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KOMATU INTEGRAL OPERATOR RELATED TO ANALYTIC FUNCTIONS

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ABSTRACT. The focus of this article is the introduction of a new sub-class of analytical functions involving Komatu integral operator and obtained coefficient bounds, distortion bounds as well as convex linear combination and partial sums for this class.

1. INTRODUCTION

Let A specify the category of analytical functions η represent on the unit disc $\Upsilon = \{w : |w| < 1\}$ with normalization $\eta(0) = 0$ and $\eta'(0) = 1$, such a function has the extension of the Taylor series on the origin in the form

$$\eta(w) = w + \sum_{\nu=2}^{\infty} a_\nu w^\nu. \tag{1.1}$$

Indicated by S, the subclass of A be composed of functions that are univalent in Υ.

Then a $\eta(w)$ function of A is known as starlike and convex of order ϑ if it delights the pursing

$$\Re\left\{\frac{w\eta'(w)}{\eta(w)}\right\} > \vartheta,\ (w \in \Upsilon), \tag{1.2}$$

$$\text{and } \Re\left\{1 + \frac{w\eta''(w)}{\eta'(w)}\right\} > \vartheta,\ (w \in \Upsilon), \tag{1.3}$$

for specific $\vartheta(0 \leq \vartheta < 1)$ respectively and we express by $S^*(\vartheta)$ and $K(\vartheta)$ the subclass of A be expressed by aforesaid functions respectively. Also, indicate by T the subclass of A made up of functions of this form

$$\eta(w) = w - \sum_{\nu=2}^{\infty} a_\nu w^\nu,\ (a_\nu \geq 0,\ w \in \Upsilon) \tag{1.4}$$

and let $T^*(\vartheta) = T \cap S^*(\vartheta), C(\vartheta) = T \cap K(\vartheta)$. There are interesting properties in the $T * (\vartheta)$ and $C(\vartheta)$ classes and were thoroughly studied by Silverman [14] and others.

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Recently, Komatu [5] added an integral operator $I_{\wp}^{\ell}$ indicated by

$$I_{\wp}^{\ell}\eta(w)=\frac{\wp^{\ell}}{\Gamma(\ell)}\int_{0}^{1}t^{\wp-2}\left(log\frac{1}{t}\right)^{\ell-1}\eta(wt)dt, \tag{1.5}$$

where $\wp>0,\ell\geq 0$ and $w\in\Upsilon$.

Thus, $\eta\in A$ is of the form (1.1), so it's easy to find out (1.5) that (see [5])

$$I_{\wp}^{\ell}\eta(w)=w+\sum_{\nu=2}^{\infty}\Omega_{\nu}(\wp,\ell)a_{\nu}w^{\nu}, \tag{1.6}$$

where $\Omega_{\nu}(\wp,\ell)=\left(\frac{\wp}{\wp+\nu-1}\right)^{\ell}$.

From (1.5), it is quite obvious that

$$w\left(I_{\wp}^{\ell}\eta(w)\right)'=(\wp+1)I_{\wp}^{\ell-1}\eta(w)-\wp I_{\wp}^{\ell}\eta(w)$$

and

$$w^{2}\left(I_{\wp}^{\ell}\eta(w)\right)''=(\wp+1)^{2}I_{\wp}^{\ell-2}\eta(w)-(2\wp+1)(\wp+1)I_{\wp}^{\ell-1}\eta(w)+\wp(\wp+1)I_{\wp}^{\ell}\eta(w).$$

We point out that

(i). for $\wp=1$ and $\ell=\gamma(\gamma$ is an integer), the multiplier transformation $I_{1}^{\gamma}\eta(w)=I^{\gamma}\eta(w)$ was studied by Flett [2]

(ii). for $\wp=1$ and $\ell=-\gamma(\gamma\in N)$, the differential operator $I_{1}^{-\gamma}\eta(w)=D^{\gamma}\eta(w)$ was examined by Salagean [12]

(iii). for $\wp=2$ and $\ell=\gamma(\gamma$ is an integer), the operator $I_{2}^{\ell}\eta(w)=I^{\gamma}\eta(w)$ was considered [16]

(iv). for $\wp=2$, $I_{2}^{\ell}\eta(w)=I^{\ell}\eta(w)$ was examined by Jung et al. [4].

As a result of the work of see ([1, 6, 7, 9, 10]), we propose a new subclass $\phi_{\wp}^{\ell}(\hbar,\vartheta)$ of A concerning Komatu integral operator [5] as below:

Definition 1.1. For $0\leq\hbar<1, 0\leq\vartheta<1,\wp>0,\ell>0$, we say $\eta(w)\in A$ is in $\phi_{\wp}^{\ell}(\hbar,\vartheta)$ if it fulfils the requirement

$$\Re\left(\frac{w\left(I_{\wp}^{\ell}\eta(w)\right)'+\hbar w^{2}\left(I_{\wp}^{\ell}\eta(w)\right)''}{I_{\wp}^{\ell}\eta(w)}\right)>\vartheta,\ (w\in\Upsilon). \tag{1.7}$$

Also we indicate by $T\phi_{\wp}^{\ell}(\hbar,\vartheta)=\phi_{\wp}^{\ell}(\hbar,\vartheta)\cap T$.

2. Coefficient Inequalities

This section gives us an adequate requirement for a function η given by (1.1) to be in $\phi_{\wp}^{\ell}(\hbar,\vartheta)$.

Theorem 2.1. *A function $\eta\in A$ is assigned to the class $\phi_{\wp}^{\ell}(\hbar,\vartheta)$ if*

$$\sum_{\nu=2}^{\infty}[\nu+\hbar\nu(\nu-1)-\vartheta]\Omega_{\nu}(\wp,\ell)|a_{\nu}|\leq 1-\vartheta. \tag{2.1}$$

Proof. Since $0 \leq \vartheta < 1$ and $\hbar \geq 0$, now if we put

$$\varrho(w) = \frac{w\left(I_{\wp}^{\ell}\eta(w)\right)' + \hbar w^2\left(I_{\wp}^{\ell}\eta(w)\right)''}{I_{\wp}^{\ell}\eta(w)},\ (w \in \Upsilon)$$

Then it's just a matter of proving it $|\varrho(w) - 1| < 1 - \vartheta, (w \in \Upsilon)$.
Indeed if $\eta(w) = w\ (w \in \Upsilon)$, then we have $\varrho(w) = w (w \in \Upsilon)$.
Implies (2.1) holds.
If $\eta(w) \neq w\ (|w| = r < 1)$, then there exist a coefficient $\Omega_\nu(\wp, \ell)a_\nu \neq 0$ for some $\nu \geq 2$. The consequence is that $\sum_{\nu=2}^{\infty} \Omega_\nu(\wp, \ell)|a_\nu| > 0$. Further note that

$$\sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp, \ell)|a_\nu| > (1-\vartheta)\sum_{\nu=2}^{\infty}\Omega_\nu(\wp, \ell)|a_\nu|$$

$$\Rightarrow \sum_{\nu=2}^{\infty}\Omega_\nu(\wp, \ell)|a_\nu| < 1.$$

By (2.1), we obtain

$$\begin{aligned}
|\varrho(w) - 1| &= \left|\frac{\sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - 1]\Omega_\nu(\wp, \ell)a_\nu w^{\nu-1}}{1 + \sum_{\nu=2}^{\infty}\Omega_\nu(\wp, \ell)a_\nu w^{\nu-1}}\right| \\
&< \frac{\sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - 1]\Omega_\nu(\wp, \ell)|a_\nu|}{1 - \sum_{\nu=2}^{\infty}\Omega_\nu(\wp, \ell)|a_\nu|} \\
&\leq \frac{\sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp, \ell)|a_\nu| - (1-\vartheta)\Omega_\nu(\wp, \ell)|a_\nu|}{1 - \sum_{\nu=2}^{\infty}\Omega_\nu(\wp, \ell)|a_\nu|} \\
&\leq \frac{(1-\vartheta) - (1-\vartheta)\sum_{\nu=2}^{\infty}\Omega_\nu(\wp, \ell)|a_\nu|}{1 - \sum_{\nu=2}^{\infty}\Omega_\nu(\wp, \ell)|a_\nu|} \\
&= 1 - \vartheta,\ (w \in \Upsilon).
\end{aligned}$$

Hence we obtain

$$\Re\left(\frac{w\left(I_{\wp}^{\ell}\eta(w)\right)' + \hbar w^2\left(I_{\wp}^{\ell}\eta(w)\right)''}{I_{\wp}^{\ell}\eta(w)}\right) = \Re(\varrho(w)) > 1 - (1-\vartheta) = \vartheta, .$$

Then $\eta \in \phi_{\wp}^{\ell}(\hbar, \vartheta)$. . ∎

Theorem 2.2. *Let η be given by (1.4). Then the function $\eta \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$*

$$\Leftrightarrow \sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp, \ell)|a_\nu| \leq 1 - \vartheta. \tag{2.2}$$

Proof. In view of Theorem 2.1, to examine it $\eta \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ fulfils the coefficient inequality (2.1). If $\eta \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ then the function

$$\varrho(w) = \frac{w\left(I_{\wp}^{\ell}\eta(w)\right)' + \hbar w^2\left(I_{\wp}^{\ell}\eta(w)\right)''}{I_{\wp}^{\ell}\eta(w)}, \ (w \in \Upsilon)$$

satisfies $\Re(\varrho(w)) > \vartheta$. This implies that

$$I_{\wp}^{\ell}\eta(w) = w - \sum_{\nu=2}^{\infty} \Omega_\nu(\wp, \ell)|a_\nu| w^\nu \neq 0; (w \in \Upsilon \setminus \{0\}).$$

Noting that $\frac{I_{\wp}^{\ell}\eta(r)}{r}$ in the open interval $(0, 1)$, this is the real continuous function with $\eta(0) = 1$, we have

$$\frac{I_{\wp}^{\ell}\eta(r)}{r} = 1 - \sum_{\nu=2}^{\infty} \Omega_\nu(\wp, \ell)|a_\nu| r^{\nu-1} > 0, \quad (0 < r < 1). \tag{2.3}$$

Now $\vartheta < \varrho(r) = \frac{1 - \sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1)]\Omega_\nu(\wp,\ell)|a_\nu| r^{\nu-1}}{1 - \sum_{\nu=2}^{\infty}\Omega_\nu(\wp,\ell)|a_\nu| r^{\nu-1}}$ and consequently by (2.3),

we get $\sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp, \ell)|a_\nu| r^{\nu-1} \leq 1 - \vartheta$.

Letting $r \to 1$, we get $\sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp, \ell)|a_\nu| \leq 1 - \vartheta$.

This proves the converse part. □

Remark 2.3. If a function η of the form (1.2) belongs to the class $T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ then

$$|a_\nu| \leq \frac{1 - \vartheta}{[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp, \ell)}, \quad (\nu \geq 2).$$

The equality holds for the functions

$$\eta_\nu(w) = w - \frac{1 - \vartheta}{[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp, \ell)} w^n, (w \in \Upsilon, \nu \geq 2). \tag{2.4}$$

3. Distortion Theorem

In the section, the distortion limits of the functions owned by the class $T\phi_{\wp}^{\ell}(\hbar, \vartheta)$.

Theorem 3.1. *Let $\eta \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ and $|w| = r < 1$. Then*

$$r - \frac{1 - \vartheta}{[2\hbar - \vartheta + 2]\Omega_\nu(\wp, \ell)} r^2 \leq |\eta(w)| \leq r + \frac{1 - \vartheta}{[2\hbar - \vartheta + 2]\Omega_\nu(\wp, \ell)} r^2 \tag{3.1}$$

and

$$1 - \frac{2(1 - \vartheta)}{[2\hbar - \vartheta + 2]\Omega_\nu(\wp, \ell)} r \leq |\eta'(w)| \leq 1 + \frac{2(1 - \vartheta)}{[2\hbar - \vartheta + 2]\Omega_\nu(\wp, \ell)} r. \tag{3.2}$$

The approximation is sharp, with the $\eta_2(w)$ extreme function indicated by (2.4).

Proof. Since $\eta \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$, we apply Theorem 2.2 to attain

$$[2\hbar - \vartheta + 2]\Omega_\nu(\wp, \ell) \sum_{\nu=2}^{\infty} |a_\nu| \leq \sum_{\nu=2}^{\infty} [\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp, \ell)|a_\nu| \leq 1 - \vartheta.$$

$$\text{Thus } |\eta(w)| \leq |w| + |w|^2 \sum_{\nu=2}^{\infty} |a_\nu| \leq r + \frac{1-\vartheta}{[2\hbar - \vartheta + 2]\Omega_\nu(\wp, \ell)}\ r^2.$$

$$\text{Also we have, } |\eta(w)| \leq |w| - |w|^2 \sum_{\nu=2}^{\infty} |a_\nu| \leq r - \frac{1-\vartheta}{[2\hbar - \vartheta + 2]\Omega_\nu(\wp, \ell)}\ r^2,$$

and (3.1) follows. In similar way for η', the inequalities

$$|\eta'(w)| \leq 1 + \sum_{\nu=2}^{\infty} \nu|a_\nu||w|^{\nu-1} \leq 1 + |w| \sum_{\nu=2}^{\infty} \nu|a_\nu|$$

and

$$\sum_{\nu=2}^{\infty} \nu|a_\nu| \leq \frac{2(1-\vartheta)}{[2\hbar - \vartheta + 2]\Omega_\nu(\wp, \ell)}$$

are satisfied, which leads to (3.2). □

4. Radii of close-to-convexity and starlikeness

A close-to-convex and star-like radius of this class $T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ is obtained in this section.

Theorem 4.1. *Let η be specified by (1.4) is in $T\phi_{\wp}^{\ell}(\hbar, \vartheta)$. Then η is the order of close-to-convex ℓ $(0 \leq \ell < 1)$ in the disc $|w| < t_1$, where*

$$t_1 = \inf_{\nu \geq 2} \left[\frac{(1-\ell)[\nu + \nu\hbar(\nu-1) - \vartheta]\Omega_\nu(\wp, \ell)}{\nu(1-\vartheta)} \right]^{\frac{1}{\nu-1}}. \tag{4.1}$$

The estimate is sharp with the extremal function $\eta(w)$ is indicated by (2.4).

Proof. If $\eta \in T$ and η isorder of close-to-convex ℓ then we get

$$|\eta'(w) - 1| \leq 1 - \ell. \tag{4.2}$$

For the L.H.S of (4.2), we obtain

$$|\eta'(w) - 1| \leq \sum_{\nu=2}^{\infty} \nu a_\nu |w|^{\nu-1} < 1 - \ell$$

$$\Rightarrow \sum_{\nu=2}^{\infty} \frac{\nu}{1-\ell} a_\nu |w|^{\nu-1} \leq 1.$$

We know that $\eta(w) \in T\phi_{\wp}^{\ell}(\hbar, \vartheta) \Leftrightarrow$

$$\sum_{\nu=2}^{\infty} \frac{[\nu + \nu\hbar(\nu-1) - \vartheta]\Omega_\nu(\wp, \ell)}{(1-\vartheta)} a_\nu \leq 1.$$

Thus (4.2) holds true if

$$\frac{\nu}{1-\ell}|w|^{\nu-1} \leq \frac{[\nu+\nu\hbar(\nu-1)-\vartheta]\Omega_\nu(\wp,\ell)}{(1-\vartheta)}$$

$$\Rightarrow \quad |w| \leq \left[\frac{(1-\ell)[\nu+\nu\hbar(\nu-1)-\vartheta]\Omega_\nu(\wp,\ell)}{\nu(1-\vartheta)}\right]^{\frac{1}{\nu-1}}$$

hence the proof. □

Theorem 4.2. *Let $\eta \in T\phi_\wp^\ell(\hbar,\vartheta)$. Then η is order of starlike ℓ, $(0 \leq \ell < 1)$ in the disc $|w| < t_2$, where*

$$t_2 = \inf_{\nu\geq 2}\left[\frac{(1-\ell)[\nu+\nu\hbar(\nu-1)-\vartheta]\Omega_\nu(\wp,\ell)}{(\nu-\ell)(1-\vartheta)}\right]^{\frac{1}{\nu-1}}. \tag{4.3}$$

The estimate is sharp with the extremal function $\eta(w)$ is indicated by (2.4).

Proof. We have $\eta \in T$ and η is order of starlike ℓ, we have

$$\left|\frac{w\eta'(w)}{\eta(w)}-1\right| < 1-\ell. \tag{4.4}$$

For the L.H.S of (4.4), we have

$$\left|\frac{w\eta'(w)}{\eta(w)}-1\right| \leq \frac{\sum\limits_{\nu=2}^{\infty}(\nu-1)a_\nu|w|^{\nu-1}}{1-\sum\limits_{\nu=2}^{\infty}a_\nu|w|^{\nu-1}}$$

$(1-\ell)$ is bigger than the R.H.S of the left relation if

$$\sum_{\nu=2}^{\infty}\frac{\nu-\ell}{1-\ell}a_\nu|w|^{\nu-1} < 1.$$

We know that $\eta \in T\phi_\wp^\ell(\hbar,\vartheta)\nu$

$$\Leftrightarrow \sum_{\nu=2}^{\infty}\frac{[\nu+\nu\hbar(\nu-1)-\vartheta]\Omega_\nu(\wp,\ell)}{(1-\vartheta)}a_\nu \leq 1.$$

Thus (4.4) is true if

$$\frac{\nu-\ell}{1-\ell}|w|^{\nu-1} \leq \frac{[\nu+\nu\hbar(\nu-1)-\vartheta]\Omega_\nu(\wp,\ell)}{(1-\vartheta)}$$

$$\Rightarrow |w| \leq \left[\frac{(1-\ell)[\nu+\nu\hbar(\nu-1)-\vartheta]\Omega_\nu(\wp,\ell)}{(\nu-\ell)(1-\vartheta)}\right]^{\frac{1}{\nu-1}}.$$

It yield starlikeness of the family. □

5. Convex Linear combinations

Theorem 5.1. *Let* $\eta_1(w) = w$ *and*

$$\eta_\nu(w) = w - \frac{1-\vartheta}{[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp,\ell)} w^\nu, \quad (w \in \Upsilon, \nu \geq 2). \tag{5.1}$$

Then $\eta \in T\phi_\wp^\ell(\hbar,\vartheta) \Leftrightarrow \eta$ *in the way it can be expressed*

$$\eta(w) = \sum_{\nu=1}^{\infty} \mu_\nu \eta_\nu(w), \ (\mu_\nu \geq 0) \tag{5.2}$$

and $\sum\limits_{\nu=1}^{\infty} \mu_\nu = 1.$

Proof. If a function η is of the form $\eta(w) = \sum\limits_{\nu=1}^{\infty} \mu_\nu \eta_\nu(w), \mu_\nu \geq 0$ and $\sum\limits_{\nu=1}^{\infty} \mu_\nu = 1$ then

$$\begin{aligned}
&\sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp,\ell)|a_\nu| \\
=&\sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp,\ell)\frac{(1-\vartheta)\mu_\nu}{[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp,\ell)} \\
=&\sum_{\nu=2}^{\infty}(1-\vartheta)\mu_\nu = (1-\mu_1)(1-\vartheta) \\
\leq&(1-\vartheta)
\end{aligned}$$

which provides (2.2), hence $\eta \in T\phi_\wp^\ell(\hbar,\vartheta)$, by Theorem 2.2.

On the other hand, if η is in the class $\eta \in T\phi_\wp^\ell(\hbar,\vartheta)$, then we may set

$$\mu_\nu = \frac{[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp,\ell)}{1-\vartheta}|a_\nu|, \quad (\nu \geq 2),$$

and $\mu_1 = 1 - \sum\limits_{\nu=2}^{\infty} \mu_\nu$.

Then the function η is of the form (5.2) .

6. Partial Sums

Silverman [14] examined partial sums η for the function $\eta \in A$ given by (1.1) Established Through

$$\eta_1(w) = w \text{ and } \eta_m(w) = w + \sum_{\nu=2}^{m} a_\nu w^\nu, m = 2,3,4,\cdots. \tag{6.1}$$

In this paragraph, In the class $\phi_\wp^\ell(\hbar,\vartheta)$, partial function sums can be considered and sharp lower limits can be reached for the function. True component ratios of η to η_m and η' to η'_m.

Theorem 6.1. *Let* $\eta \in \phi_{\wp}^{\ell}(\hbar, \vartheta)$ *and fulfils (2.1). Then*

$$\Re\left(\frac{\eta(w)}{\eta_m(w)}\right) \geq 1 - \frac{1}{d_{m+1}}, (w \in \Upsilon, m \in N), \tag{6.2}$$

where

$$d_\nu = \frac{[\nu + \hbar\nu(\nu-1) - \vartheta]}{1-\vartheta}. \tag{6.3}$$

Proof. Clearly, $d_{\nu+1} > d_\nu > 1, \nu = 2, 3, 4, \cdots$.

Thus by Theorem 2.1 we get,

$$\sum_{\nu=2}^{\infty} |a_\nu| + d_{m+1} \sum_{\nu=2}^{\infty} |a_\nu| \leq \sum_{\nu=2}^{\infty} d_\nu |a_\nu| \leq 1. \tag{6.4}$$

Setting $g(w) = d_{m+1}\left\{\frac{\eta(w)}{\eta_m(w)} - \left(1 - \frac{1}{d_{m+1}}\right)\right\}$

$$g(w) = 1 + \frac{d_{m+1} \sum\limits_{\nu=m+1}^{\infty} a_\nu w^{\nu-1}}{1 + \sum\limits_{\nu=2}^{m} a_\nu w^{\nu-1}} \tag{6.5}$$

it be good enough to show $\Re(g(w)) > 0, w \in \Upsilon$. Applying (6.4) we think that

$$\left|\frac{g(w)-1}{g(w)+1}\right| \leq \frac{d_{m+1} \sum\limits_{\nu=2}^{\infty} |a_\nu|}{2 - 2\sum\limits_{\nu=2}^{m} |a_\nu| - d_{m+1} \sum\limits_{\nu=m+1}^{\infty} |a_\nu|}$$
$$\leq 1,$$

which gives,

$$\Re\left(\frac{\eta(w)}{\eta_m(w)}\right) \geq 1 - \frac{1}{d_{m+1}},$$

hence the proof.

Theorem 6.2. *Let* η *in* $T\phi_{\wp}^{\ell}(\hbar, \vartheta)$ *and fulfils (2.1). Then*

$$\Re\left(\frac{\eta_m(w)}{\eta(w)}\right) \geq \frac{d_{m+1}}{1 + d_{m+1}}, \quad (w \in \Upsilon, m \in N), \tag{6.6}$$

where

$$d_\nu = \frac{[\nu + \hbar\nu(\nu-1) - \vartheta]}{1-\vartheta}. \tag{6.7}$$

Proof. Clearly, $d_{\nu+1} > d_\nu > 1, \nu = 2, 3, 4, \cdots$.

Thus by Theorem 2.1 we get,

$$\sum_{\nu=2}^{\infty}|a_\nu| + d_{m+1}\sum_{\nu=m+1}^{\infty}|a_\nu| \le \sum_{\nu=2}^{\infty} d_\nu|a_\nu| \le 1. \tag{6.8}$$

Setting $h(w) = (1+d_{m+1})\left\{\frac{\eta_m(w)}{\eta(w)} - \left(\frac{d_{m+1}}{1+d_{m+1}}\right)\right\}$

$$h(w) = 1 - \frac{(1+d_{m+1})\sum\limits_{\nu=m+1}^{\infty} a_\nu w^{\nu-1}}{1+\sum\limits_{\nu=2}^{m} a_\nu w^{\nu-1}} \tag{6.9}$$

to show $\Re(h(w)) > 0, \quad (w \in \Upsilon)$. Implementing (6.8) we attain

$$\left|\frac{h(w)-1}{h(w)+1}\right| \le \frac{(1+d_{m+1})\sum\limits_{\nu=2}^{\infty}|a_\nu|}{2-2\sum\limits_{\nu=2}^{m}|a_\nu| - (1+d_{m+1})\sum\limits_{\nu=m+1}^{\infty}|a_\nu|}$$
$$\le 1,$$

which gives,

$$\Re\left(\frac{\eta_m(w)}{\eta(w)}\right) \ge \frac{d_{m+1}}{1+d_{m+1}},$$

and hence the proof.

Theorem 6.3. *Let η in $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ and fulfils (2.1). Then*

$$\Re\left(\frac{\eta'(w)}{\eta'_m(w)}\right) \ge 1 - \frac{m+1}{d_{m+1}}, \quad (w \in \Upsilon, m \in N), \tag{6.10}$$

and

$$\Re\left(\frac{\eta'_m(w)}{\eta'(w)}\right) \ge \frac{d_{m+1}}{m+1+d_{m+1}}, \quad (w \in \Upsilon, m \in N) \tag{6.11}$$

where

$$d_\nu = \frac{[\nu + \hbar\nu(\nu-1) - \vartheta]}{1-\vartheta}. \tag{6.12}$$

Proof. By Setting

$$g(w) = d_{m+1}\left\{\frac{\eta('w)}{\eta'_m(w)} - \left(1 - \frac{m+1}{d_{m+1}}\right)\right\}, \quad (w \in \Upsilon)$$

and $h(w) = (m+1+d_{m+1})\left\{\frac{\eta'_m(w)}{\eta('w)} - \left(\frac{d_{m+1}}{m+1+d_{m+1}}\right)\right\}, \quad (w \in \Upsilon),$

The evidence is close to that of the 6.1 and 6.2 theorems, so the specifics are omitted. □

7. Convolution properties

We will prove in this section that the $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ class is closed by convolution.

Theorem 7.1. *Let $g(w)$ of the form*

$$g(w) = w - \sum_{\nu=2}^{\infty} b_\nu w^\nu$$

be regular in Υ. If $\eta \in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ then the function $\eta * g$ is in the class $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$. Here the symbol $*$ denoted to the Hadmard product .

Proof. Since $\eta \in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$, we have

$$\sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp,\ell)|a_\nu| \leq 1 - \vartheta.$$

Employing the last inequality and the fact that

$$\eta(w) * g(w) = w - \sum_{\nu=2}^{\infty} a_\nu b_\nu w^\nu.$$

We obtain

$$\begin{aligned}&\sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp,\ell)|a_\nu||b_\nu| \\ \leq &\sum_{\nu=2}^{\infty}[\nu + \hbar\nu(\nu-1) - \vartheta]\Omega_\nu(\wp,\ell)|a_\nu| \\ \leq &1 - \vartheta\end{aligned}$$

and hence, in view of Theorem 2.1, the result follows.

8. Neighbourhood for the class $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$

Following [3, 11], we defined the $\alpha-$neighbourhood of the function $\eta(w) \in T$ by

$$N_\alpha(\eta) = \left\{ g \in T : g(w) = w - \sum_{\nu=2}^{\infty} b_\nu w^\nu \text{ and } \sum_{\nu=2}^{\infty} \nu|a_\nu - b_\nu| \leq \alpha \right\}. \tag{8.1}$$

Definition 8.1. The function $\eta \in A$ is defined in the class $T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ if the function $h \in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ occurs in such a way that the function is $h \in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$

$$\left|\frac{\eta(w)}{h(w)} - 1\right| < 1 - \gamma, \quad (w \in \Upsilon,\ 0 \leq \gamma < 1). \tag{8.2}$$

Theorem 8.2. *If $h \in T\phi_{\wp}^{\ell}(\hbar,\vartheta)$ and*

$$\gamma = 1 - \frac{\alpha(2\hbar - \vartheta + 2)\Omega_2(\wp,\hbar)}{2(2\hbar - \vartheta + 2)\Omega_2(\wp,\hbar) - (1+\vartheta)}$$

then $N_\alpha(h) \subseteq T\phi_{\wp}^{\ell;\gamma}(\hbar,\vartheta)$.

Proof. Let $\eta \in N_\alpha(h)$. We then find from that

$$\sum_{\nu=2}^{\infty} \nu|a_\nu - b_\nu| \leq \alpha,$$

which is easily implies the coefficient inequality

$$\sum_{\nu=2}^{\infty} |a_\nu - b_\nu| \leq \frac{\alpha}{\nu}.$$

Since $h \in T\phi_{\wp}^{\ell}(\hbar, \vartheta)$, we have from equation (2.1) that

$$\sum_{\nu=2}^{\infty} |a_\nu| \leq \frac{1-\vartheta}{(2\hbar - \vartheta + 2)\Omega_2(\wp, \hbar)}$$

and

$$\begin{aligned}\left|\frac{\eta(w)}{h(w)} - 1\right| &< \frac{\sum\limits_{\nu=2}^{\infty} \nu|a_\nu - b_\nu|}{1 - \sum\limits_{\nu=2}^{\infty} b_\nu} \\ &\leq \frac{\alpha}{2} \, \frac{(2\hbar - \vartheta + 2)\Omega_2(\wp, \hbar)}{(2\hbar - \vartheta + 2)\Omega_2(\wp, \hbar) - (1+\vartheta)} \\ &= 1 - \gamma.\end{aligned}$$

hence the proof.

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