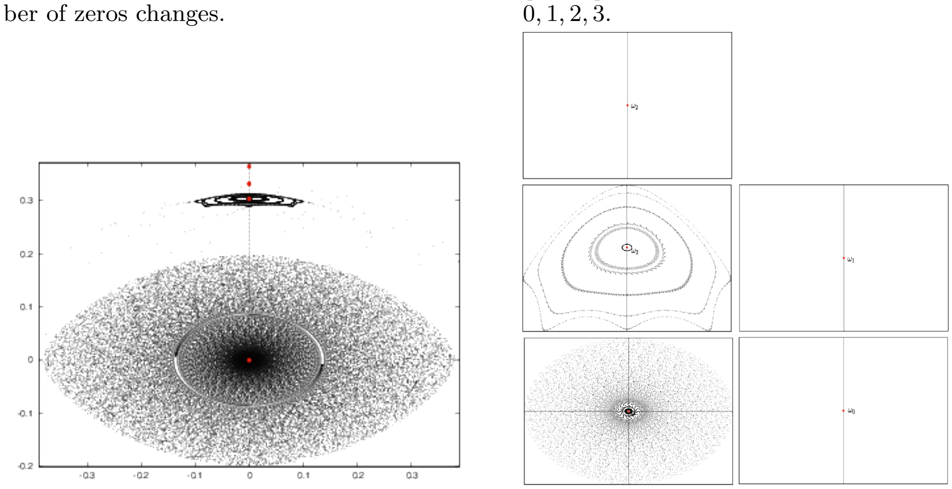
TABLE 1. Values of N_0 and N for which there exist stable odd mT-periodic solutions of (1.2) with N zeros in half period of the voltage.

N	ω_N
3	$3.032\,408\,680\,824\,740\,8 \times 10^{-1}$
2	$3.314\,532\,595\,408\,081\,8 \times 10^{-1}$
1	$3.315\,276\,475\,167\,028\,3 \times 10^{-1}$
0	$3.640\,908\,005\,638\,721\,1 \times 10^{-1}$

(A) Critical velocities ω_N for which the number of zeros changes.



(B) Solutions $\varphi_N(t)$ in the interval $\left[0, \frac{mT}{2}\right]$ with $\dot{\varphi}_N(0) = \omega_N$, $N = 0, 1, 2, 3$.



(C) Stroboscopic map for $m = 4,\ N_0 = 4$

FIGURE 4. Table and graphics about of critical velocities for $m = 4,\ N_0 = 4$.

5. Concluding Remarks

Next we discuss the the main contribution of this work regarding to the equation (1.2). Given m and n positive integers there exist a suitable adjustment for the extreme values of the input voltage, its frequency and the cubic stiffness coefficient α (the latter below certain quantities) such that the movable electrode in a transverse in-plane Comb-drive device exhibits odd mT-periodic responses with j zeros in $J_m = \left]0, \frac{mT}{2}\right[$ for each $j = 0, 1, \ldots n-1$. Additionally, the solution with no zeros in the interval J_m is unique and therefore it has minimal period mT.

Our approach is elementary because it is based on the Sturm Comparison theory and the continuity properties respect to initial conditions in ODEs. The fundamental idea is due to R. Ortega [14] and consists in the following variational principle applied to the particular truncating of (1.2) employed in our proof: the solution starting from $x = 0$ with minimum kinetic energy between those solutions with N inner zeros in a half period of the voltage, is odd and periodic. Thus, the results could depend on the particular truncation, however this approach provides some flexibility for searching odd periodic solutions.

The stability properties of these periodic solutions for a particular truncating of this type are unknown. The numerical evidence shows that some of them could be elliptic (see table 1). We will consider this problem from the analytical approach in future works.

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